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REPERCUSSIONS OF A PROBLEM OF ERDŐS AND ULAM ON DENSITY IDEALS

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By $P(\omega)$ we denote the Boolean algebra of all subsets of the set ω of natural numbers. We identify each natural number with the set of its predecessors and define:

$$I_1 = \left\{A \subset \omega : \limsup_{n\to\infty} \frac{\#(n \cap A)}{n} = 0\right\}$$

the ideal of sets of density zero, and

$$I_{\log} = \left\{A \subset \omega : \limsup_{n\to\infty} \frac{\sum_{m\in n\cap A} \frac{1}{m+1}}{\ln n} = 0\right\}$$

the ideal of sets of logarithmic density zero.

In the 1940's, P. Erdős and S. Ulam investigated the problem whether the quotient algebras $P(\omega)/I_1$ and $P(\omega)/I_{\log}$ are isomorphic. They thought they had a proof that these algebras are *not* isomorphic, but the proof was eventually lost. P. Erdős asked the mathematical community to either rediscover the proof or show that it must have been wrong (see [**1**], p. 38–39).

This request led to interesting developments. In [**6**], Adam Krawczyk and I defined the following generalization of the ideals I_1 and $I_{\log}$.

1. DEFINITION. A function $f : \omega \longrightarrow \Re^+$ is called an *EU-function* iff:
(i) $\sum_{n\in\omega} f(n) = \infty$
(ii) $\lim_{m\to\infty} \frac{f(m)}{\sum_{n=0}^{m} f(n)} = 0.$

For an EU-function f we define the ideal

$$I_f = \left\{A \subset \omega : \limsup_{n\to\infty} \frac{\sum_{m\in n\cap A} f(m)}{\sum_{m<n} f(m)} = 0\right\}.$$

It is easily seen that I_1 is I_f for the EU-function $f \equiv 1$, and $I_{\log}$ is I_g for the EU-function g that sends m to $\frac{1}{m+1}$.

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In [**6**] it was shown that the Continuum Hypothesis implies that the algebras $P(\omega)/I_f$ and $P(\omega)/I_g$ are isomorphic for every pair of EU-functions f, g. This settles the question of Erdős: the joint proof with Ulam must have been wrong.

But were Erdős and Ulam *altogether wrong*? The result of [**6**] is only a consistency result; could there perhaps be some model of ZFC in which the two algebras might be non-isomorphic? The latter question led me to develop methods sensitive enough to capture the subtle difference between I_1 and $I_{\log}$. In my doctoral dissertation at the University of Warsaw [**2**], I formulated a statement which I abbreviated CSP and proved its relative consistency with ZFC. This statement implies that the algebras $P(\omega)/I_1$ and $P(\omega)/I_{\log}$ are not isomorphic. It also decides some questions in general topology (see [**3**], [**4**]) which at first glance don't betray any kinship to the problem of Erdős and Ulam. The purpose of the present paper is to make a proof of the following theorem available to the mathematical community.

2. THEOREM [**2**]. *CSP* $\Rightarrow P(\omega)/I_1 \simeq P(\omega)/I_{\log}$.

The method of the proof of the above theorem can be used to show that CSP implies that $P(\omega)/I_f \simeq P(\omega)/I_g$ for more pairs of EU-functions than the one considered here. However, I do not know a nice characteristization of those pairs. Therefore, I decided to present the proof of the special case only, albeit in such a form that it can be easily modified to prove similar results for other pairs of EU-functions.

Throughout this paper, the letters $i, j, k, l, m, n, p, r, s, t$ denote natural numbers; the letters $a, b, c, d, u, v, w, x, y, z$ (with indices if necessary) are reserved for *finite* subsets of ω. Potentially infinite subsets of ω are denoted by letters A, B, C, D, X, Y or Z (possibly with indices).

By $\Re^+$ we denote the set of *positive* reals, i. e. *not* including zero. Letters $\alpha, \beta, \gamma, \delta, \epsilon, \mu, \nu$ denote positive reals.

The interval $[n, m)$ is the set $\{i \in \omega : n \leq i < m\}$.

Instead of e^α we write $\exp(\alpha)$.

