2

Classical Sobolev Spaces

2.1 Basic Definitions

Let $p \in [1, \infty]$ and $k, n \in \mathbb{N}$; suppose that Ω is a non-empty open subset of $\mathbb{R}^n$. Then

$$W_p^k(\Omega) := \left\{u \colon D^\alpha u \in L_p(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le k\right\} \tag{2.1.1}$$

is the classical Sobolev space of order k, based on $L_p(\Omega)$; the derivatives $D^\alpha u$ are taken in the sense of distributions. It is a linear space when endowed with addition and multiplication by scalars in the natural way; provided with the norm $\|\cdot\|_{k,p,\Omega}$ (written as $\|\cdot\|_{k,p}$ if there is no ambiguity) defined by

$$\|u\|_{k,p,\Omega} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{p,\Omega}^p\right)^{1/p} \quad \text{if } p < \infty, \tag{2.1.2}$$

and

$$\|u\|_{k,\infty,\Omega} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{\infty,\Omega} \tag{2.1.3}$$

when $p = \infty$, it is a Banach space that is uniformly convex if $p \in (1, \infty)$. Moreover, $W_2^k(\Omega)$ is a Hilbert space, with inner product $(\cdot,\cdot)_{k,2,\Omega}$ (or $(\cdot,\cdot)_{k,2}$) given by

$$(u, v)_{k,2,\Omega} := \int_\Omega \sum_{|\alpha|\le k} (D^\alpha u)\left(\overline{D^\alpha v}\right) dx. \tag{2.1.4}$$

First suppose that $\Omega = \mathbb{R}^n$. In this case, characterisations of these spaces by means of the Fourier transform F are possible and desirable. To explain this, first define the function $w_s \colon \mathbb{R}^n \to \mathbb{R}$ by

$$w_s(x) = \left(1 + |x|^2\right)^{s/2} \text{ for all } x \in \mathbb{R}^n \text{ and } s \in \mathbb{R}, \tag{2.1.5}$$

https://doi.org/10.1017/9781009254625.004 Published online by Cambridge University Press

and introduce the space

$$H_p^s(\mathbb{R}^n) := \left\{u \in \mathcal{S}'(\mathbb{R}^n) : F^{-1}(w_s Fu) \in L_p(\mathbb{R}^n)\right\}, \tag{2.1.6}$$

where $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ is its dual, the space of tempered distributions. Endowed with the norm

$$\left\|u|H_p^s(\mathbb{R}^n)\right\| := \left\|F^{-1}(w_s Fu)\right\|_{p,\mathbb{R}^n}, \tag{2.1.7}$$

it is a Banach space. In fact,

$$H_p^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n) \text{ when } p \in (1,\infty) \text{ and } s \in \mathbb{N}, \tag{2.1.8}$$

with equivalent norms. When $p = 2$ this assertion is an immediate consequence of the fact that the Fourier transform F and its inverse are unitary operators in $L_2(\mathbb{R}^n)$; for other values of p appeal to the Michlin–Hörmander Fourier multiplier theorem gives the result. We refer to [95], 3.6.1 for further details of the argument.

Now let Ω be a non-empty open subset of $\mathbb{R}^n$. In addition to the 'intrinsic' definition of $W_p^k(\Omega)$ described above it would be natural to define this space as

$$\left\{u \in L_p(\Omega) : \text{ there exists } v \in W_p^k(\mathbb{R}^n) \text{ with } v|_\Omega = u\right\}$$

and give it the norm

$$\inf\left\{\|v\|_{k,p,\mathbb{R}^n} : v \in W_p^k(\mathbb{R}^n),\ v|_\Omega = u\right\},$$

where $v|_\Omega = u$ is meant in the sense of $\mathcal{D}'(\Omega)$, so that $v(\phi) = u(\phi)$ for all $\phi \in \mathcal{D}(\Omega)$. This space, defined by restriction, coincides with $W_p^k(\Omega)$ if Ω is bounded and has a Lipschitz boundary; without some condition on the boundary the spaces may be different. Similarly, $H_p^k(\Omega)$ may be defined by restriction of elements of $H_p^k(\mathbb{R}^n)$, and coincides with $W_p^k(\Omega)$ when $p \in (1,\infty)$ and $k \in \mathbb{N}$, if Ω is bounded and has sufficiently smooth boundary. It is common to denote $H_2^k(\Omega)$ by $H^k(\Omega)$. More information on this topic is given in [95], Chapters 3 and 4.

