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2-GROUPS WITH EVERY AUTOMORPHISM CENTRAL

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Abstract

An infinite family of 2-groups is produced. These groups have no direct factors and have a non-abelian automorphism group in which all automorphisms are central.

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The automorphisms of a group G which induce the identity on $G/Z(G)$ are called *central*. This paper investigates groups which admit only central automorphisms. Various authors ([2], [5], [7], [9]) considered non-abelian p-groups with abelian automorphism groups in which necessarily every automorphism is central. Curran [1] and Malone [6] constructed non-abelian p-groups with non-abelian automorphism groups in which every automorphism is central. The groups constructed by Curran and Malone, however, all had direct factors. Malone [6] wondered if this condition were necessary. The groups G_n $(n \geqslant 4)$ described below show that direct factors are not necessary.

Define G_n to be the group

$$\Big\langle x_1, \ldots, x_n \,|\, x_i^{2^i} = 1, 1 \leqslant i \leqslant n, [x_i, x_{i+1}] = x_{i+1}^{2^i}, 1 \leqslant i < n, \\ = [x_i, x_j] = 1, 1 < i + 1 < j \leqslant n \Big\rangle.$$

Then G_n has order $2^{n(n+1)/2}$. The group G_3 was first considered by Miller [7] and is group 99 of the Hall and Senior tables [4].

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THEOREM. *For $n \geqslant 3$, G_n has no direct factors and only central automorphisms. The automorphism group of G_n has order $2^{p(n)}$, where $p(n) = (n-1)(2n^2 - n + 6)/6$, and is non-abelian for $n \geqslant 4$.*

PROOF. The centre and derived subgroups of G_n are $\langle x_2^2, x_3^2, \ldots, x_n^2 \rangle$ and $\langle x_2^2, x_3^4, \ldots, x_n^{2^{n-1}} \rangle$, respectively. Let $x(i_1, \ldots, i_n)$ denote $x_1^{i_1} \cdots x_n^{i_n}$. Then every element of G_n may be uniquely represented as $x(i_1, \ldots, i_n)$ with $0 \leqslant i_k < 2^k$. Commutator collection, together with the fact that G_n has class 2, yields the equation

$$x(i_1, \ldots, i_n)^2 = x\big(0, 2i_2(1 + i_1), 2i_3(1 + 2i_2), \ldots, 2i_n(1 + 2^{n-2}i_{n-1})\big).$$

Thus the square of every element is central. Since the centre has exponent 2^{n-1}, G_n has exponent 2^n. All elements of maximal order must have i_n odd.

Suppose G_n is the direct product $A \times B$, where A has exponent 2^n. If $y = x_n^{2^{n-1}}$, then the 2^{n-1}th power of every element of order 2^n is y. Thus y is an element of A, and no element of B has order 2^n. Hence every element of order 2^n is contained in A, and so A contains $x_1 x_n, \ldots, x_{n-1}x_n$, x_n and must therefore equal G_n.

If $i_{k,j}$ are integers, then the map taking x_1 to $x(1, 2i_{1,2}, 4i_{1,3}, \ldots, 2^{n-1}i_{1,n})$ and x_k to $x(2i_{k,1}, \ldots, 2i_{k,k-1}, 2i_{k,k} + 1, 2^2 i_{k,k+1}, \ldots, 2^{n-k+1}i_{k,n})$, $1 < k \leqslant n$, defines a homomorphism of G_n which is the identity on $G_n/Z(G_n)$. Indeed, because the centre of G_n equals the Frattini subgroup, these homomorphisms are automorphisms. There are $2^{p(n)}$ distinct automorphisms of the above form.

As G_n has no abelian direct factors, [8, Theorem 1] may be used to show that G_n has $2^{p(n)}$ central automorphisms. We shall prove by induction that all automorphisms of G_n are central for $n \geqslant 3$, and thus all have the above form. The case $n = 3$ is easy because $p(3) = 7$ and the Miller group, G_3, has 2^7 automorphisms (see [4, 7]).

