THE CATEGORICAL PRODUCT OF GRAPHS

DONALD J. MILLER*

1. Introduction. Undirected graphs and graph homomorphisms as introduced by Sabidussi (**6**, p. 386), form a category that admits a categorical product. For the category of graphs and full graph homomorphisms, the categorical product was introduced by Čulik (**1**) under the name *cardinal product*. It was independently defined by Weichsel (**8**) who called it the *Kronecker product* and investigated the connectedness of products of finitely many factors. Hedetniemi (**4**) was the first to make use of the fact that the cardinal product is categorical. Connectedness studies were recently carried out for products of directed graphs by McAndrew (**5**) and Harary and Trauth (**2**). In the present paper, we are concerned with the connectedness of products of arbitrary families of graphs, and the question, first considered in (**1**, p. 152), of the uniqueness of the decomposition of a graph into indecomposable factors. We also show that the strong product introduced by Sabidussi (**6**) is naturally related to a categorical product, and investigate the relationship between the cardinal and strong product.

The graphs we consider are undirected and have no multiple edges and no loops. $E(X)$ and $V(X)$ will denote the edge set and vertex set of a graph X, respectively. If X is a graph and $e \in E(X)$, we denote by (e) the graph consisting of the edge e and its incident vertices. If Y is a subgraph of X, we define $X\backslash Y$ to be the smallest subgraph with $E(X\backslash Y) = E(X) - E(Y)$.

Let X and Y be graphs. By a *homomorphism* of X into Y we mean a function ϕ: $V(X) \to V(Y)$ such that $[\phi x, \phi y] \in E(Y)$ whenever $[x, y] \in E(X)$. For a homomorphism ϕ: $V(X) \to V(Y)$ we shall write ϕ: $X \to Y$. *A monomorphism* of X into Y is a one-to-one homomorphism. If A is a subgraph of X, we let ϕA denote that subgraph of Y defined by

$$V(\phi A) = \phi(V(A)), \quad E(\phi A) = \{[\phi x, \phi x'] \in E(Y) \mid [x, x'] \in E(A)\}.$$

ϕ: $X \to Y$ is an *epimorphism* if $\phi X = Y$; it is an *isomorphism* if it is both a monomorphism and an epimorphism.

Paths and *circuits* in a graph X will be regarded as subgraphs of X. Connectedness, components and distance are defined, as usual, in terms of paths. The distance between the vertices x and y of X will be denoted by $d_X(x, y)$.

Received June 27, 1967. The content of this paper is based on part of the author's doctoral thesis written at McMaster University under the direction of Professor G. Sabidussi.

*I wish to express my gratitude to Professor G. Sabidussi for his valuable suggestions and assistance in the preparation of this work.

The *diameter* of a connected graph X is

$$\sup_{x,y \in V(X)} d_X(x, y).$$

For any graph X, the number of vertices will be denoted by $|X|$.

A graph X is *bipartite* if and only if $E(X) \neq \emptyset$ and X contains no circuit of odd order. X will be called *non-bipartite* if it contains a circuit of odd order. For a non-bipartite graph, we define the *odd mesh* of X to be $\min|C|$, the minimum taken over all circuits of odd order.

For $x \in V(X)$ we let $V(X; x) = \{y|\ [x, y] \in E(X)\}$. $|V(X; x)|$ is called the *degree* of x and is denoted by $d(x; X)$. X is said to be of *bounded degree* if and only if

$$\sup_{x \in V(X)} d(x;X) < \infty.$$

Given any family (X_a), $a \in A$, of graphs, we define the *cardinal product* $X = \Pi_{a \in A} X_a$ by

$$V(X) = \prod_{a \in A} V(X_a),$$

$$E(X) = \{[x, y]|\ x, y \in V(X), [\mathrm{pr}_a x, \mathrm{pr}_a y] \in E(X_a) \text{ for all } a \in A\}$$

(here, pr_a: $V(X) \rightarrow V(X_a)$ denotes the projection of the cartesian product onto its ath factor). X is easily seen to be categorical: the projections pr_a: $X \rightarrow X_a$ are homomorphisms; hence, if Y is any graph and ϕ_a: $Y \rightarrow X_a$, $a \in A$, a family of homomorphisms, then ϕ: $Y \rightarrow X$, defined by

$$\mathrm{pr}_a(\phi y) = \phi_a y, \quad y \in V(Y), a \in A,$$

is a homomorphism and $\mathrm{pr}_a \phi = \phi_a$ for each $a \in A$. As is customary, we shall denote ϕ by $\Pi_{a \in A} \phi_a$. The product of two graphs will be denoted by $X_1 \times X_2$.

