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CYCLIC SUBGROUP SEPARABILITY OF GENERALIZED FREE PRODUCTS

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ABSTRACT. We derive a criterion for a generalized free product of groups to be cyclic subgroup separable. We see that most of the known results for cyclic subgroup separability are covered by this criterion, and we apply the criterion to polygonal products of groups. We show that a polygonal product of finitely generated abelian groups, amalgamating cyclic subgroups, is cyclic subgroup separable.

1. **Introduction.**

1.1. *Notation.* Let G be a group. Then we use $N \triangleleft_f G$ to denote that N is a normal subgroup of finite index in G. We denote by $A *_H B$ the generalized free product of A and B with the subgroup H amalgamated. If $G = A *_H B$ and $x \in G$, then $||x||$ denotes the amalgamated free product length of x in G. If $\overline{G}$ is a homomorphic image of G, then we use $\bar{x}$ to denote the image of $x \in G$ in $\overline{G}$.

Let H be a subgroup of a group G. Then G is said to be *H-separable* if, for each $x \in G \setminus H$, there exists $N \triangleleft_f G$ such that $x \notin NH$. A group G is *subgroup separable* if G is H-separable for all finitely generated (f.g.) subgroups H of G. A group G is *residually finite* (RF) if G is $\langle 1 \rangle$-separable. In particular, a group G is said to be *cyclic subgroup separable* (π_c) if G is $\langle x \rangle$-separable for each $x \in G$. Clearly, every subgroup separable group is π_c, and every π_c group is RF.

1.2. *Residual finiteness of generalized free products.* In [4, Proposition 2], G. Baumslag proved a residual finiteness criterion for the generalized free product of two residually finite (RF) groups. For the generalized free product amalgamating a cyclic subgroup, Allenby and Tang [3] introduced a simple criterion, using potency, to derive the residual finiteness of the generalized free product with a cyclic subgroup amalgamated. Their idea motivated Wehrfritz [14] to find a residual finiteness criterion for the generalized free product with any subgroup amalgamated. Baumslag's criterion has been used extensively in the study of the residual finiteness of generalized free products.

1.3. *Statement of results.* The object of this paper is to study the cyclic subgroup separability of generalized free products of groups. The following theorem plays an important role in this study.

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THEOREM 1.1. *Let* $G = E*_H F$ *and let* $\Lambda = \{(P,Q) : P \lhd_f E, Q \lhd_f F$ *and* $P\cap H = Q\cap H\}$.

(1) $\bigcap_{(P,Q)\in\Lambda} PH = H$ *and* $\bigcap_{(P,Q)\in\Lambda} QH = H$,

(2) $\bigcap_{(P,Q)\in\Lambda} P\langle x\rangle = \langle x\rangle$ *and* $\bigcap_{(P,Q)\in\Lambda} Q\langle y\rangle = \langle y\rangle$ *for all* $x \in E$, $y \in F$.

Then G is π_c.

We note that G. Baumslag [4, Proposition 2] proved that the group G is RF if we replace (2) above by $\bigcap_{(P,Q)\in\Lambda} P = 1 = \bigcap_{(P,Q)\in\Lambda} Q$. From Theorem 1.1, it is not difficult to derive the following:

PROPOSITION 1.2. *Let* $G = E *_H F$. *Suppose that*

(a) E *and* F *are* π_c *and* H*-separable,*

(b) for each $N \lhd_f H$ *there exist* $N_E \lhd_f E$ *and* $N_F \lhd_f F$ *such that* $N_E \cap H = N_F \cap H \subset N$.

Then G is π_c.

In [14], Wehrfritz showed that the group G in Proposition 1.2 is RF if we substitute "E and F are RF and H-separable" for (a) in the proposition.

Let G and Λ be as in Theorem 1.1. Then, for each $(P,Q) \in \Lambda$, we have a homomorphism

$$\psi_{P,Q}\colon E *_H F \longrightarrow E/P *_{\overline{H}} F/Q, \tag{1}$$

where $\overline{H} = HP/P = HQ/Q$. Using this notation, Shirvani [13] proved that $G = E *_H F$ is RF if, and only if, $\bigcap_{(P,Q)\in\Lambda} \operatorname{Ker}\psi_{P,Q} = \langle 1\rangle$. As an easy generalization of this, we find

THEOREM 1.3. *Let* $G = E *_H F$ *and let* Λ *be as in Theorem 1.1. For a given* f.g. *subgroup L of G, G is L-separable if, and only if,* $\bigcap_{(P,Q)\in\Lambda}(\operatorname{Ker}\psi_{P,Q})L = L$.

