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A CONSTRUCTION OF SUPERNILPOTENT RADICAL CLASSES

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Abstract

In a recent paper van Leeuwen and Heyman constructed a supernilpotent radical class using the class of almost nilpotent rings. Using a similar construction, for any class C satisfying the following four properties we obtain a supernilpotent radical class containing C.

(N1) C contains the class Z of all zero rings.

(N2) C is hereditary.

(N3) C is homomorphically closed.

(N4) If A and A/I are elements of C for some ideal I of a ring A, then $A \in C$.

Every supernilpotent radical class P clearly satisfies these conditions. For any such radical class we define the class of almost radical rings and use these to construct a new radical class P_2 which contains the given one. Also, we give a characterization for dual supernilpotent radicals.

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I

In a 1975 paper [7] van Leeuwen and Heyman introduced the class of almost nilpotent rings. These are rings each proper homomorphic image of which is nilpotent. Using this class they construct a supernilpotent radical L_2 which is independent of the Jacobson radical.

In this paper we give a general construction which yields the results of van Leeuwen and Heyman as a special case. For every class C which satisfies the properties:

(N1) C contains the class of all zero rings, (N2) C is hereditary,

(N3) C is homomorphically closed,

(N4) C has the extension property (that is, if $A/I \in C$ and $I \in C$ then $A \in C$), we give a construction of a supernilpotent radical which contains C. It is clear that the class of all nilpotent rings is an example of a class satisfying (N1) through (N4). As a matter of fact, any class C satisfying (N1) through (N4) also satisfies:

(N1)′ C contains the class of all nilpotent rings.

Properties (N1) through (N4) are clearly equivalent to (N1)′, (N2), (N3), and (N4).

Any supernilpotent radical class is also an example of a class satisfying (N1) through (N4). Any such radical class P will generate the construction of a supernilpotent radical class $P_2 \supset P$.

Throughout this text all rings considered will be associative, and the terminology and basic radical theoretic results used may be found in Divinsky [2].

II

DEFINITION 1. For any class C satisfying properties (N1) through (N4) the class C_1 is defined as follows:

1) If A is not subdirectly irreducible then $A \in C_1$ and only if each proper homomorphic image of A is in C.

2) If A is subdirectly irreducible then $A \in C_1$ if and only if $A \in C$.

C_1 will be called the class of *almost C-rings*.

Since C is homomorphically closed, it is clear that $C \subseteq C_1$ and that C_1 is also homomorphically closed. Note that for a subdirectly irreducible ring in C_1, it is also true that each of its proper homomorphic images is in C. It is not clear in general whether C_1 contains any rings which are not in C; however, the class C of all nilpotent rings does furnish us with an example where the containment is proper ([7], page 259).

LEMMA 2. *If $A \in C_1$, then either $A \in C$ or A has no nonzero C-ideals.*

PROOF. Let $A \in C_1$, $A \notin C$, and suppose A has a nonzero C-ideal I. Then A cannot be subdirectly irreducible so A/I must be in C. Property (N4) then forces $A \in C$, and we have our contradiction.

LEMMA 3. *C_1 is hereditary.*

PROOF. Let A be an element of C_1 and I an ideal of A such that $(0) \neq I \neq A$. If A is subdirectly irreducible, then $A \in C$, which implies $I \in C$ by (N2), and thus $I \in C \subseteq C_1$. So assume that A is not subdirectly irreducible and that $I \notin C_1$. Then note that $A \notin C$, for if so, $I \in C \subseteq C_1$ since C is hereditary. There are two possibilities for I.

1) I is not subdirectly irreducible. Then there exists an ideal $J \neq (0)$ of I such that $I/J \notin C$. Let J' be the ideal of A generated by J. Then $(0) \neq J' \subseteq I$ and by Andrunakievič's Lemma ([1], Lemma 4), we have $(J')^3 \subseteq J$. If $(J')^3 \neq (0)$, then $A/(J')^3 \in C$ and by property (N2), $I/(J')^3 \in C$. The natural map from $I/(J')^3 \to I/J$ then forces $I/J \in C$ by property (N3). This is impossible however. If $(J')^3 = (0)$, then $(J')^2$ is a zero ring and hence in C by (N1). If $(J')^2 \neq 0$, then $A/(J')^2 \in C$, so by property (N4) $A \in C$, which yields $I \in C$ since C is hereditary. But this cannot happen. Finally, if $(J')^2 = (0)$, then $J' \in C$ since it is a zero ring. But then $A/J' \in C$, forcing $A \in C$, which is again an impossibility.