2. Connectedness of the cardinal product. In §§ 2–4, unless otherwise stated, it will always be assumed that $E(X) \neq \emptyset$ for all graphs X. We shall need the following two propositions.

PROPOSITION 1 (**8**, Theorem 1). *Let X_1 and X_2 be connected graphs. Then the following statements are equivalent:*
(i) *$X_1 \times X_2$ is disconnected (consisting of exactly two components);*
(ii) *X_1 and X_2 are both bipartite.*

PROPOSITION 2 (**4**, Corollary 1.26b). *Let X_1 be a bipartite graph and X_2 any graph. Then $X_1 \times X_2$ is bipartite.*

In view of the intimate relationship between the connectedness of a cardinal product and the non-bipartiteness of its factors, we first prove a general converse of Proposition 2.

THEOREM 1. *For each* $a \in A$, *let* X_a *be a non-bipartite graph with odd mesh equal to* n_a. *Then the cardinal product* $\Pi_{a\in A}X_a$ *is non-bipartite* (*with odd mesh equal to* $\sup_{a\in A}n_a$) *if and only if* $\sup_{a\in A}n_a < \infty$.

Proof. Let $X = \Pi_{a\in A}X_a$. First, assume that $\sup_{a\in A}n_a = n < \infty$. For each $a \in A$, let C_a be a circuit of odd order n_a in X_a. Then there exists an $a_0 \in A$ with $|C_{a_0}| = n$. For each $a \in A$, n_a, n odd and $n_a \leqq n$ imply that there exists an epimorphism ϕ_a: $C_{a_0} \to C_a$. Since ϕ_{a_0} is a monomorphism, $\phi = \Pi_{a\in A}\phi_a$ is a monomorphism from C_{a_0} to X. Hence, $\phi C_{a_0} \subseteq X$ is an odd circuit of order n, i.e., X is non-bipartite.

Now, let $C \subseteq X$ be an odd circuit. For $a \in A$, pr_a: $X \to X_a$ being a homomorphism implies that $\mathrm{pr}_a C$ is a non-bipartite subgraph of X_a. Hence,

$$n_a \leqq |\mathrm{pr}_a C| \leqq |C| \quad \text{for all } a \in A.$$

This proves the necessity part of the theorem, as well as, in combination with the first part of the proof, the fact that n is the odd mesh of X.

Theorem 1 allows us to make a comment on a conjecture advanced by Hedetniemi (**4**, Conjecture 1.2). Let A be an index set and for each $a \in A$ let X_a be a graph with chromatic number $\chi(X_a) = n_a$, i.e., n_a is the least cardinal for which there exists a homomorphism ϕ_a: $X_a \to K_{n_a}$, where K_{n_a} is the complete n_a-graph. Since pr_b: $\Pi_{a\in A}X_a \to X_b$ is a homomorphism for each $b \in A$, we have that $\phi_b\, \mathrm{pr}_b$: $\Pi_{a\in A}X_a \to K_{n_b}$ is also a homomorphism, i.e.,

$$\chi\Big(\prod_{a\in A} X_a\Big) \leqq \min_{a\in A} \chi(X_a). \tag{1}$$

Hedetniemi's conjecture is that equality holds for A finite. The question is still open. However, the following example shows that the conjecture cannot be extended to infinite index sets. For $n \geqq 1$, let C_{2n+1} be a circuit of order $2n + 1$. Then $\chi(C_{2n+1}) = 3$, $n \geqq 1$. The odd mesh of C_{2n+1} is $2n + 1$; hence, by Theorem 1, $\Pi_{n\geqq 1}C_{2n+1}$ is bipartite, i.e.,

$$\chi\Big(\prod_{n\geqq 1} C_{2n+1}\Big) = 2,$$

so that (1) is a strict inequality.

We now turn to the connectedness of the cardinal product of a family of connected non-bipartite graphs.

