The Conditions for the Reality of the Roots of an n-ic.

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The homogeneous real linear transformation in n variables is such that, when these variables are used as a set of mutually rectangular coordinates, an n-dimensional sphere is transformed into an n-dimensional ellipsoid; n mutually rectangular radii of the sphere become the n, mutually rectangular, principal radii of the ellipsoid. When these principal radii have not been rotated from their original directions, the transformation is said to be *pure*, or *irrotational*. Since these radii are necessarily real, the roots of the n-ic for the determination of the n principal elongations are necessarily real. If, therefore, we find the conditions which must subsist amongst the n^2 constants which define the transformation, in order that it may be pure, we have got conditions which are *sufficient* to ensure that the roots of the n-ic for the determination of the elongations along non-rotated lines shall be real.

Tait (*Proc. R.S.E.*, 1896, *Scientific Papers*, CXX.) first showed that these conditions are not necessary, and proved that a sufficient condition, in three dimensions, is that the transformation shall be decomposable into two superposed pure transformations. The statement is true generally. He gave the single relation which must subsist amongst the nine constants in order that there should be three real non-rotated lines, which are not in general mutually rectangular. He explicitly confined his investigation to transformations capable of representing strain possible in ordinary matter, but he remarked that it was easy to remove that restriction.

Muir (*Phil. Mag.*, 1896, Vol. XLIII.), not noticing Tait's restriction, added explicitly to Tait's single condition other conditions regarding signs which are implicitly involved in the restriction. He applied Tait's process to the n-dimensional transformation, giving $(n-1)(n-2)/2$ equations of the same form as Tait's, and a rule by

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which they could be readily written down. In the investigation given below, $(n-1)(n-2)/2$ equations of a different type are found, and a compact general statement, which includes them all, is given. Muir's equations, along with other necessary but not independent relations, can of course be deduced from them. It is found also that Muir's conditions regarding signs are necessary if we remove the analogue of Tait's restriction, which is their equivalent.

The well-known condition that the transformation shall be pure is that the determinantal equation shall be axi-symmetrical. This is proved readily by considering the n coordinates of a point, referred to n rectangular axes, in n-dimensional space, say $x_1, \ldots, x_n$. Assume another set of such axes, with coordinates $\xi_1, \ldots, \xi_n$, as the set of non-rotated lines. Under these conditions ξ_p becomes $\xi_p' = e_p \xi_p$, where e_p is a constant. Let l_{pq} denote the cosine of the angle between the axes of ξ_p and x_q. Referring the point $(x_1, \ldots, x_n)$ to the ξ axes, imposing the transformation which changes ξ_p to ξ_p', etc., and referring back to the x axes, we get

$$x_r' = \sum_{p=1}^{p=n} . \, e_p l_{pr} \sum_{q=1}^{q=n} l_{pq} x_q,$$

$$x_q' = \sum_{p=1}^{p=n} . \, e_p l_{pq} \sum_{r=1}^{r=n} l_{pr} x_r,$$

so that the coefficient of x_q in x_r' is equal to the coefficient of x_r in x_q'.

In accordance with Tait's method, we now seek to obtain the conditions which must hold amongst the coefficients, when the equation is not axi-symmetrical, in order that it shall be capable of being changed into an axi-symmetrical equation by a process which does not alter the roots.

Let the coefficient of x_q in x_r' be c_{rq} while the coefficient of x_r in x_q' is c_{qr}. Tait's process consists in multiplying the rth row by a_r, and dividing the qth column by a_q, and so on. The coefficients now become

$$c_{rq} a_r / a_q, \quad c_{(r+1)q} a_{r+1} / a_q, \quad c_{r(q+1)} a_r / a_{q+1}, \quad c_{(r+1)(q+1)} a_{r+1} / a_{q+1},$$

with the corresponding set in which q and r are interchanged.

Hence, postulating axi-symmetry, and eliminating the a's, we find

$$c_{qr}\, c_{(q+1)(r+1)}\, c_{r(q+1)}\, c_{(r+1)q} = c_{rq}\, c_{(r+1)(q+1)}\, c_{q(r+1)}\, c_{(q+1)r}.$$

The $(n-1)(n-2)/2$ equations of this type are conditions under which the roots shall be real.

If we use the term "image minors" with reference to any minor $c_{qr}\, c_{(q+1)(r+1)} - c_{q(r+1)}\, c_{(q+1)r}$ and the corresponding one in which q and r are interchanged, and if we use the term "cross products" with reference to the two quadruple products, we can say that

The roots of an n-*ic are real if the cross products of any pair of image minors, in the determinantal form of the equation, are equal.*

When $q+1=r$, the coefficient c_{rr} is common to both cross products, so that the above condition includes Tait's condition for the case $n=3$. When $r=q$, the condition becomes a mere identity.

From these conditions it is easy to prove Muir's conditions, such as $c_{12}c_{24}c_{41} = c_{21}c_{14}c_{42}$; as also more complicated relations, such as

$$c_{n(n-1)}\, c_{(n-2)(n-2)} \ldots c_{21}c_{1n} = c_{n1}c_{12} \ldots c_{(n-2)(n-1)}c_{(n-1)n}.$$

Since we have $c_{pq}a_p^2 = c_{qp}a_q^2$, we see that, so far as this condition goes, c_{pq} and c_{qp} might be of opposite signs if we regard a_p *or* a_q as an imaginary quantity. But, since $c_{pq}a_p/a_q$ and $c_{qp}a_q/a_p$ must be equal and real, all the a's must be imaginary if one a is so. Therefore c_{pq} and c_{qp} must have like signs when we consider real coefficients only. In the case in which the roots represent, in proper units, the squared frequencies of the fundamental vibrations of a system of n masses, under the action of forces which are homogeneous and linear in the coordinates, c_{pq} and c_{qp} are necessarily of like sign since the masses are positive and the third law of motion holds.

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