Suppose $n > 3$. The above remarks show that y is fixed by all automorphisms of G_n. There is an isomorphism $'$ from $G_n/\langle y \rangle$ to $G_{n-1} \times Z$ such that the elements $x_1', \ldots, x_{n-1}'$ generate G_{n-1}, and such that the element x_n' of order 2^{n-1} generates Z. The homomorphism $*$ from $\mathrm{Aut}(G_n)$ to $\mathrm{Aut}(G_{n-1} \times Z)$ and the inductive hypothesis give information about $\mathrm{Aut}(G_n)$.

Let ι be the inclusion map from G_{n-1} to $G_{n-1} \times Z$, π the projection map from $G_{n-1} \times Z$ to G_{n-1}, and ϕ an automorphism of G_n. Then, arguing as in Malone [6], we see that $\iota\phi^*\pi$ is an automorphism of G_{n-1}. So any automorphism of G_n takes x_1 to $x(1, 2i_{1,2}, \ldots, 2^{n-2}i_{1,n-1}, j_{1,n})$ and x_k to $x(2i_{k,1}, \ldots, 2i_{k,k-1}, 2i_{k,k} + 1, 2^2 i_{k,k+1}, \ldots, 2^{n-k}i_{k,n-1}, j_{k,n})$, $1 < k < n$. But x_k is mapped to an element of order 2^k, so 2^{n-k} divides $j_{k,n}$ for $1 \leqslant k < n$. Since x_n' is central and x_n is not, x_n is mapped to $x(2i_{n,1}, \ldots, 2i_{n,n-1}, 2i_{n,n} + 1)$. Thus all automorphisms of G_n are central, which completes the inductive proof.

Finally, $\mathrm{Aut}(G_n)$ is non-abelian for $n \geqslant 4$. Let ϕ be the automorphism which maps x_4 to $x_3^2x_4$ and which fixes the remaining generators. Let ψ map x_3 to $x_3x_4^4$ and fix the other generators. Then $(x_4)\phi\psi = x_3^2x_4^9$ and $(x_4)\psi\phi = x_3^2x_4$. Thus ϕ and ψ do not commute, and so the proof of the theorem is complete.

Further Examples. We exhibit some further examples of 2-groups with all automorphisms central. Let H_n and K_n denote the groups

$$H_n = \langle x_1, \ldots, x_n \mid x_i^4 = 1, 1 \leqslant i \leqslant n, [x_i, x_n] = x_{i+1}^2, 1 \leqslant i < n \rangle, \text{ and}$$

$$K_n = \langle x_0, \ldots, x_n \mid x_0^2 = x_i^4 = x_n^2 = 1, 1 \leqslant i < n, [x_0, x_i] = x_{i+1}^2, \\ 1 \leqslant i < n-1, [x_0, x_{n-1}] = x_n \rangle,$$

where the commutators $[x_i, x_j]$ $(1 \leqslant i < j \leqslant n)$ not shown above are trivial. A proof of the statements made below appears in [3], a copy of which may be obtained from the author. The groups H_n and K_n have order 2^{2n}, and their automorphism groups are elementary abelian of order 2^{n^2}. Furthermore, H_n and K_n are not isomorphic either to themselves or to any of Jonah and Konvisser's groups [5].

The groups G_n, like the groups H_n and K_n, may be generalized to give different groups whose automorphisms are all central. Let $G(l_1, \ldots, l_n)$ denote the group

$$\langle x_1, \ldots, x_n \mid x_i^{m_i} = 1, 1 \leqslant i \leqslant n, [x_{i-1}, x_i] = x_i^{m_i/2}, 1 < i \leqslant n, \\ [x_i, x_j] = 1, 1 < i+1 < j \leqslant n \rangle,$$

where $n \geqslant 3$, $m_i = 2^{l_i}$ and $0 < l_1 < \cdots < l_n$. Then every automorphism of $G(l_1, \ldots, l_n)$ is central if and only if $l_1 = 1$ and $l_2 = 2$. The groups $G(1, 2, l)$ are isomorphic to those considered by Struik [9], and the group $G(1, \ldots, n)$ is isomorphic to G_n. Finally, $G(1, 2, \ldots, l_n)$ has no direct factors and, when $n > 3$, has a non-abelian automorphism group of order 2^p, where $p = 2(n-1) + \sum_{k=3}^{n}[(l_1 + l_2 + \cdots + l_k) - k + (n-k)(k-1)]$.

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