LEMMA 1. *Let* X *be a connected non-bipartite graph of diameter* d, $x, y \in V(X)$ *not necessarily distinct, and* $P = [p_0, \ldots, p_s]$ *a path of length* $s \geqq 4d$. *Then there exists a homomorphism* ϕ: $P \to X$ *such that* $\phi p_0 = x$ *and* $\phi p_s = y$.

Proof. Let C be a circuit of least odd order, $e = [x_0, y_0] \in E(C)$. Note that $|C| \leqq 2d + 1$. Let R_1 be a shortest path joining x and x_0 in X of length r_1, and R_3 a shortest path joining y_0 and y in X of length r_3. Let

$$R_2 = \begin{cases} C\backslash(e) & \text{if } r_1 + r_3 \equiv s \pmod 2, \\ (e) & \text{otherwise,} \end{cases}$$

and let r_2 be the length of R_2. Then

$$r = r_1 + r_2 + r_3 \equiv s \pmod 2$$

and $r \leqq 4d$. Let $P' = [p_0, \ldots, p_r]$. Clearly, there exists a homomorphism $\psi: P' \to R_1 \cup R_2 \cup R_3$ such that $\psi p_0 = x$ and $\psi p_r = y$. But $r \equiv s \pmod 2$ and $r \leqq s$ imply that there exists a homomorphism $\nu: P \to P'$ such that $\nu p_0 = p_0$ and $\nu p_s = p_r$. Then $\psi\,\nu: P \to X$ is the desired homomorphism.

THEOREM 2. *The cardinal product of a family* (X_a), $a \in A$, *of connected non-bipartite graphs is connected if and only if*

$$B = \{b \in A |\ \operatorname{diam} X_b = \infty\}$$

is finite, and

$$D = \{\operatorname{diam} X_a |\ a \in A - B\}$$

is bounded.

Proof. Let $X = \Pi_{a\in A} X_a$ and assume that B is finite and D is bounded. Let $X_1 = \Pi_{b\in B} X_b$ and $X_2 = \Pi_{a\in A-B} X_a$; then $X \cong X_1 \times X_2$. B finite implies, by Proposition 1, that X_1 is connected and, by Theorem 1, that X_1 is non-bipartite. Hence, to show that X is connected, it suffices, by Proposition 1, to show that X_2 is connected.

Let $x, y \in V(X_2)$ and let $P = [p_0, \ldots, p_{4s}]$ be a path of length $4s$, where $s = \sup_{a\in A-B} \operatorname{diam} X_a$. By the lemma, there exists a homomorphism $\phi_a: P \to X_a$ such that

$$\phi_a p_0 = \operatorname{pr}_a x \quad \text{and} \quad \phi_a p_{4s} = \operatorname{pr}_a y, \qquad a \in A - B.$$

Let $\phi = \Pi_{a\in A-B} \phi_a: P \to X_2$. Then

$$\phi p_0 = x \quad \text{and} \quad \phi p_{4s} = y.$$

Since ϕP is a connected subgraph of X_2 and $x, y \in \phi P$, we have that X_2 is connected, and therefore X is connected.

Conversely, assume that X is connected. If B is infinite or D is unbounded, then for $a \in A$ there exist $x_a, y_a \in V(X_a)$ such that

$$\sup_{a\in A} d_{X_a}(x_a, y_a) = \infty. \tag{2}$$

Define $x, y \in V(X)$ by $\operatorname{pr}_a x = x_a$, $\operatorname{pr}_a y = y_a$, $a \in A$. X connected implies that there exists a path P joining x and y in X. $\operatorname{pr}_a P$ is a connected subgraph of X_a containing x_a and y_a, and hence contains a path joining x_a and y_a. Therefore,

$$d_{X_a}(x_a, y_a) \leqq |\operatorname{pr}_a P| \leqq |P|, \qquad a \in A,$$

contradicting (2).

As an immediate corollary of Theorem 2 we have that if (X_a), $a \in A$, is a family of connected non-bipartite graphs such that $X = \Pi_{a\in A} X_a$ is connected, then X is non-bipartite. This follows from $n_a \leqq 2 \operatorname{diam} X_a + 1$ for all $a \in A - B$, where n_a is the odd mesh of X_a. Hence, $\sup_{a \in A} n_a < \infty$, and therefore X is non-bipartite by Theorem 1. The converse of this corollary is obviously not true.