This result and Theorem 1.1 directly imply the following:

COROLLARY 1.4. *Let* $G = E *_H F$ *and* Λ *be as in Theorem 1.1. Assume that* $\bigcap_{(P,Q)\in\Lambda} PH = H = \bigcap_{(P,Q)\in\Lambda} QH$. *Then G is* π_c *if, and only if,* $\bigcap_{(P,Q)\in\Lambda}(\operatorname{Ker}\psi_{P,Q})\langle x\rangle = \langle x\rangle$, *for all* $x \in A \cup B$.

Finally, we apply our result to a special kind of generalized free products, known as polygonal products of groups, and we generalize some results found in [2], [11].

Let P be a polygon. Assign a group G_v to each vertex v and a group G_e to each edge e of P. Let α_e and β_e be monomorphisms which embed G_e as a subgroup of the two vertex groups at the ends of the edge e. Then the *polygonal product G* is defined to be the group generated by the generators and relations of the vertex groups G_v together with the extra relations obtained by identifying $g_e\alpha_e$ and $g_e\beta_e$ for each $g_e \in G_e$. By abuse of language, we say that G is the polygonal product of the (vertex) groups $G_0, G_1, \ldots, G_n$, amalgamating the (edge) subgroups $H_0, H_1, \ldots, H_n$ with *trivial intersections*, if $G_i \cap G_{i+1} = H_i$ and $H_i \cap H_{i+1} = 1$, where $0 \le i \le n$ and the subscripts i are taken modulo $n+1$.

THEOREM 1.5. *Let P be the polygonal product of the polycyclic-by-finite groups* A, B, C, D *amalgamating the subgroups* $\langle b\rangle, \langle c\rangle, \langle d\rangle, \langle a\rangle$ *with trivial intersections. If* a, b, c, d *are in the centers of the vertex groups containing them, then P is* π_c.

A similar result for the polygonal product of more than four f.g. abelian groups, amalgamating any subgroups with trivial intersections, will be considered in a later paper. But, if amalgamated cyclic subgroups in a polygonal product are finite, then we have the following result which is an extension of [2, Theorem 4.4.].

THEOREM 1.6. *Let P_0 be the polygonal product of* f.g. *nilpotent groups $A_0, A_1, \ldots, A_n$ ($n \geq 3$), amalgamating finite cyclic subgroups $\langle h_0\rangle, \langle h_1\rangle, \ldots, \langle h_n\rangle$ with trivial intersections. If there exist two vertex groups A_i, A_j (say $i < j$) such that h_{i-1}, h_j are of prime orders, then P_0 is π_c.*

Residual finiteness of the polygonal product in the next theorem is known [11]. We may prove the next result by following the proof of Theorem 1.1.

THEOREM 1.7 ([10]). *Let P_0 be the polygonal product of the* f.g. *nilpotent groups A_0, B_0, C_0, D_0, amalgamating $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$, with trivial intersections. If a and c are of prime orders p and q, respectively, then P_0 is π_c.*

For the polygonal product of f.g. nilpotent groups, amalgamating arbitrary cyclic subgroups, the situation is not as simple as it is in the above theorems. Considering the simplest polygonal product of four torsion-free nilpotent groups, we may prove the following result.

THEOREM 1.8. *Let P be the polygonal product of the four* f.g. *torsion-free nilpotent groups $\langle a, b\rangle$, $\langle b, c\rangle$, $\langle c, d\rangle$, $\langle d, a\rangle$, amalgamating the cyclic subgroups $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$, with trivial intersections. Then P is π_c.*

2. **Proofs and applications.** In this section, we prove our results and apply them to the known results. We begin by proving Theorem 1.1.

