Bull. Aust. Math. Soc. **97** (2018), 308–312
doi:10.1017/S0004972717000946

PRODUCTS OF ROTATIONS BY A GIVEN ANGLE IN THE ORTHOGONAL GROUP

M. G. MAHMOUDI

(Received 24 July 2017; accepted 16 August 2017; first published online 2 November 2017)

Abstract

For every rotation ρ of the Euclidean space $\mathbb{R}^n$ ($n \geq 3$), we find an upper bound for the number r such that ρ is a product of r rotations by an angle α ($0 < \alpha \leq \pi$). We also find an upper bound for the number r such that ρ can be written as a product of r full rotations by an angle α.

2010 *Mathematics subject classification*: primary 51F25; secondary 15A23, 51N30.
Keywords and phrases: rotation group, reflection.

1. Introduction

Let $\mathrm{O}(n)$ denote the *orthogonal group* of $\mathbb{R}^n$, that is, the group of all linear isomorphisms of $\mathbb{R}^n$ which preserve the Euclidean distance; equivalently,

$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) : A^T A = I\}.$$

The subgroup consisting of all elements $A \in \mathrm{O}(n)$ whose determinant is 1 is called the *special orthogonal group* (or the *rotation group*) of $\mathbb{R}^n$ and is denoted by $\mathrm{SO}(n)$,

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R}).$$

Elements of $\mathrm{SO}(n)$ are called *rotations*. A *reflection* σ_u along a nonzero vector $u \in \mathbb{R}^n$ is the linear isomorphism $\sigma_u : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\sigma_u(x) = x - 2\frac{\langle x, u\rangle}{\langle u, u\rangle}u,$$

where $\langle\, ,\rangle$ denotes the ordinary scalar product of $\mathbb{R}^n$. It can be easily checked that $\sigma_u \in \mathrm{O}(n)$, $\det \sigma_u = -1$ and $\sigma_u \circ \sigma_u = \mathrm{id}$.

A *half-turn* (or a 180° rotation) is an element $\rho \in \mathrm{SO}(n)$ for which there exists a subspace $W \subseteq \mathbb{R}^n$ of dimension 2 such that $\rho|_W = -\mathrm{id}$ and $\rho|_{W^\perp} = \mathrm{id}$. Alternatively, a half-turn can be defined as an element of $\mathrm{SO}(n)$ which can be expressed as $\sigma_u\sigma_v$,

The support from Sharif University of Technology is gratefully acknowledged.

where $u, v \in \mathbb{R}^n$ are two nonzero orthogonal vectors. More generally, the composition of two reflections σ_u and σ_v is a rotation by twice the angle between u and v (see [2, Proposition 8.7.7.8]). Anticlockwise and clockwise rotations by an angle α are treated equally. In particular, the angle α of a rotation is considered to be between 0 and π.

A well-known theorem due to Artin [1, page 134] states that for $n \geq 3$ every element of $\mathrm{SO}(n)$ is a product of at most n half-turns. A natural question to ask is how this theorem can be adapted for rotations by an arbitrary angle α.

Intuitively, when α is very small, a rotation by a relatively large angle cannot be expressed as a product of n rotations by the angle α. Hence, in a possible statement of this theorem for an arbitrary angle α, the number of rotations needed for the angle α (that is, *α-rotations*) should depend on α. Note that when π is an integer multiple of α, that is, $\pi = k\alpha$ for some positive integer k, then Artin's theorem directly implies that every element of $\mathrm{SO}(n)$ is a product of at most kn rotations by the angle α. The reason is that in this situation, every rotation by the angle π is a product of k identical rotations by the angle α. But, in general, determining the smallest positive integer m so that every element of $\mathrm{SO}(n)$ can be expressed as the product of at most m rotations by a given angle α cannot be obtained directly from Artin's theorem.

By generalising Artin's original argument and using a simple geometric idea, we prove the following result, which gives an upper bound for the required number of α-rotations.