3. Modification of Lemma 1. Lemma 1, which is crucial for the proof of Theorem 2, can be rephrased as a statement concerning homomorphisms of odd circuits into X. At the same time, we shall show that the bound on the length of P in Lemma 1 can be substantially decreased.

PROPOSITION 3. *Let X be a connected non-bipartite graph of finite diameter d, C a circuit of odd length greater than or equal to $3d$. Then, given any $x, y \in V(X)$, there is a homomorphism $\phi: C \to X$ such that $x, y \in V(\phi C)$, Moreover, $3d$ is best possible.*

Proof. Since X is non-bipartite, there exist $z_1, z_2 \in V(X)$ such that $e = [z_1, z_2] \in E(X)$ and $d_X(x, z_1) = d_X(x, z_2)$. Let P be a shortest path joining x and y, Q a shortest path joining y and z_1, R_i a shortest path joining z_i and x, $i = 1, 2$. Then $S = P \cup Q \cup R_1$ is the homomorphic image of a circuit of length $s = d_X(x, y) + d_X(y, z_1) + d_X(z_1, x)$. Trivially, $s \leqq 3d$ (since all paths involved are shortest) and $x, y \in V(S)$. If s is odd, the proof is complete. Assume that s is even.

Case (i): d is odd. s being even means that $s \leqq 3d - 1$. Let

$$S' = \begin{cases} P \cup (Q \backslash e) \cup R_2 & \text{if } e \in E(Q), \\ P \cup Q \cup (e) \cup R_2 & \text{if } e \notin E(Q). \end{cases}$$

S' is the homomorphic image of a circuit of odd length. $s' = s \pm 1 \leqq 3d$, which completes the case that d is odd.

Case (ii): d is even. Construct S' as before. Then $s' \leqq 3d + 1$ (and note that $3d + 1$ is the smallest odd length greater than or equal to $3d$).

To see that $3d$ and $3d + 1$, respectively, are best possible, consider a $(2n + 3)$-circuit D and at each of two adjacent vertices of D attach a path of length n. Let x and y be the two vertices of degree 1 of the resulting graph X. The diameter of X is $2n + 1$ and the shortest odd circuit C that will map properly has length $6n + 3$. This disposes of the case that d is odd.

If d is even, we consider a $(2n + 1)$-circuit D. At one vertex of D we attach two paths of length n and denote the resulting graph by Y and the vertices of degree 1 by x and y. The diameter of Y is $2n$ and the shortest path that will map properly has length $6n + 1$.

Note that Proposition 3 implies that the lower bound on the length s of the path in Lemma 1 can be reduced to $3d$. A straightforward minimality argument will show, in fact, that the length of P in Lemma 1 can further be reduced to $2d$. As a corollary to the proof of Theorem 2, we would then have that

$$\operatorname{diam} \prod_{a \in A} X_a \leqq 2\left(\sup_{a \in A} \operatorname{diam} X_a\right).$$

In the case of the two examples cited above to show that $3d$ and $3d + 1$ are best possible, it is interesting to compare our result with one of Hedetniemi's (**3**), where he is concerned with the existence of circuits that can be mapped epimorphically onto the graph. His result predicts the existence of a circuit of order $q + d(x, y)$ which maps onto the graphs, where q is the number of edges.

4. Decomposition into indecomposable factors. Our aim in this section is to prove a general theorem which shows that the decomposition of a graph into a cardinal product of indecomposable factors is non-unique even for finite connected graphs.

Definition 1. A graph X is called *indecomposable* (or prime) with respect to cardinal multiplication if and only if there do not exist graphs X_1 and X_2 such that $X_1 \times X_2 \cong X$.

It should be pointed out that unit graphs (i.e., graphs consisting of a single vertex and no edges) do not act as identity elements relative to cardinal multiplication. More precisely, if X is any graph and $|Y| = 1$, then $X \times Y \cong X$ if and only if $E(X) = \emptyset$.

Definition 2. Let X and X_0 be graphs. X will be called X_0-*admissible* if and only if there exists a graph X_1 such that

(i) $X_0 \times X_1$ is a spanning subgraph of X;

(ii) $[(x_0, x_1), (x_0', x_1')] \in E(X)$ implies that $[x_0, x_0'] \in E(X_0)$, and $[x_1, x_1'] \in E(X_1)$ or $x_1 = x_1'$;

(iii) if $[(x_0, x_1), (x_0', x_1)] \in E(X)$ for some $[x_0, x_0'] \in E(X_0)$, then $[(y_0, x_1), (y_0', x_1)] \in E(X)$ for all $[y_0, y_0'] \in E(X_0)$.