2) I is subdirectly irreducible with heart H. If $H^2 = (0)$ then $H \in C$. If $H^2 = H$, then by Lemma 77 page 137 in [2], A/I^* is subdirectly irreducible with heart $\overline{H} \cong H$, where I^* is the annihilator of I. If $I^* = (0)$ then A is subdirectly irreducible, which we have assumed is not the case. If $I^* \neq (0)$ then $A/I^* \in C$, since $A \in C_1$, and $\overline{H}$ is in C by property (N2). But $\overline{H} \cong H$, so $H \in C$ by (N3). In any case, we see that the heart, H, of I is in C. If $I = H$ we are done, so suppose $I \neq H$. If $I/H \in C$, then $H \in C$ implies $I \in C$ by (N4) and we are done. Suppose $I/H \notin C$. Then, if we construct H', the ideal of A generated by H, and consider $(H')^3$, then, since H is simple, Andrunakievič's Lemma forces either $(H')^3 = H$ or $(H')^3 = (0)$. If $(H')^3 = H$ then H is an ideal of A, which forces A/H, and consequently I/H, to be in C. If $(H')^3 = 0$ (in which case H is a simple nil ring), we may proceed as in part 1 of the proof. Then $(H')^2$ is a zero ring in C, and if $(H')^2 \neq 0$ then $A/(H')^2 \in C$, which forces $A \in C$. If $(H')^2 = 0$ then H' is a zero ring in C, and $A/H' \in C$ forces A again to be in C. These final contradictions conclude the proof.

There is no reason to believe that C_1 is a radical class, since it may not be closed under taking direct sums. However, with the following definition, we do get a radical class.

DEFINITION 4. If C satisfies properties (N1)–(N4) and C_1 is as given in Definition 1, we define C_2 to be the class of all rings each nonzero homomorphic image of which contains a nonzero C_1-ideal.

From results of Sulinski, Anderson and Divinsky [6], we see that, since C_1 contains all zero rings, the construction of the lower radical stops at the second step. Hence C_2 is merely the lower radical generated by C_1. Also Hoffman and Leavitt [3] have shown that, since C_1 is hereditary, then so must be C_2.

III

C_2 is then a hereditary radical class containing C. Property (N1)′ assures that C_2 is supernilpotent. We arrive at this conclusion in an alternative fashion and also realize C_2 as an upper radical as follows. Let $\mathfrak{M}$ be the class of all rings with no nonzero C_1-ideals. If C_1 is a radical class then $\mathfrak{M}$ is the semisimple class of C_1.

THEOREM 5. *$\mathfrak{M}$ is a weakly special class of rings* [5].

PROOF. a) If $A \in \mathfrak{M}$ and I is an ideal of A such that $I^2 = (0)$, then $I \in C \subseteq C_1$ and consequently $I = (0)$. A can thus have no nonzero nilpotent ideals and hence is semiprime.

b) Let $A \in \mathfrak{M}$ and I be a nonzero ideal of A. Suppose $I \notin \mathfrak{M}$. Then I has a nonzero C_1-ideal J. Let J' be the ideal of A generated by J. Recall that $J' \subseteq I$ and $(J')^3 \subseteq J$. Also $(J')^3 \neq (0)$ since A is semiprime. Thus, since $J \in C_1$, we must have $(J')^3 \in C_1$ by Lemma 3. But this is a nonzero C_1-ideal of A, which is impossible. Thus I can have no nonzero C_1-ideals and must be in $\mathfrak{M}$ as desired.

c) Let $B \in \mathfrak{M}$ with B an ideal of a ring A such that $B^* = (0)$. Suppose $A \notin \mathfrak{M}$. Then A has a nonzero C_1-ideal I. Now $I \cap B$ is an ideal of I and, by Lemma 3, $I \cap B \in C_1$. However, $I \cap B$ is also an ideal of B which has no nonzero C_1-ideals. Thus $I \cap B = (0)$, $I \subseteq B^* = (0)$, which is a contradiction, and we see that $A \in \mathfrak{M}$ as desired.

Rjabuhin [5] has shown that $\mathfrak{M}$ must generate an upper radical $\mathfrak{U}\mathfrak{M}$ which is supernilpotent. Recall that $\mathfrak{U}\mathfrak{M}$ consists of all rings which have no nonzero homomorphic image in $\mathfrak{M}$.

THEOREM 6. $C_2 = \mathfrak{U}\mathfrak{M}$.

PROOF. $A \in \mathfrak{U}\mathfrak{M}$ if and only if each nonzero homomorphic image of A is not in $\mathfrak{M}$. This is equivalent to each nonzero homomorphic image of A having a nonzero C_1-ideal which requires, by definition, that $A \in C_2$.

A corollary to this is the previously mentioned:

COROLLARY 7. *C_2 is a supernilpotent radical class.*

Rjabuhin has also shown that such an upper radical must satisfy the intersection property of Leavitt [4]. We state the result as a corollary.

COROLLARY 8. *For any ring A, $C_2(A)$ is the intersection of all ideals I of A such that A/I has no nonzero C_1-ideals.*

It is clear that if C is the class of nilpotent rings, then C_1 is the class of almost nilpotent rings and C_2 is the radical L_2 of van Leeuwen and Heyman. Since our construction is general, however, we may consider different classes to play the part of C.