2.2 Fundamental Results

Here we list, for ease of reference, some of the most useful results concerning Sobolev spaces. In what follows the closure of $C_0^\infty(\Omega)$ in $W_p^k(\Omega)$ will be denoted by $\overset{0}{W}{}_p^k(\Omega)$. Proofs may be found in [64] and [61]; see also [32] and [141]. We begin with embeddings.

Theorem 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$, let $k \in \mathbb{N}$ and suppose that $p \in [1,\infty)$.*

https://doi.org/10.1017/9781009254625.004 Published online by Cambridge University Press

(i) Assume that Ω has Lipschitz boundary and that $kp < n$. Then

$$W_p^k(\Omega) \hookrightarrow L_s(\Omega) \text{ if } s \in \left[p, np/(n-kp)\right],$$

and this embedding is compact if $s \in [p, np/(n-kp))$. If for some $l \in \mathbb{N}_0$ and $\gamma \in (0,1]$ the inequality $(k-l-\gamma)p \geq n$ holds, then

$$W_p^k(\Omega) \hookrightarrow C^{l,\gamma}\left(\overline{\Omega}\right),$$

and the embedding is compact if $(k-l-\gamma)p > n$. These results hold without any condition on $\partial\Omega$ if $W_p^k(\Omega)$ is replaced by $\overset{0}{W}{}_p^k(\Omega)$.

(ii) If $k, l \in \mathbb{N}_0$, $l > k$ and $\partial\Omega$ is of class C, then $W_p^l(\Omega) \hookrightarrow\hookrightarrow W_p^k(\Omega)$; the condition on $\partial\Omega$ may be dropped if $W_p^l(\Omega)$ and $W_p^k(\Omega)$ are replaced by $\overset{0}{W}{}_p^l(\Omega)$ and $\overset{0}{W}{}_p^k(\Omega)$, respectively.

(iii) If $p \in (1,\infty)$, then $W_p^k(\Omega) \hookrightarrow\hookrightarrow W_q^{k-1}(\Omega)$ whenever $q \in [1,p)$. Note that no condition on $\partial\Omega$ is required.

The theorem shows that the embedding $I_{p,q}$ of $W_p^1(\Omega)$ in $L_q(\Omega)$ is compact whenever $p \in (1,\infty)$ and $q \in [1,p)$, no matter how unpleasant $\partial\Omega$ may be. There is a dramatic change when $q = p$, for while $I_{p,p}$ is compact when $\partial\Omega$ is of class C, the 'rooms and passages' example (see Theorem V.4.18 of [64]) shows that in the absence of any condition on the boundary of Ω, $I_{p,p}$ may be noncompact. Indeed, [68] contains an example in which $I_{p,p}$ is not even strictly singular.

It turns out that these results may be refined by use of the Lorentz spaces introduced in Section 1.3.3. For example, as regards (i), it can be shown that when $1 \leq p < n$, the smallest r.i. space $X(\Omega)$ such that $\overset{0}{W}{}_p^1(\Omega) \hookrightarrow X(\Omega)$ is $L_{p^*,p}(\Omega)$, where $p^* = np/(n-p)$, so that the embedding $\overset{0}{W}{}_p^1(\Omega) \hookrightarrow L_{p^*,p}(\Omega)$ is optimal in the class of r.i. spaces, so far as the target space is concerned. The more complicated question of optimality of the domain space in such embeddings is briefly discussed in p. 23 of [65], where references to further work on this topic can be found.