In view of (iii) we can introduce, for convenience, the following subset $V \subseteq V(X_1)$:

$$x_1 \in V \text{ if and only if } [(x_0, x_1), (x_0', x_1)] \in E(X) \text{ for some } [x_0, x_0'] \in E(X_0).$$

Condition (iii) can then be restated as: for each $[x_0, x_0'] \in E(X_0)$ and each $x_1 \in V$, $[(x_0, x_1), (x_0', x_1)] \in E(X)$. We shall also apply the term X_0-admissible to any graph Y isomorphic to a graph X which is X_0-admissible in the sense just defined. X will be called *properly* X_0-*admissible* if it is X_0-admissible and does not have X_0 as a factor with respect to cardinal multiplication.

Note that condition (ii) implies that if X is X_0-admissible, then $\mathrm{pr}_0 \colon X \to X_0$ is a homomorphism.

Remark. The definition of admissibility can be phrased in terms of another graph multiplication as follows. Let X_0 and X_1 be graphs and $V \subseteq V(X_1)$, V possibly empty. Define $X_0 \otimes_V X_1$ by

$$V(X_0 \otimes_V X_1) = V(X_0) \times V(X_1),$$

$$E(X_0 \otimes_V X_1) = E(X_0 \times X_1) \cup \{[(x_0, x_1), (x_0', x_1)] \mid [x_0, x_0'] \in E(X_0) \text{ and } x_1 \in V\}.$$

For $V = V(X_1)$, we shall denote $X_0 \otimes_V X_1$ by $X_0 \otimes X_1$. (The symbol $\otimes$ should not be confused with the "Kronecker product" of Weichsel (**7**) or the "tensor product" of Harary and Trauth (**2**), both of which are our product, $\times$.) Then, a graph X is X_0-admissible if and only if there exists a graph X_1 and a subset $V \subseteq V(X_1)$ such that $X = X_0 \otimes_V X_1$.

Example. For any non-zero cardinals m, n, and r, the complete bipartite graph $K_{mr,nr}$ is properly $K_{m,n}$-admissible. This follows from

$$K_{m,n} \otimes K_r \cong K_{mr,nr}$$

and the fact that every complete bipartite graph is indecomposable with respect to cardinal multiplication. This can be seen as follows. If $K_{m,n} \cong X_1 \times X_2$, then each factor is a homomorphic image of $K_{m,n}$. But, trivially, any homomorphic image of $K_{m,n}$ is of the form $K_{r,s}$, with $r \leqq m$, $s \leqq n$, and hence bipartite. By Proposition 1, this implies that $K_{m,n}$ is disconnected, a contradiction. Hence, $K_{m,n}$ is indecomposable.

We shall investigate the existence of further properly X_0-admissible graphs after proving the following theorem.

THEOREM 3. *Let X, Y, and Z be arbitrary graphs and $V \subseteq V(Z)$. Then*

$$X \times (Y \otimes_V Z) \cong Y \times (X \otimes_V Z).$$

Proof. Let ϕ: $X \times (Y \otimes_V Z) \to Y \times (X \otimes_V Z)$ be defined by

$$\phi(x, (y, z)) = (y, (x, z)).$$

Obviously, ϕ is one-to-one and onto. To show that ϕ is a homomorphism, let $[(x, (y, z)), (x', (y', z'))] \in E(X \times (Y \otimes_V Z))$. Hence, $[x, x'] \in E(X)$ and $[(y, z), (y', z')] \in E(Y \otimes_V Z)$. Then $[y, y'] \in E(Y)$, and $[z, z'] \in E(Z)$ or $z = z' \in V$. If $[z, z'] \in E(Z)$, then $[(y, (x, z)), (y', (x', z'))]$ obviously belongs to $E(Y \times (X \otimes_V Z))$. If $z = z' \in V$, then $[(x, z), (x', z')] \in E(X \otimes_V Z)$, and therefore $[(y, (x, z)), (y', (x', z'))]$ again belongs to $E(Y \times (X \otimes_V Z))$. Hence, ϕ is a homomorphism.