Whenever we use Landau's symbols $o(1)$ or $O(1)$ it is understood that the independent variable approaches $+\infty$. E. g., instead of: "$\exists\gamma \in \Re^+ \ \forall n > 0 \ |\sum_{m=0}^{n} \frac{1}{m+1} - \ln n| < \gamma$" we write: "$\sum_{m=0}^{n} \frac{1}{m+1} - \ln n = O(1)$".

Fin denotes the ideal of finite subsets of ω. We write $A =_I B$ to indicate that the symmetric difference $A \triangle B \in I$.

The difference of two sets will be denoted by $A \setminus B$. Note that e. g., $5 - 3 = 2$, but $5 \setminus 3 = [3, 5) = \{3, 4\}$.

Before we plunge into the technical details of the proof of Theorem 2, let us try for a moment to prove in ZFC that $P(\omega)/I_1 \simeq P(\omega)/I_{\log}$. It will be instructive to see where we fail. We could try to split ω into two sequences of consecutive intervals $\big([n_k, n_{k+1})\big)_{k\in\omega}$ and $\big([m_k, m_{k+1})\big)_{k\in\omega}$ such that

$$\lim_{k\to\infty} \frac{\sum_{i=n_k}^{n_{k+1}-1} i}{\sum_{j=m_k}^{m_{k+1}-1} \frac{1}{j+1}} = \lim_{k\to\infty} \frac{n_{k+1}-n_k}{\ln\left[\frac{m_{k+1}}{m_k}\right]} = 1,$$

choose a sequence of mappings $(H_k)_{k\in\omega}$ where $H_k\colon P\big([n_k, n_{k+1})\big) \longrightarrow P\big([m_k, m_{k+1})\big)$, and define for $X \subset \omega\colon H(X) = \bigcup_{k=0}^{\infty} H_k\big(X \cap [n_k, n_{k+1})\big)$.

With some care, we *can* arrange that there is an isomorphic embedding $\underline{H}$ such that the following diagram commutes:

$$\begin{array}{cccc} H: & P(\omega) & \longrightarrow & P(\omega) \\ & \Big\downarrow \pi_{I_1} & & \Big\downarrow \pi_{I_{\log}} \\ \underline{H}: & P(\omega)/I_1 & \longrightarrow & P(\omega)/I_{\log}, \end{array}$$

where π_{I_1}, $\pi_{I_{\log}}$ are the canonical projections. In the sequel, we shall refer to a situation like the above by saying that H is a *lifting* of $\underline{H}$.

There remains however one problem: since the intervals $[m_k, m_{k+1})$ tend to be much longer than the intervals $[n_k, n_{k+1})$, we cannot expect H to be a surjection. Well, it doesn't have to be one. Remember that we care about $\underline{H}$ rather than H. All we need is that for every $\epsilon > 0$ and sufficiently large k, if $y \subset [m_k, m_{k+1})$, then there is some $x \subset [n_k, n_{k+1})$ so that

$$\frac{\sum_{j\in H_k(x)\triangle z} \frac{1}{j+1}}{\ln m_{k+1} - \ln m_k} < \epsilon.$$

It turns out that the latter is impossible. In a sense, that is what the second half of the proof of Theorem 2 (i. e., Lemma 21) is all about. The first half of the proof of Theorem 2 is devoted to showing that if CSP holds, then every isomorphism of $P(\omega)/I_1$ and $P(\omega)/I_{\log}$ must resemble the function $\underline{H}$ described above.

We don't need the full force of CSP here. So we formulate only its consequence CSPD and prove the following:

3. THEOREM. *CSPD* $\Rightarrow P(\omega)/I_1 \simeq P(\omega)/I_{\log}$.

CSPD is an immediate consequence of both CSP and the statement AT whose relative consistency with ZEC is proved in [**5**]. The definition of the full CSP can be found in [**5**].

The statement CSPD can be formulated most conveniently if we treat $P(\omega)$ as a metric space with the metric defined by: $\rho(X, Y) = 2^{-\min(X\triangle Y)}$. For every $A \subset \omega$

we shall treat $P(A)$ as a metric subspace of $P(\omega)$. It thus makes sense to speak about continuous functions from $P(A)$ in $P(\omega)$, analytic subsets of $P(\omega)$, etc.