PROOF OF THEOREM 1.1. Let $g \notin \langle x\rangle$, where $g, x \in G$. Since we want to find $N \triangleleft_f G$ such that $g \notin N\langle x\rangle$, we may assume that x is cyclically reduced. As we noted, G is RF by Baumslag [4, Proposition 2]. Hence, we also may assume that $x \neq 1$. Clearly $g \neq 1$.

CASE 1. Suppose $g \notin \langle x\rangle$ is implied by the syllable lengths of g and x; that is,
Subcase 1: $||x|| = 0$ and $||g|| \geq 1$,
Subcase 2: $||x|| = 1$, say, $x \in E \setminus H$ and
(i) $||g|| \geq 2$, or
(ii) $||g|| = 1$ and $g \in F \setminus H$,
Subcase 3: $||x|| \geq 2$ and
(i) $||g|| = 0$, or
(ii) $||g|| \neq 0$ and $||x||$ does not divide $||g||$.

If $||g|| \geq 1$, say, $g = a_1 b_1 \cdots a_m b_m$ where $a_i \in E\setminus H$ and $b_i \in F\setminus H$ (the other cases being similar), then by (1) we can find (P_i, Q_i), $(P_i', Q_i') \in \Lambda$ such that $a_i \notin P_i H$ and $b_i \notin Q_i' H$ for all i. Let $P_0 = \bigcap_{i=1}^m (P_i \cap P_i')$ and $Q_0 = \bigcap_{i=1}^m (Q_i \cap Q_i')$. Then $(P_0, Q_0) \in \Lambda$. If $1 \neq g \in H$ then we choose, from (2), $(P_0, Q_0) \in \Lambda$ such that $g \notin P_0$. Note that $||g\psi_{P_0,Q_0}|| = ||g||$ and $g\psi_{P_0,Q_0} \neq 1$, where ψ_{P_0,Q_0} is as in (1). In a similar way, we can find $(P_0', Q_0') \in \Lambda$ such that

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$\|x\psi_{P_0',Q_0'}\| = \|x\|$ and $x\psi_{P_0',Q_0'} \neq 1$. Let $P = P_0 \cap P_0'$ and $Q = Q_0 \cap Q_0'$. Then $(P, Q) \in \Lambda$, and $\bar{g} \neq 1 \neq \bar{x}$, $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$, where $\overline{G} = G\psi_{P,Q} = E/P *_{\overline{H}} F/Q$. Note that $\bar{g} \notin \langle \bar{x} \rangle$. Since $\overline{G} = E/P *_{\overline{H}} F/Q$ is free-by-finite, hence it is subgroup separable [8], and since $\bar{g} \notin \langle \bar{x} \rangle$, there exists $\bar{N} \lhd_f \overline{G}$ such that $\bar{g} \notin \bar{N}\langle \bar{x} \rangle$. Let N be the preimage of $\bar{N}$ in G. Then $g \notin N\langle x \rangle$ and $N \lhd_f G$ as required.

CASE 2. Suppose that g and x are in the same factor, say, E. Since g and x are in E, by assumption (2), there exists $(P, Q) \in \Lambda$ such that $g \notin P\langle x \rangle$. It follows that $\bar{g} \notin \langle \bar{x} \rangle$, where $\overline{G} = E/P *_{\overline{H}} F/Q$. Now, as in Case 1, we can find $N \lhd_f G$ such that $g \notin N\langle x \rangle$.