A similar argument shows that ϕ is an epimorphism, and hence we have that ϕ is an isomorphism.

We now return to the question of the existence of properly X_0-admissible graphs.

LEMMA 2. *Let X_1 and X_2 be finite graphs and let $V \subseteq V(X_2)$ with $|V|$ odd. Then $X_1 \otimes_V X_2$ is properly X_1-admissible.*

Proof. Let X_3 be any graph. Then

$$|E(X_1 \otimes_V X_2)| = m_1(2m_2 + n_2), \qquad |E(X_1 \times X_3)| = 2m_1m_3,$$

where $m_i = |E(X_i)|$, $i = 1, 2, 3$, and $n_2 = |V|$. Hence, if $X_1 \otimes_V X_2 \cong X_1 \times X_3$, then $2m_2 + n_2 = 2m_3$, contrary to n_2 being odd.

Now take X_1, X_2, and V as in Lemma 2, X_0 any finite graph. Then

$$X_0 \times (X_1 \otimes_V X_2) \cong X_1 \times (X_0 \otimes_V X_2),$$

and by Lemma 2, we have that $X_0 \otimes_V X_2$ is properly X_0-admissible.

This shows that the decomposition of connected graphs into a cardinal product of indecomposable factors is non-unique in a very strong sense. For, if we take X_0 and X_1 to be indecomposable and non-isomorphic as well, then X_0 does not occur as a factor in either X_1 or $X_0 \otimes_V X_2$ since $X_0 \otimes_V X_2$ is properly X_0-admissible, and X_1 does not appear as a factor in either X_0 or $X_1 \otimes_V X_2$. A simple illustration of this situation is the following. Take positive integers m, n, r, and s. Then, by the example preceding Theorem 3,

$$K_{mr,nr} \times K_s \cong (K_{m,n} \times (K_s \otimes K_r)).$$

For $s \geqq 3$, this is a connected graph, and in all cases, the four factors $K_{mr,nr}$, K_s, $K_{m,n}$, and $K_s \otimes K_r$ are indecomposable.

5. The strong product. In this section we no longer require that $E(X) \neq \emptyset$.

We define the *strong product* $X^* = \Pi_{a \in A}{}^* X_a$ of a family of graphs (X_a), $a \in A$, by:

(i) $V(X^*) = \Pi_{a \in A} V(X_a)$;

(ii) For $x, y \in V(X^*)$, $[x, y] \in E(X^*)$ if and only if there exists a non-empty subset B of A such that

$$[\mathrm{pr}_b x, \mathrm{pr}_b y] \in E(X_b), \qquad b \in B,$$

and

$$\mathrm{pr}_a x = \mathrm{pr}_a y, \qquad a \in A - B.$$

For strong multiplication, the unit graphs do act as identity elements; however, for $a_0 \in A$, the projection mapping $\mathrm{pr}_{a_0}\colon X^* \to X_{a_0}$ is not a homomorphism provided that one of the factors X_a, $a \neq a_0$, has an edge. The strong product of two graphs will be denoted by $X_1 * X_2$.

THEOREM 4. *The strong product of a family* (X_a), $a \in A$, *of connected graphs is connected if and only if*

$$B = \{b \in A \mid \operatorname{diam} X_b = \infty\}$$

is finite and

$$D = \{\operatorname{diam} X_a \mid a \in A - B\}$$

is bounded.

Proof. Let $X = \Pi_{a \in A}{}^* X_a$ and assume that X is connected. If B is infinite or D is unbounded, then, for $a \in A$, there exist $x_a, y_a \in V(X_a)$ such that

$$\sup_{a \in A} d_{X_a}(x_a, y_a) = \infty. \tag{3}$$

Define $x, y \in V(X)$ by $\mathrm{pr}_a x = x_a$, $\mathrm{pr}_a y = y_a$, $a \in A$. X connected implies that there exists a path P joining x and y in X. $\mathrm{pr}_a P$ is a connected subgraph of

X_a containing x_a and y_a, and hence contains a path joining x_a and y_a. Therefore,

$$d_{X_a}(x_a, y_a) \leqq |\mathrm{pr}_a P| \leqq |P|, \qquad a \in A,$$

contradicting (3).