4. DEFINITION. By CSPD we abbreviate the following statement: "For every pair (f, g) of EU-functions, every homomorphism $\underline{F}: P(\omega)/I_f \longrightarrow P(\omega)/I_g$, and every sequence $(a_k)_{k\in\omega}$ of pairwise disjoint finite subsets of ω, there exists an $A \subset \omega$ which contains infinitely many of the sets a_k and such that the restrictions of $\underline{F}$ to $P(A)/I_f$ has a continuous lifting."

5. REMARK. Obviously, a lifting of $\underline{F}|P(A)/I_f$ is a function $F: P(A) \longrightarrow P(\omega)$.

The remainder of this paper is devoted to the proof of 3. As indicated above, the first part of this proof will be presented in a more general fashion than strictly necessary.

6. DEFINITION. Let f be an EU-function, $A \subset \omega$, and $n \in \omega$. We denote:

(a) $S_f(n) = \sum_{m<n} f(m)$,

(b) $ld_f(A, n) = \frac{\sum_{m\in A\cap n} f(m)}{S_f(n)}$,

(c) $c_f(A) = \sup\{ ld_f(A, n) : n \in \omega\}$,

(d) $d_f(A) = \limsup_{n\to\infty} ld_f(A, n)$.

These definitions are best remembered if we think that ld stands for "local density", c for "concentration", and d for "density". Somewhat abusing terminology we shall write S_1, ld_1, c_1, d_1, if $f \equiv 1$, and $S_{\log}, ld_{\log}, c_{\log}, d_{\log}$ if g is the EU-function that sends m to $\frac{1}{m+1}$. Clearly, $I_{\log} = \{A \subset \omega : d_{\log}(A) = 0\}$.

Now we summarize basic properites of the newly defined functions.

7. PROPOSITION. *Let f be an EU-function.*

(a) The function $ld_f(\cdot, n)$ is a probability measure for every n.

(b) If $X \subset Y$, then $c_f(X) \le c_f(Y)$.

(c) If $(u_k)_{k\in\omega}$ is a sequence of pairwise disjoint finite subsets of ω, and $A \subset \omega$, then $d(A) \ge \limsup_{k\to\infty} c_f(A_k \cap u_k)$.

(d) If $A_k \subset \omega$ for every k, then $c_f(\bigcup_{k=0}^{\infty} A_k) \le \sum_{k=0}^{\infty} c_f(A_k)$.

(e) If a_k is a finite subset of ω for every k, and $\sum_{k=0}^{\infty} c_f(a_k) < +\infty$, then $d_f(\bigcup_{k=0}^{\infty} a_k) = 0$.

Proof. Only (e) requires a proof. For every $\epsilon > 0$ there exists such a number $i(\epsilon)$ such that $\sum_{k=i(\epsilon)}^{\infty} c_f(a_k) < \epsilon$. Let ϵ be fixed and denote: $b = \bigcup_{k=0}^{i(\epsilon)-1} a_k$ and $\max b = l$.

If n is large enough so that $S_f(n) \ge \frac{S_f(l+1)}{\epsilon}$, then: $ld_f(\bigcup_{k=0}^{\infty} a_k, n) \le ld_f(b, n) + ld_f(\bigcup_{k=i(\epsilon)}^{\infty} a_k, n) \le ld_f(l+1, n) + \sum_{k=i(\epsilon)}^{\infty} c_f(a_k) < 2\epsilon$. Since ϵ was chosen arbitrarily, (e) follows.

8. LEMMA. *Suppose f, g are EU-functions, $\underline{F}\colon P(\omega)/I_f \longrightarrow P(\omega)/I_g$ is an isomorphic embedding, $A \in \omega$, and $F\colon P(A) \longrightarrow P(\omega)$ is a continuous lifting of $\underline{F}|P(A)/I_g$. Moreover, let $(y_l)_{l\in\omega}$ be a sequence of pairwise disjoint finite subsets of A. Then there exist: sequences $(u_k)_{k\in\omega}, (v_k)_{k\in\omega}$ of pairwise disjoint finite subsets of ω, and a sequence $(H_k)_{k\in\omega}$ such that:*

(1) $\forall k\, \exists l\; u_k = y_l$

(2) $H_k\colon P(u_k) \longrightarrow P(v_k)$

(3) $H_k(u_k) = v_k$

(4) If we define $B = \bigcup_{k=0}^{\infty} u_k$ and $F^*\colon P(B) \longrightarrow P(\omega)$ by $F^*(X) = \bigcup_{k=0}^{\infty} H_k(X \cap u_k)$, then $F^*(X) =_{I_g} F(X)$ for all $X \subset B$.