Theorem 1.1. *Let α be an angle with $0 < \alpha \leq \pi$ and let $n \geq 3$. Let $\rho \in \mathrm{SO}(n)$ be an element which can be written as a product of $2k$ reflections. Then ρ is a product of at most $2mk$ rotations by the angle α, where $m = \frac{1}{2}[\![\pi/\alpha]\!]$. More precisely, ρ can be expressed as a product of $2mk$ elements of the form $\sigma_x\sigma_y$, so that the angle between x and y is $\alpha/2$.*

Here, for a real number x, the expression $[\![x]\!]$ denotes the smallest even integer n such that $x \leq n$. When $\alpha = \pi$, from the above result, we recover Artin's theorem.

Note that by the Cartan–Dieudonné theorem, every rotation can be expressed as a product of at most $t \leq n$ reflections. The minimal number of reflections required to express any element σ of $\mathrm{O}(n)$ as a product of t reflections was determined by Scherk in [6] and is equal to the rank of the linear map $\sigma - \mathrm{id}$. The minimal number of half-turns needed to express any element of the rotation group of an arbitrary quadratic space as a product of t half-turns was determined in various situations in [3, 5] and [4].

Next, we study products of *full α-rotations* in $\mathrm{SO}(n)$. By a full α-rotation, we mean an element $\rho \in \mathrm{SO}(n)$ such that for every nonzero $x \in \mathbb{R}^n$, the angle between x and $\rho(x)$ is α. Alternatively, a full α-rotation $\rho \in \mathrm{SO}(n)$ can be characterised by the identity $\langle x, \rho(x)\rangle = \cos(\alpha)\langle x, x\rangle$ for all $x \in \mathbb{R}^n$. There is no difference between an ordinary α-rotation and a full α-rotation in $\mathbb{R}^2$.

Trivial examples of full rotations on $\mathbb{R}^n$ are id, the identity map, and $-\mathrm{id}$ (for even n), which are full rotations by 0 and π, respectively. A nontrivial example is an *almost-complex structure J* on $\mathbb{R}^n$, a full rotation by $\pi/2$. We recall that an almost-complex

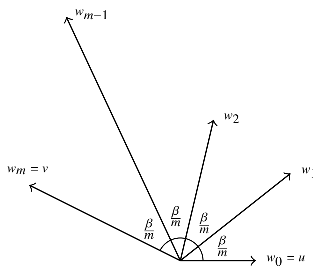

FIGURE 1. Division of β into m equal parts.

structure on $\mathbb{R}^n$ is a map $J \in \mathrm{SO}(n)$ such that $\langle x, J(x)\rangle = 0$ for all $x \in \mathbb{R}^n$. Almost-complex structures on $\mathbb{R}^n$ only exist if n is even.

By a standard argument in linear algebra (see Section 3), we can easily see that for $0 < \alpha < \pi$, a full α-rotation on $\mathbb{R}^n$ only exists if n is even and we prove the following theorem.

THEOREM 1.2. *Let $n \geq 4$ be an even integer and let $0 < \alpha < \pi$. Let $\rho \in \mathrm{SO}(n)$ be an element which can be written as a product of at most $2k$ reflections. Then ρ can be written as a product of at most $2m'k$ full α-rotations, where $m' = \frac{1}{2}⟦\pi/2\alpha⟧$ if $0 < \alpha \leq \frac{1}{2}\pi$ and $m' = \frac{1}{2}⟦\pi/(2\alpha - \pi)⟧$ if $\frac{1}{2}\pi < \alpha < \pi$.*

2. Products of α-rotations in SO(n)

PROOF OF THEOREM 1.1. By assumption, we can write

$$\rho = \sigma_{u_1}\sigma_{v_1}\cdots\sigma_{u_k}\sigma_{v_k},$$

where $u_i, v_i \in \mathbb{R}^n$ $(i = 1, \ldots, k)$ are nonzero and $k \geq 1$ is minimal. Moreover, we may assume that u_i and v_i are linearly independent for $i = 1, \ldots, k$. Hence, it is enough to show that every expression $\sigma_u\sigma_v$ $(u, v \in \mathbb{R}^n\backslash\{0\})$ is a product of at most $2m$ rotations by the angle α. As we have mentioned in Section 1, $\sigma_x\sigma_y$ is a rotation by the angle α if and only if the angle between x and y is $\alpha/2$. Let $w_0 = u, w_1, \ldots, w_m = v$ be nonzero vectors so that the angle between w_i and w_{i+1} is β/m (see Figure 1).