Suppose that B is finite and D is bounded. Take any $x, y \in V(X)$. Since X_a is connected for each $a \in A$, $\mathrm{pr}_a x$ and $\mathrm{pr}_a y$ can be joined in X_a by a shortest path $P_a = [\mathrm{pr}_a x = x_0{}^a, \; x_1{}^a, \ldots, x_{n(a)}{}^a = \mathrm{pr}_a y]$. Since B is finite, $k_1 = \max_{b \in B} n(b)$ exists and since D is bounded, $k_2 = \max_{a \in A-B} n(a)$ exists. Let $k = \max\{k_1, k_2\}$.

For $0 \leqq i \leqq k$, define $x_i \in V(X)$ as follows:

$$\mathrm{pr}_a x_i = \begin{Bmatrix} x_i{}^a, & 0 \leqq i \leqq n(a), \\ x_{n(a)}{}^a, & n(a) \leqq i \leqq k, \end{Bmatrix}, \qquad a \in A.$$

To show that $[x_i, x_{i+1}] \in E(X)$, $0 \leqq i \leqq k - 1$, we first note that for $a \in A$, either

$$[\mathrm{pr}_a x_i, \mathrm{pr}_a x_{i+1}] \in E(X_a)$$

or

$$\mathrm{pr}_a x_i = \mathrm{pr}_a x_{i+1}.$$

Since $k = \max\{k_1, k_2\}$, there exists an $a_0 \in A$ such that $n(a_0) = k$, and thus

$$[\mathrm{pr}_{a_0} x_i, \mathrm{pr}_{a_0} x_{i+1}] \in E(X_{a_0}).$$

Hence, $[x_i, x_{i+1}] \in E(X)$ and $P = [x_0, \ldots, x_n]$ is a path joining x and y in X. This completes the proof.

The similarity between Theorems 2 and 4 leads one to suspect that the strong product may be related in a natural way to a categorical product in some particular category of graphs. This is precisely the situation.

Let $\mathscr{K}_1$ denote the category of undirected graphs and graph homomorphisms, and let $\mathscr{K}_2$ denote the category of undirected graphs with loops at each vertex and graph homomorphisms. The categorical product is defined in $\mathscr{K}_2$ in an analogous way in which the categorical product is defined in $\mathscr{K}_1$. The strong product in $\mathscr{K}_1$ is related to the categorical product in $\mathscr{K}_2$ in the following manner: For any graph X in $\mathscr{K}_1$, let $R(X)$ denote the graph in $\mathscr{K}_2$ obtained from X by adjoining a loop at each vertex and, for any graph Y in $\mathscr{K}_2$, let $S(Y)$ denote the graph in $\mathscr{K}_1$ obtained from Y by deleting all loops. Then the strong product of a family (X_a), $a \in A$, of graphs in $\mathscr{K}_1$ is related to the categorical product in $\mathscr{K}_2$ by

$$\prod_{a \in A}{}^* X_a = S\Big(\prod_{a \in A} R(X_a)\Big).$$

We now show how various graph multiplications are related. We define the *cartesian product* $X^0 = \Pi_{a \in A}{}^0 X_a$ of a family of graphs (X_a), $a \in A$, as follows:

$$V(X^0) = \prod_{a \in A} V(X_a),$$

https://doi.org/10.4153/CJM-1968-151-x Published online by Cambridge University Press

$E(X^0) = \{[x, y] | \; x, y \in V(X^0), [\mathrm{pr}_a x, \mathrm{pr}_a y] \in E(X_a)$ for exactly one $a \in A$, $\mathrm{pr}_b x = \mathrm{pr}_b y$ for all $b \in A - \{a\}\}$.

The cartesian product of two graphs will be denoted by $X_1 \circ X_2$.

For two factors, the strong, cardinal, and cartesian products are related by

$$X_1 * X_2 = (X_1 \times X_2) \cup (X_1 \circ X_2).$$

LEMMA 3. *The cartesian product of an arbitrary family of connected graphs is connected if and only if the number of factors is finite.*

The proof is trivial.

As a consequence of the following proposition we have that, if X_1 and X_2 are connected non-trivial graphs of bounded degree, then there exists an automorphism ϕ of $X_1 * X_2$ such that $\phi(X_1 \times X_2) = X_1 \circ X_2$ if and only if $X_1 \cong X_2 \cong C_n$ is an n-circuit of odd order.