IV

In this section we shall assume that $\mathcal{P}$ will always refer to a supernilpotent radical class and, as mentioned previously, $\mathcal{P}$ then satisfies properties (N1) through (N4). In this case the rings in $\mathcal{P}_1$ shall be called *almost radical* rings. Lemma 2 and Corollary 8 can now be restated to read:

LEMMA 2′. *If $A \in \mathcal{P}_1$ then either $A \in \mathcal{P}$ or A is $\mathcal{P}$-semisimple.*

COROLLARY 8′. *For any ring A, $\mathcal{P}_2(A)$ is the intersection of all ideals I of A such that A/I has no almost radical ideals.*

It is clear that for any $\mathcal{P}$, $\mathcal{P}_1$ will contain the almost nilpotent rings and then $\mathcal{P}_2$ will contain the radical L_2 of van Leeuwen and Heyman. One can also see that if $\mathcal{P} \neq \mathcal{P}_1$, then $\mathcal{P}_2$ is a larger radical than $\mathcal{P}$. For an example of such a situation, let $\mathcal{B}$ be the Baer radical. Divinsky ([2], Example 10, page 103) gives an example of a ring W which is almost nilpotent but $\mathcal{B}$-semisimple. W would then be in $\mathcal{B}_1$ but not in $\mathcal{B}$. Hence $\mathcal{B} \neq \mathcal{B}_2$.

At this point it is natural to ask under what conditions on $\mathcal{P}$ it is necessary that $\mathcal{P} = \mathcal{P}_1 = \mathcal{P}_2$. We find this to be true for dual radicals.

Recall that a radical $\mathcal{P}$ is called *dual* if $\mathcal{P} = \mathcal{P}_\phi$, where $\mathcal{P}_\phi$ is the upper radical generated by the class $\mathfrak{M}$ of all subdirectly irreducible rings with $\mathcal{P}$-semisimple hearts [2].

LEMMA 9. *For any supernilpotent radical $\mathcal{P}$, we have $\mathcal{P} \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_\phi$.*

PROOF. Clearly $\mathcal{P} \subseteq \mathcal{P}_2$, so let $A \notin \mathcal{P}_\phi$. Then A has a nonzero homomorphic image A/K which is subdirectly irreducible with $\mathcal{P}$-semisimple heart H/K. But

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then if A/K has a nonzero $\mathcal{P}_1$-ideal B/K, we have that B/K is subdirectly irreducible with heart H/K. By definition, then $B/K \in \mathcal{P}$. By (N2), then $H/K \in \mathcal{P}$. But this is impossible, since H/K is $\mathcal{P}$-semisimple, unless $B/K = (0)$. Thus A has a nonzero homomorphic image with no nonzero $\mathcal{P}_1$-ideal and, as a result, $A \notin \mathcal{P}_2$. Consequently $\mathcal{P}_2 \subseteq \mathcal{P}_\phi$ as desired.

COROLLARY 10. *If $\mathcal{P}$ is a dual supernilpotent radical, then $\mathcal{P} = \mathcal{P}_1 = \mathcal{P}_2$.*

PROOF. The definition of dual radical, together with Lemma 9, requires that $\mathcal{P} \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_\phi = \mathcal{P}$.

It is interesting to note that if $\mathcal{P}$ is the Jacobson radical J, then since J_2 must contain the van Leeuwen-Heyman radical, L_2, and L_2 is independent of the Jacobson radical, we may assert anew that the Jacobson radical is not a dual radical. This result may be seen to be true for any radical independent of L_2.

We leave for further study the problem of deciding where the radicals C_2 fit into the hierarchy of radical classes, as well as whether the construction $C_2 \subseteq (C_2)_2 \subseteq \ldots$ must terminate.

References

[1] V. A. Andrunakievič, 'Radicals in associative rings *I*', *Mat. Sb.* **44** (1958), 179–212. (Russian)

[2] N. Divinsky, *Rings and radicals* (University of Toronto Press, 1965).

[3] A. E. Hoffman and W. G. Leavitt, 'Properties inherited by the lower radical', *Portugal. Math.* **27** (1968), 63–68.

[4] W. G. Leavitt, 'The intersection property of an upper radical', *Arch. Math. (Basel)* **24** (1973), 486–492.

[5] Ju. M. Rjabuhin, 'On hypernilpotent and special radicals', *Studies in Algebra and Math. Analysis*, 65–72 (Izdat., "Karta Modlovenjaska", Kishinev, 1965).

[6] A. Sulinski, T. Anderson and N. Divinsky, 'Lower radical properties for associative and alternative rings', *J. London Math. Soc.* **41** (1966), 417–424.

[7] L. C. A. Van Leeuwen and G. A. P. Heyman, 'A class of almost nilpotent rings', *Acta. Math. Sci. Hungar.* **26** (1975), 259–262.

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