Proof. First observe that $P(A)$ is compact. The continuous function F is therefore uniformly continuous. This means that there are increasing sequences of natural numbers $(n_k)_{k\in\omega}, (m_k)_{k\in\omega}$, and a sequence of functions $(G_k)_{k\in\omega}$ such that

$$G_k\colon P(n_k \cap A) \longrightarrow P(m_k) \quad \text{for every } k,$$

$$G_{k+1}(x) \cap m_k = G_k(x \cap n_k) \quad \text{for every and } x \subset n_{k+1} \cap A,$$

$$\text{and } F(X) = \bigcup_{k=0}^{\infty} G_k(X \cap n_k) \quad \text{for every } X \subset A.$$

Passing to a subsequence if necessary, we may without loss of generality assume that for every k there is an l such that $y_l \subset [n_k, n_{k+1})$.

We fix such sequences throughout the proof of 7.

9. DEFINITION. Let $\epsilon > 0$, and $k < k^+$. A set $c \subset [n_k, n_{k^+}) \cap A$ is called an (ϵ, k, k^+)-*stabilizer*, if for every $j > k^+$, arbitrary $a, b \subset n_k \cap A$, and $d \subset [n_{k^+}, n_j) \cap A$, the following inequality holds for every $p \geq m_{k^+}$: $ld_g\big(G_j(a \cup c \cup d) \triangle G_j(b \cup c \cup d), p\big) < \epsilon$.

10. PROPOSITION. *For every $\epsilon > 0$ and $k \in \omega$ there exists a number $k^+ > k$ and an (ϵ, k, k^+)-stabilizer c.*

Proof. Assume that for certain k_0 and ϵ no such objects exist. Then there are increasing sequences $(k_i)_{i\in\omega}, (p_i)_{i\in\omega}$ such that for every i:

(a) $d_i \subset [n_{k_0}, n_{k_{i+1}})$,

(b) $d_i \subset d_{i+1} \subset A$,

(c) $d_{i+1} \cap n_{k_{i+1}} = d_i$,

(d) $a_i, b_i \subset n_{k_0}$,

(e) $m_{k_i} \leq p_i < m_{k_{i+1}}$,

(f) $ld_g\big(G_{k_{i+1}}(a_i \cup d_i) \triangle G_{k_{i+1}}(b_i \cup d_i), p_i\big) \geq \epsilon$.

By Dirichlet's pigeonhole principle, we may without loss of generality assume that $a_i = a$ and $b_i = b$ for all i and fixed $a, b \subset n_{k_0}$.

It follows that

$$d_g\Big(F(a \cup \bigcup_{i=0}^{\infty} d_i) \triangle F(b \cup \bigcup_{i=0}^{\infty} d_i)\Big)$$
$$= d_g\Big(\bigcup_{i=0}^{\infty}\Big(G_{k_{i+1}}(a \cup d_i) \triangle G_{k_{i+1}}(b \cup d_i)\Big)\Big)$$
$$\geq \limsup_{i\to\infty} ld_g\Big(G_{k_{i+1}}(a \cup d_i) \triangle G_{k_{i+1}}(b \cup d_i), p_i\Big)$$
$$\geq \epsilon.$$

Hence, $F(a \cup \bigcup_{i=0}^{\infty} d_i) \triangle F(b \cup \bigcup_{i=0}^{\infty} d_i) \notin I_g$. This contradicts the fact that F is a lifting of a function from $P(A)/I_f$ into $P(\omega)/I_g$.

Proposition 10 guarantees that there exist: an increasing sequence $(k_i)_{i\in\omega}$ and a sequence $(c_i)_{i\in\omega}$ such that $c_i \subset [n_{k_i}, n_{k_{i+1}}) \cap A$ is a $(2^{-i}, k_i, k_{i+1})$-stabilizer. To simplify the notation, we assume that $k_i = i$ for all i.

Denote: $n_{-1} = m_{-1} = 0$.