CASE 3. Suppose $\|x\| \geq 2$, $\|g\| \neq 0$ and $\|x\|$ divides $\|g\|$. Since x is cyclically reduced, we may assume that $x = e_1 f_1 \cdots e_n f_n$, where $e_i \in E \setminus H$ and $f_i \in F \setminus H$. Since $\|x\|$ divides $\|g\|$, we may write $g = a_1 b_1 \cdots a_m b_m$ or $g = b_1 a_1 \cdots b_m a_m$, where $a_j \in E \setminus H$, $b_j \in F \setminus H$, and $m = ns$ for some integer s. As in Case 1, we can find $(P_1, Q_1) \in \Lambda$ such that $a_j, e_i \notin P_1 H$ and $b_j, f_i \notin Q_1 H$ for all i, j. Now $g^{-1}x^s \neq 1 \neq gx^s$ and G is RF by [4]. Hence there exists $M \lhd_f G$ such that $g^{-1}x^s \notin M$ and $gx^s \notin M$. Note that $(M \cap E, M \cap F) \in \Lambda$. Let $P = P_1 \cap M \cap E$, and $Q = Q_1 \cap M \cap F$, then $(P, Q) \in \Lambda$. Hence, in $\overline{G} = G\psi_{P,Q} = E/P *_{\overline{H}} F/Q$, we have $\|\bar{g}\| = \|g\|$ and $\|\bar{x}\| = \|x\|$. By the choice of M, $\bar{g} \neq \bar{x}^s$ and $\bar{g} \neq \bar{x}^{-s}$, thus $\bar{g} \notin \langle \bar{x} \rangle$ in $\overline{G}$. Now, as before, we can find $N \lhd_f G$ such that $g \notin N\langle x \rangle$. This completes the proof. ∎

It is not difficult to see that (a) and (b) in Proposition 1.2 imply (1) and (2) in Theorem 1.1. Hence, we omit the proof of Proposition 1.2. Now we list some known results which follow from Proposition 1.2. For the proofs, we refer the reader to [10, §2.2].

COROLLARY 2.1 ([1]). *Let E and F be π_c and let H be finite. Then $E *_H F$ is π_c.*

COROLLARY 2.2 ([7]). *Let A and B be π_c groups and $A \cap B = \langle a \rangle$. Assume that there exists an integer k such that, for each integer n, we can find $N \lhd_f A$ satisfying $N \cap \langle a \rangle = \langle a^{nk} \rangle$. Then $A *_{\langle a \rangle} B$ is π_c.*

In [6, p.42], Dyer mentioned that $A *_H A$ is not RF, if A is not H-separable. Hence, we have the following from Theorem 1.1.

COROLLARY 2.3. *Let A be π_c (or* RF*) and H be a subgroup of A. Then A is H-separable if, and only if, $A *_H A$ is π_c (or* RF*).*

Next result is a generalization of Boler and Evans' result [5] and Allenby and Gregorac [1] mentioned the result for the generalized free product of two π_c groups amalgamating a retract. A subgroup H of a group G is called a *retract* if there exists $G_1 \lhd G$ such that $G = G_1 H$ and $G_1 \cap H = 1$. In this case, we denote $G = G_1 \cdot H$.

COROLLARY 2.4 ([1]). *Let G_i be a π_c group with a retract H for each $i \in I$. Then the generalized free product Q_I of the G_i ($i \in I$) amalgamating H is π_c.*

Now we prove Theorem 1.3. We recall the homomorphism $\psi_{P,Q}$ from (1).

PROOF OF THEOREM 1.3. ($\Longleftarrow$) Let $g \in G \setminus L$. Then, by assumption, there exists $(P,Q) \in \Lambda$ such that $g \notin (\operatorname{Ker}\psi_{P,Q})L$, where $\psi_{P,Q}$ is as in (1). Hence $g\psi_{P,Q} \notin L\psi_{P,Q}$. Since $G\psi_{P,Q} = E/P *_{\overline{H}} F/Q$ is subgroup separable by [8], we can find $N \triangleleft_f G$ such that $g \notin NL$.

($\Longrightarrow$) Let $g \in G \setminus L$. Since G is L-separable, there exists $N \triangleleft_f G$ such that $g \notin NL$. Let $P = N \cap E$ and $Q = N \cap F$. Then clearly $P \triangleleft_f E$, $Q \triangleleft_f F$, and $P \cap H = N \cap H = Q \cap H$; hence $(P,Q) \in \Lambda$. Moreover, $\operatorname{Ker}\psi_{P,Q} = \langle P,Q\rangle^G \subset N$, hence $g \notin (\operatorname{Ker}\psi_{P,Q})L$. This proves that $\bigcap_{(P,Q)\in\Lambda}(\operatorname{Ker}\psi_{P,Q})L \subset L$; hence $\bigcap_{(P,Q)\in\Lambda}(\operatorname{Ker}\psi_{P,Q})L = L$. ∎

We note that $A *_{\langle c\rangle} B$ has solvable power problem whenever A and B have solvable power problems (Lipschutz, [12]). On the other hand, it is not known whether $A *_{\langle c\rangle} B$ is π_c whenever A and B are π_c. However, for residual finiteness, the Higman's group $\langle a,c; a^{-1}ca = c^2\rangle *_{\langle c\rangle} \langle b,c; b^{-1}cb = c^2\rangle$ is not RF [9], but its factors are RF.