Next we list some very useful inequalities. We say that an open set $\Omega \subset \mathbb{R}^n$ supports the *p-Friedrichs inequality* if there is a constant $c > 0$ such that for all $u \in C_0^\infty(\Omega)$,

$$\|u\|_{p,\Omega} \leq c\, \||\nabla u|\|_{p,\Omega}.$$

An example of such a set is given in the next result: note that the set Ω need not be bounded.

https://doi.org/10.1017/9781009254625.004 Published online by Cambridge University Press

Theorem 2.2 *Let Ω be an open subset of $\mathbb{R}^n$ that lies between two parallel co-ordinate hyperplanes at a distance l apart, and suppose that $p \in [1, \infty)$. Then for all $u \in \overset{0}{W}{}^1_p(\Omega)$,*

$$\|u\|_{p,\Omega} \leq l \, \| |\nabla u| \|_{p,\Omega} .$$

If the parallel hyperplanes are not parallel to co-ordinate hyperplanes the inequality still holds but with l replaced by Cl for some constant C independent of u. When Ω is bounded we have the following.

Theorem 2.3 *Let Ω be a bounded open subset of $\mathbb{R}^n$ and suppose that $p \in [1, \infty]$. Then for all $u \in \overset{0}{W}{}^1_p(\Omega)$,*

$$\|u\|_{p,\Omega} \leq (|\Omega| / \omega_n)^{1/n} \, \| |\nabla u| \|_{p,\Omega} .$$

If $p \in [1, n)$ and $q \in \left[p, np/(n-p)\right]$, then there is a constant C such that for all $u \in \overset{0}{W}{}^1_p(\Omega)$,

$$\|u\|_{q,\Omega} \leq C \, |\Omega|^{1/n+1/q-1/p} \, \| |\nabla u| \|_{p,\Omega} .$$

It is sometimes convenient to use the *homogeneous Sobolev space* $\overset{0}{\mathcal{D}}{}^1_p(\Omega)$, where

$$\overset{0}{\mathcal{D}}{}^1_p(\Omega) := \text{ completion of } C_0^\infty(\Omega) \text{ with respect to the norm } u \longmapsto \| |\nabla u| \|_{p,\Omega} .$$

This coincides with $\overset{0}{W}{}^1_p(\Omega)$ when Ω supports the p-Friedrichs inequality, in which case

$$\overset{0}{\mathcal{D}}{}^1_p(\Omega) \subset \left\{ u \in W^1_p(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega \right\},$$

with equality when $\partial\Omega$ is of class C (see [90], Theorem 1.4.2.2).
For elements of the whole space $W^1_p(\Omega)$ there is

Theorem 2.4 (The Poincaré inequality) *Let Ω be a bounded, convex open subset of $\mathbb{R}^n$ with diameter d and let $p \in [1, \infty]$. Then for all $u \in W^1_p(\Omega)$,*

$$\|u - u_\Omega\|_{p,\Omega} \leq (\omega_n / |\Omega|)^{1-1/n} \, d^n \, \| |\nabla u| \|_{p,\Omega},$$

where $u_\Omega = |\Omega|^{-1} \int_\Omega u(x) \, dx$.

The following results concerning $\overset{0}{W}{}^1_p(\Omega)$ are useful.

Theorem 2.5 *Let Ω be an open subset of $\mathbb{R}^n$ and let $p \in (1, \infty)$.*

(i) If $u \in W^1_p(\Omega)$ and supp *u is a compact subset of Ω, then $u \in \overset{0}{W}{}^1_p(\Omega)$.*

(ii) Suppose that $u \in W_p^1(\Omega) \cap C(\overline{\Omega})$. *If* $u = 0$ *on* $\partial\Omega$, $u \in \overset{0}{W}{}_p^1(\Omega)$; *if* $\partial\Omega \in C^1$ *and* $u \in \overset{0}{W}{}_p^1(\Omega)$, *then* $u = 0$ *on* $\partial\Omega$.

Next we give some details of the behaviour under translation of functions in $W_p^1(\Omega)$ that will be used extensively later on.