By changing u to $\pm u$ and v to $\pm v$ if necessary, we may assume that $\beta \leq \pi/2$ (note that multiplying the axis u of a reflection σ_u by a scalar does not change σ_u). Since $2m = ⟦\pi/\alpha⟧$, we have $\pi/\alpha \leq 2m$ and hence $\beta/m \leq \pi/(2m) \leq \alpha$. We can write

$$\sigma_u\sigma_v = \sigma_{w_0}\sigma_{w_m} = (\sigma_{w_0}\sigma_{w_1})(\sigma_{w_1}\sigma_{w_2})\cdots(\sigma_{w_{m-1}}\sigma_{w_m}).$$

Hence, it is enough to prove that for $i = 1, \ldots, m-1$, the expression $\sigma_{w_i}\sigma_{w_{i+1}}$ can be written as a product of at most two rotations by the angle α. Since the angle between w_i and w_{i+1} is $\beta/m \leq \alpha$, the problem is reduced to proving that if the angle β between u and v is at most α, then $\sigma_u\sigma_v$ is a product of at most two rotations by the angle α.

https://doi.org/10.1017/S0004972717000946 Published online by Cambridge University Press

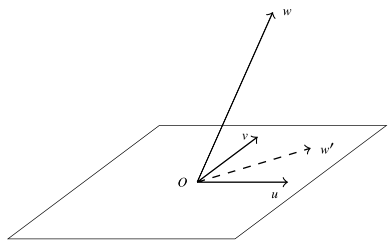

FIGURE 2. The ray w in the plane perpendicular to u and v.

We claim that we can find a nonzero vector w such that the angle between w and both u and v is $\alpha/2$. In this way, we may write $\sigma_u\sigma_v = (\sigma_u\sigma_w)(\sigma_w\sigma_v)$ and the conclusion follows.

To find w, consider the bisector w' of u and v. Consider a plane perpendicular to the subspace generated by u and v and passing through w' (since $n \geq 3$, this is always possible). In this plane, every ray w passing through the origin makes equal angles with u and v (see Figure 2). This angle attains its minimum when w lies on w'. Hence, the minimum value of this angle is $\beta/2$ and it can take an arbitrary value $\geq \beta/2$. In particular, the value $\alpha/2$ is attained since $\beta \leq \alpha$. □

3. Products of full α-rotations in SO(n)

LEMMA 3.1. *Let $\rho \in \mathrm{SO}(n)$. The following conditions are equivalent:*

(a) $\langle x, \rho(x)\rangle = \cos(\alpha)\langle x, x\rangle$ *for every* $x \in \mathbb{R}^n$*;*

(b) $\langle x, \rho(y)\rangle + \langle \rho(x), y\rangle = 2\cos(\alpha)\langle x, y\rangle$ *for every* $x, y \in \mathbb{R}^n$*;*

(c) $\rho^2 - 2\cos(\alpha)\rho + \mathrm{id} = 0$.

PROOF. To prove the implication (a) ⇒ (b), it suffices to replace x by $x + y$ in (a). For the implication (b) ⇒ (a), put $x = y$ in (b). For the equivalence (b) ⇔ (c), note that (b) is equivalent to the identity $\langle \rho^{-1}(x) + \rho(x) - 2\cos(\alpha)x, y\rangle = 0$ and this is also equivalent to (c) since $\langle\ ,\ \rangle$ is nondegenerate. □

LEMMA 3.2. *The following statements hold.*

(a) *If ρ (respectively ρ') is a full α-rotation on $\mathbb{R}^n$ (respectively $\mathbb{R}^m$), then $\rho \oplus \rho'$ is a full α-rotation on $\mathbb{R}^{n+m}$.*

(b) *If ρ is a full α-rotation on $\mathbb{R}^n$, then ρ^{-1} is a full α-rotation on $\mathbb{R}^n$.*

(c) *Let $0 < \alpha < \pi$. Then $\mathrm{SO}(n)$ contains an α-rotation if and only if n is even.*