Finally, we prove our results on polygonal products.

PROOF OF THEOREM 1.5. Let $P = E *_H F$ where $E = A *_{\langle b\rangle} B$, $F = D *_{\langle d\rangle} C$ and $H = \langle a\rangle * \langle c\rangle$. To apply Proposition 1.2, we first note that E and F are subgroup separable [1, Theorem 5]. Hence (a) in the proposition holds. For (b) in the proposition, let $N \triangleleft_f H$. Then there exists a natural homomorphism $\pi\colon E \longrightarrow (A/\langle b\rangle)*(B/\langle b\rangle)$. Let $\bar{E} = E\pi = \bar{A}*\bar{B}$, where $\bar{A} = A/\langle b\rangle$ and $\bar{B} = B/\langle b\rangle$. We note that $\langle\bar{a}\rangle * \langle\bar{c}\rangle \cong H$ and $N \cong \bar{N} \triangleleft_f \langle\bar{a}\rangle * \langle\bar{c}\rangle$. Now, considering $\bar{A} * \bar{B} = \bar{A} *_{\langle\bar{a}\rangle} (\langle\bar{a}\rangle * \langle\bar{c}\rangle) *_{\langle\bar{c}\rangle} \bar{B}$, we have a homomorphism $\phi\colon \bar{E} \longrightarrow (\bar{A}/\bar{N}\cap\langle\bar{a}\rangle) *_{\langle\tilde{a}\rangle} (\langle\bar{a}\rangle * \langle\bar{c}\rangle/\bar{N}) *_{\langle\tilde{c}\rangle} (\bar{B}/\bar{N}\cap\langle\bar{c}\rangle)$, where $\langle\tilde{a}\rangle = \langle\bar{a}\rangle/\bar{N}\cap\langle\bar{a}\rangle = \bar{N}\langle\bar{a}\rangle/\bar{N}$ and $\langle\tilde{c}\rangle = \bar{N}\langle\bar{c}\rangle/\bar{N} = \langle\bar{c}\rangle/\bar{N}\cap\langle\bar{c}\rangle$. Since $\langle\tilde{a}\rangle$ and $\langle\tilde{c}\rangle$ are finite, therefore, $\bar{E}\phi$ is RF. Note that $(\langle\bar{a}\rangle*\langle\bar{c}\rangle)/\bar{N}$ is finite. It follows that there exists $\tilde{M} \triangleleft_f \bar{E}\phi$ such that $\tilde{M}\cap\big((\langle\bar{a}\rangle*\langle\bar{c}\rangle)/\bar{N}\big) = 1$. Now, let N_E be the preimage of $\tilde{M}$ in E under the homomorphism $\pi \circ \phi$. Then $N_E \triangleleft_f E$ and $N_E \cap H = N$. Similarly, we can find $N_F \triangleleft_f F$ such that $N_F \cap H = N$. This proves (b) in Proposition 1.2. Therefore, P is π_c by the proposition. ∎

As a consequence of Theorem 1.5, we have the next result which is a generalization of [2, Theorem 3.4.].

COROLLARY 2.5. *Let P be the polygonal product of the* f.g. *abelian groups A, B, C, D amalgamating the subgroups $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$, $\langle a\rangle$ with trivial intersections. Then P is π_c.*

It is easy to prove the next lemma.