Proposition 2.6 *Let* Ω *be an open subset of* $\mathbb{R}^n$, *let* $p \in (1, \infty)$ *and suppose that* $u \in W_p^1(\Omega)$. *Then for every open subset* ω *of* $\mathbb{R}^n$ *with compact closure contained in* Ω, *and all* $h \in \mathbb{R}^n$ *with* $|h| < dist(\omega, \partial\Omega)$,

$$\int_\omega |u(x+h) - u(x)|^p \, dx \le |h|^p \, \||\nabla u|\|_{p,\Omega}^p . \tag{2.2.1}$$

If $\Omega = \mathbb{R}^n$, *then for all* $h \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx \le |h|^p \, \||\nabla u|\|_{p,\mathbb{R}^n}^p . \tag{2.2.2}$$

Proof First suppose that $u \in C_0^\infty(\mathbb{R}^n)$. Let $h, x \in \mathbb{R}^n$ and set $v(t) = u(x+th)$ $(t \in \mathbb{R})$. Then $v'(t) = h \cdot \nabla u(x+th)$ and

$$u(x+h) - u(x) = v(1) - v(0) = \int_0^1 h \cdot \nabla u(x+th)\, dt,$$

so that

$$|u(x+h) - u(x)|^p \le |h|^p \int_0^1 |\nabla u(x+th)|^p \, dt. \tag{2.2.3}$$

Thus

$$\begin{aligned}\int_\omega |u(x+h) - u(x)|^p \, dx &\le |h|^p \int_\omega dx \int_0^1 |\nabla u(x+th)|^p \, dt \\ &= |h|^p \int_0^1 dt \int_{\omega+th} |\nabla u(y)|^p \, dy.\end{aligned}$$

If $|h| < \operatorname{dist}(\omega, \partial\Omega)$, there is an open subset ω' of $\mathbb{R}^n$ with compact closure contained in Ω such that $\omega + th \subset \omega'$ for all $t \in [0, 1]$. Hence

$$\int_\omega |u(x+h) - u(x)|^p \, dx \le |h|^p \int_{\omega'} |\nabla u(y)|^p \, dy, \tag{2.2.4}$$

which gives (2.2.1) when $u \in C_0^\infty(\mathbb{R}^n)$. If $u \in W_p^1(\Omega)$, then by Theorem 1.3.14 of [65], there is a sequence $\{u_k\}$ of functions in $C_0^\infty(\mathbb{R}^n)$ such that as $k \to \infty$, $u_k \to u$ in $L_p(\Omega)$ and $\nabla u_k \to \nabla u$ in $\left(L_p(\omega')\right)^n$ for all ω' with compact closure contained in Ω. Application of (2.2.4) to (u_k) and letting $k \to \infty$ gives (2.2.1) for all $u \in W_p^1(\Omega)$; and (2.2.2) follows immediately. □

https://doi.org/10.1017/9781009254625.004 Published online by Cambridge University Press

To conclude this chapter, we give the result that the composition of a bounded Sobolev embedding with an almost compact embedding is compact, following the presentation of Slavíková [159]. Given an open subset Ω of $\mathbb{R}^n$, suppose that X is a Banach function space on (Ω, μ_n), where μ_n is a Lebesgue n-measure, and let

$$W^1X := \{f \in X\colon\ |\nabla f| \in X\},$$

where the functions involved are real-valued and weakly differentiable. We endow W^1X with the norm $\|\cdot\|_X + \||\nabla\cdot|\|_X$. For example, when $X = L_p(\Omega)$ the space just constructed is simply the usual Sobolev space $W_p^1(\Omega)$.

Theorem 2.7 *Let Ω be an open subset of R^n and suppose that X, Y, Z are Banach function spaces over (Ω, μ_n) such that $W^1X \hookrightarrow Y$ and $Y \overset{*}{\hookrightarrow} Z$. Then $W^1X \hookrightarrow\hookrightarrow Z$.*

In Chapter 3 we give an adaptation of this result of Slavíková to deal with fractional Sobolev spaces.

Further illustrations and consequences of this line of thought are given in [145]. In [67], Theorem 2.7 is used to obtain concrete conditions sufficient to ensure the compactness of embeddings of Sobolev spaces based on spaces with variable exponent.