PROPOSITION 4. *Let X_1 and X_2 be connected non-trivial graphs of bounded degree. Then $X_1 \circ X_2 \cong X_1 \times X_2$ if and only if $X_1 \cong X_2 \cong C_n$, where C_n is an n-circuit of odd order.*

Proof. If $X_1 \times X_2 \cong X_1 \circ X_2$, X_i connected, $i = 1, 2$, we have, by Proposition 1 and Lemma 3, that at least one of the X_i's is non-bipartite, say X_1. If X_2 is bipartite, then $X_1 \times X_2$ is also bipartite by Proposition 2, contrary to $X_1 \circ X_2$ being non-bipartite. Hence, both X_1 and X_2 are non-bipartite. Let the odd mesh of X_1 and X_2 be k_1 and k_2, respectively. Clearly, $X_1 \circ X_2$ has odd mesh equal to $\min\{k_1, k_2\}$ and, by Theorem 1, the odd mesh of $X_1 \times X_2 = \max\{k_1, k_2\}$. Therefore, $k_1 = k_2$.

We now use the fact that X_1 and X_2 are of bounded degree. For $i = 1, 2$, let

$$d_i = \sup_{x \in X_i} d(x; X_i).$$

By hypothesis, $0 < d_i < \infty$, $i = 1, 2$. Then

$$\sup_{x \in X_1 \times X_2} d(x; X_1 \times X_2) = d_1 d_2 \quad \text{and} \quad \sup_{x \in X_1 \circ X_2} d(x; X_1 \circ X_2) = d_1 + d_2.$$

Since $X_1 \times X_2 \cong X_1 \circ X_2$, we have that $d_1 d_2 = d_1 + d_2$, i.e., $d_1 = 2 = d_2$. This, together with X_1 and X_2 being non-bipartite graphs of the same odd mesh, implies that $X_1 \cong X_2 \cong C_n$, where C_n is an odd circuit.

To prove the converse, let $C_n = [x_0, x_1, \ldots, x_{n-1}]$ and define

$$\phi: C_n \circ C_n \to C_n \times C_n$$

as follows: for $0 \leqq i \leqq n - 1$, $0 \leqq j \leqq n - 1$, define

$$\phi(x_i, x_j) = (x_{j+i}, x_{j-i}),$$

where the subscripts are taken mod n.

Since n is odd, we have that ϕ: $V(C_n \circ C_n) \to V(C_n \times C_n)$ is one-to-one and onto. Moreover, it is easily verified that ϕ: $C_n \circ C_n \to C_n \times C_n$ is an isomorphism.

From the previous proposition, we immediately have that *the cardinal product of two non-trivial connected graphs may be a decomposable graph with respect to cartesian multiplication*. The following proposition shows that the situation is quite different for the decomposability of the strong product with respect to either cardinal or cartesian multiplication. We do not include the proof since it is essentially straightforward but tedious.

PROPOSITION 5. *The strong product of two non-trivial connected graphs is indecomposable with respect to cardinal* (*cartesian*) *multiplication.*

REFERENCES

1. K. Čulik, *Zur Theorie der Graphen*, Časopis Pěst. Mat. *83* (1958), 133–155.

2. F. Harary and C. A. Trauth, Jr., *Connectedness of products of two directed graphs*, Siam J. Appl. Math. *14* (1966), 250–254.

3. Stephen T. Hedetniemi, *Homomorphisms of graphs* (University of Michigan Technical Report 03105-42-T, December, 1965; it is stated there that "the complete report is available in the major Navy technical libraries and can be obtained from the Defense Documentation Center").

4. ——— *Homomorphisms of graphs and automata* (University of Michigan Technical Report, Project 03105-44-T, July, 1966; it is stated there that "the complete report is available in the major Navy technical libraries and can be obtained from the Defense Documentation Center").

5. M. H. McAndrew, *On the product of directed graphs*, Proc. Amer. Math. Soc. *14* (1963), 600–606.

6. G. Sabidussi, *Graph derivatives*, Math. Z. *76* (1961), 385–401.

7. ——— *Graph multiplication*, Math. Z. *72* (1960), 446–457.

8. P. M. Weichsel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc. *13* (1962), 47–52.

McMaster University,
Hamilton, Ontario;
University of Waterloo,
Waterloo, Ontario