PROOF. To verify (a), note that

$$\begin{aligned}\langle x \oplus y, (\rho \oplus \rho')(x \oplus y)\rangle &= \langle x, \rho(x)\rangle + \langle y, \rho(y)\rangle = \cos(\alpha)(\langle x, x\rangle + \langle y, y\rangle)\\ &= \cos(\alpha)\langle x \oplus y, x \oplus y\rangle.\end{aligned}$$

For (b), note that $\langle x, \rho^{-1}(x)\rangle = \langle \rho(x), x\rangle = \cos(\alpha)\langle x, x\rangle$ since ρ is an isometry.

https://doi.org/10.1017/S0004972717000946 Published online by Cambridge University Press

To prove (c), consider a unit vector $x \in \mathbb{R}^n$. Since ρ is an α-rotation, we have $\langle x, \rho(x)\rangle = \cos\alpha$. As $\cos(\alpha) \neq \pm 1$, the vectors x and $\rho(x)$ are linearly independent. By Lemma 3.1(c), the subspace W generated by x and $\rho(x)$ is stable under ρ. Since the restriction of ρ to $W^\perp$ (whose dimension is $n-2$) is also a full α-rotation, we can use induction to conclude that n is even. Conversely, let ρ be an ordinary α-rotation on $\mathbb{R}^2$. By (a), the map $\bigoplus_{i=1}^{n/2} \rho$ is a full rotation on $\mathbb{R}^n$. □

Proof of Theorem 1.2. By assumption, we can write

$$\rho = \sigma_{u_1}\sigma_{v_1}\cdots\sigma_{u_k}\sigma_{v_k},$$

where $u_i, v_i \in \mathbb{R}^n$ are nonzero and $k \geq 1$. By Theorem 1.1, every expression $\sigma_u\sigma_v$ can be written as a product of $2m'$ expressions of the form $\sigma_x\sigma_y$, where the angle between x and y is α. We may assume that x and y are linearly independent. Let W be the two-dimensional subspace generated by x and y.

It is enough to express $\sigma_x\sigma_y$ as a product of at most two full α-rotations on $\mathbb{R}^n$. Let w be a nonzero vector in W so that the angles between w and x and between w and y are both equal to $\alpha/2$ (in other words, w is a bisector for x and y). By Lemma 3.2, there exists a full α-rotation J on $W^\perp$. Hence, we can write $\sigma_x\sigma_y = (\sigma_x\sigma_w)(\sigma_w\sigma_y) = (\sigma_x\sigma_w|_W \oplus J)(\sigma_w\sigma_y|_W \oplus J^{-1})$ and the claim is proved. □

Corollary 3.3. *If $n \geq 4$ is even, then every element of* SO(n) *is a product of at most* $2n$ *almost-complex structures.*

Acknowledgements

I thank my colleague Mostafa Esfahanizadeh for inspiring discussions during the course of this work. I also thank the referee for a timely review of the manuscript.

References

[1] E. Artin, *Geometric Algebra* (Interscience, New York, London, 1957).
[2] M. Berger, *Geometry I*, Universitext (Springer, Berlin, 2009). (Translated from the 1977 French original by M. Cole and S. Levy; fourth printing of the 1987 English translation.)
[3] E. W. Ellers, 'Products of half-turns', *J. Algebra* **99**(2) (1986), 275–294.
[4] E. W. Ellers and O. Villa, 'Half turns in characteristic 2', *Linear Algebra Appl.* **483** (2015), 221–226.
[5] H. Ishibashi, 'On some systems of generators of the orthogonal groups', *Sci. Rep. Tokyo Kyoiku Daigaku Sect. A* **11** (1972), 96–105.
[6] P. Scherk, 'On the decomposition of orthogonalities into symmetries', *Proc. Amer. Math. Soc.* **1** (1950), 481–491.

M. G. MAHMOUDI, Department of Mathematical Sciences,
Sharif University of Technology, PO Box 11155-9415, Tehran, Iran
e-mail: mmahmoudi@sharif.ir

https://doi.org/10.1017/S0004972717000946 Published online by Cambridge University Press