For every k we choose a $y_{1_k} \subset [n_{2k-1}, n_{2k}) \cap A$ and put: $u_k = y_{1_k}$, $\tilde{v} = [m_{2k-1}, m_{2k+1})$, $B = \bigcup_{k=0}^{\infty} u_k$, $C = \bigcup_{k=0}^{\infty} c_{2k}$, $z_k = \bigcup_{i=0}^{k-1}(u_i \cup c_{2i})$. For $a \subset u_k$ we define: $\tilde{H}_k(a) = G_{2k+1}(z_k \cup a \cup c_{2k}) \cap \tilde{v}_k \cap F(A)$, and let $v_k = \tilde{H}_k(u_k) \cap \tilde{v}_k$, $H_k(a) = \tilde{H}_k(a) \cap v_k$.

Clearly, points (1)–(3) of Lemma 8 are satisfied. We check (4). Let $X \subset B$. We have to show that $F^*(X) = \bigcup_{k=0}^{\infty} H_k(X \cap u_k) =_{I_g} F(X)$. Observe that since c_{2k-2} was chosen a $(2^{-2k+2}, 2k-2, 2k-1)$-stabilizer, we have $c_g\Big(G_{2k+1}\big((X \cup C) \cap n_{2k+1}\big) \triangle G_{2k+1}\big(z_k \cup (X \cap u_k) \cup c_{2k}\big) \cap \tilde{v}_k\Big) \leq 2^{-2k-2}$.

Since $\bigcup_{k=0}^{\infty} \tilde{v}_k = \omega$, it follows from the choice of $(G_k)_{k\in\omega}$ that

$$F(X \cup C) = \bigcup_{k=0}^{\infty} G_{2k+1}\big((X \cup C) \cap n_{2k+1}\big)$$
$$=_{I_g} \bigcup_{k=0}^{\infty}\Big(G_{2k+1}\big(z_k \cup (X \cap u_k) \cup c_{2k}\big) \cap \tilde{v}_k\Big).$$

Since F is a lifting of a homomorphism, the following holds: $F(X) =_{I_g} F(X \cup C) \cap F(A) \cap F(B \cup C) = \bigcup_{k=0}^{\infty} H_k(X \cap u_k) = F^*(X)$. This concludes the proof of Lemma 8.

11. LEMMA. *Let $\underline{F}$, F be as in the hypothesis of 8, and assume $(H_k)_{k\in\omega}$, $(u_k)_{k\in\omega}$, $(v_k)_{k\in\omega}$, B and F^* satisfy (2)–(4) of 8. Then there are functions $L, M, N\colon \Re^+ \longrightarrow \omega$ and $\alpha\colon \Re^+ \longrightarrow \Re^+$ such that for all $\epsilon > 0$ and $k \in \omega$:*

(a) If $k \geq L(\epsilon)$, then $c_g\big(H_k(\emptyset)\big) < \epsilon$,

(b) If $k \geq M(\epsilon)$, and $a, b \subset u_k$, then $c_g\Big(\big(H_k(a) \cap H_k(b)\big) \triangle H_k(a \cap b)\Big) < \epsilon$,

(c) If $k \geq N(\epsilon)$, and $a \subset u_k$ is such that $c_f(a) \geq \epsilon$, then $c_g\big(H_k(a)\big) \geq \alpha(\epsilon)$.

Proof. Point (a) follows immediately from 6(c) and the fact that $d_g\big(F^*(\emptyset)\big) = 0$. Point (b) follows from 6(c) and the fact that $F^*(X \cap Y) \triangle \big(F^*(X) \cap F^*(Y)\big) \in I_g$ for $X, Y \subset B$.

Assume now that (c) fails, i. e., that there are: an $\epsilon > 0$ and sequences $(k_i)_{i \in \omega}$, $(x_i)_{i \in \omega}$ such that for all i:

(i) $x_i \subset u_{k_i}$,

(ii) $c_f(x_i) \geq \epsilon$,

(iii) $c_g\big(H_{k_i}(x_i)\big) \leq 2^{-i}$.

Let $X = \bigcup_{i=0}^{\infty} x_i$ and $Y = \bigcup_{i=0}^{\infty} H_{k_i}(x_i)$. It follows from 7(e) that $d_g(Y) = 0$. On the other hand, $F^*(X) \triangle Y = \bigcup \{ H_k(\emptyset) : k \in \omega \text{ and } \forall_i\, k \neq k_i \} \subset F^*(\emptyset) \in I_g$. Hence, $F^*(X) \in I_g$. But from 7(c) we infer that $X \notin I_f$. Therefore, $\mathrm{Ker}(\underline{F}) \neq 0$, contradicting our assumption that $\underline{F}$ is an isomorphic embedding of $P(\omega)/I_f$ into $P(\omega)/I_g$.