LEMMA 2.6 ([10]). *Let $E = E_1 \cdot H$ be a* RF *group with a retract H. Then E is H-separable.*

LEMMA 2.7. *Let $E = E_1 \cdot H$ be a π_c group with a retract H, where H is* f.g.*, and let F be subgroup separable. Then $E *_H F$ is π_c.*

PROOF. By Lemma 2.6, E is H-separable. Clearly F is H-separable. Thus, to apply Proposition 1.2, we need only consider (b) in Proposition 1.2. For this, let $N_H \triangleleft_f H$ be given. Choose $T = \{h_0, h_1, \ldots, h_r\}$ to be a complete set of coset representatives of N_H in H, where $h_0 = 1$. Note that N_H is f.g. Since F is N_H-separable, and since $h_k \notin N_H$,

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for $1 \leq k \leq r$, there exists $M \triangleleft_f F$ such that $h_k \notin MN_H$, for all $k \neq 0$. This implies that $M \cap H \subset N_H$. Since H is a retract of E and $M \cap H \triangleleft_f H$, there exists $N \triangleleft_f E$ such that $N \cap H = M \cap H$. This proves (b) in Proposition 1.2. Thus $E *_H F$ is π_c by Proposition 1.2. ∎

PROOF OF THEOREM 1.6. Without loss of generality, we let $i = 0$.

CASE 1. All A_i are finite. We put $E = \langle h_n, h_0 \rangle *_{\langle h_0 \rangle} A_1 *_{\langle h_1 \rangle} \cdots *_{\langle h_{j-2} \rangle} A_{j-1} *_{\langle h_{j-1} \rangle} \langle h_{j-1}, h_j \rangle$ and $F = A_n *_{\langle h_{n-1} \rangle} A_{n-1} *_{\langle h_{n-2} \rangle} \cdots *_{\langle h_{j+1} \rangle} A_{j+1}$. Then $P_0 = \big((E *_H F) *_S A_0\big) *_T A_j$, where $H = \langle h_n \rangle * \langle h_j \rangle$, $S = \langle h_n, h_0 \rangle$, and $T = \langle h_{j-1}, h_j \rangle$. Since all A_k are finite, E and F are subgroup separable [1]. Now A_0, A_j are nilpotent, and orders of h_n, h_j are primes. It follows that $\langle h_n \rangle \cap \langle h_0 \rangle^S = 1 = \langle h_j \rangle \cap \langle h_{j-1} \rangle^T$. Thus, there exists a homomorphism $\pi \colon E \longrightarrow \langle h_n \rangle * \langle h_j \rangle$ defined by $x\pi = 1$ for all $x \in A_1 \cup A_2 \cup \cdots \cup A_{j-1}$ (if $j \geq 2$) (or $x\pi = 1$ for all $x \in \langle h_0 \rangle$ (if $j = 1$)) and $y\pi = y$ for all $y \in \langle h_n \rangle \cup \langle h_j \rangle$. Briefly, $H = \langle h_n \rangle * \langle h_j \rangle$ is a retract of E. Hence, by Lemma 2.7, $E *_H F$ is π_c. Note that the S, T are finite. It follows that $P_0 = \big((E *_H F) *_S A_0\big) *_T A_j$ is π_c by Corollary 2.1.

CASE 2. The A_k are not necessarily finite. Note that each $B_k = \langle h_{k-1}, h_k \rangle$ is finite, and note that the polygonal product P, of the subgroups B_k, of the A_k, amalgamating the $\langle h_k \rangle$, is π_c by the above case, where the subscripts k are taken modulo $n + 1$. It follows that P_0 is π_c, since $P_0 = P *_{B_0} A_0 *_{B_1} A_1 *_{B_2} \cdots *_{B_n} A_n$, and since each B_k is finite. ∎

PROOF OF THEOREM 1.8. Let $A = \langle a, b \rangle$, $B = \langle b, c \rangle$, $C = \langle c, d \rangle$ and $D = \langle d, a \rangle$. Then $P = E *_H F$, where $E = A *_{\langle b \rangle} B$, $F = D *_{\langle d \rangle} C$ and $H = \langle a \rangle * \langle c \rangle$. Note that $\langle a \rangle^A \cap \langle b \rangle = 1 = \langle c \rangle^B \cap \langle b \rangle$. It follows that $\langle b \rangle$ is a retract of both A and B. Hence, E is π_c by Corollary 2.4. Similarly F is π_c. Since $\langle b \rangle^A \cap \langle a \rangle = 1 = \langle b \rangle^B \cap \langle c \rangle$, H is a retract of E. Similarly H is a retract of F. Hence, the theorem follows from Corollary 2.4.

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