Before we can formulate the next lemma, we must introduce another bit of terminology.

12. DEFINITION . Let f be an EU-function, and suppose $\mu \in (0, 1)$. We define inductively a function $\mathrm{acc}_f(\cdot, \mu, \cdot)$ as follows: $\mathrm{acc}_f(t, \mu, 0) = t$, $\mathrm{acc}_f(t, \mu, s + 1) = \min\{ p : ld_f\big([\mathrm{acc}_f(t, \mu, s), p), p\big) \geq \mu \}$. In particular, $\mathrm{acc}_f(t, \mu, 1) = \min\{ p : \frac{\sum_{m=t}^{p-1} f(m)}{\sum_{m<p} f(m)} \geq \mu \}$.

13. EXAMPLE .

(a) $\mathrm{acc}_f(t, \frac{1}{2}, s) = t \cdot 2^s$ for all s, t.

(b) $\mathrm{acc}_{\log}(t, \mu, 1) = \min\{ p : \frac{\ln p - \ln t}{\ln p} \geq \mu \} + O(1) = \min\{ p : p \geq t^{\frac{1}{1-\mu}} \} + O(1)$.

It follows that $\mathrm{acc}_{\log}(t, \mu, s)$ is of the order of magnitude of $\exp\Big(\big(\frac{1}{1-\mu}\big)^s \cdot \ln t\Big)$.

14. LEMMA. *Suppose $\underline{F}$, F are as in the hypothesis of Lemma 8, and that $(H_k)_{k \in \omega}$, $(u_k)_{k \in \omega}$, $(v_k)_{k \in \omega}$, B and F^* satisfy (2)–(4) of Lemma 8. Let $t \in \omega$, and assume that every u_k is of the form $\big[t_k, \mathrm{acc}_f(t_k, \frac{1}{2}, t_k)\big)$. Then there exist $\mu \in \Re^+$ and k_0, $r \in \omega$ such that for every $k > k_0$ there exists $w_k \subset v_k$ such that $\min w_k \geq \mathrm{acc}_g(t, \mu, \delta_k)$ and $c_g(w_k) \geq \mu$, where $\delta_k = \lfloor t_k / 2\lceil \frac{1}{\mu} \rceil \rfloor - 1$.*

Proof. Let α, L, M, N be functions which satisfy (a)–(c) of Lemma 11. Let $\mu = \frac{\alpha(\frac{1}{2})}{2}$ and $\beta = \frac{\mu}{(2\lceil \frac{1}{\mu} \rceil + 1)^2}$. Furthermore, we denote: $a_{k,i} = \big[\mathrm{acc}_f(t_k, \frac{1}{2}, i), \mathrm{acc}_f(t_k, \frac{1}{2}, i + 1)\big)$ and $b_{k,i} = H_k(a_{k,i})$ for $i < t_k$. We fix a number $l > \max\{ L(\frac{\beta}{2}), M(\frac{\beta}{2}), N(\frac{1}{2}) \}$ such that $\min v_k > t$ whenever $k \geq l$.

15. CLAIM. *If $k \geq 1$ and $i < j < t_k$, then*
(a) $c_f(a_{k,i}) \geq \frac{1}{2}$,
(b) $c_g(b_{k,i}) \geq 2\mu$,
(c) $c_g(b_{k,i} \cap b_{k,j}) \leq \beta$.

Proof. Point (a) follows from the definition of $a_{k,i}$, point (b) from the fact that $k \geq l > N(\frac{1}{2})$ and $\alpha(\frac{1}{2}) = 2\mu$. The following inequalities prove (c):

$$\begin{aligned} c_g(b_{k,i} \cap b_{k,j}) &= c_g\big(H_k(a_{k,i}) \cap H_k(a_{k,j})\big) \\ &\leq c_g\Big(\big(H_k(a_{k,i}) \cap H_k(a_{k,j}) \triangle H_k(a_{k,i} \cap a_{k,j})\big) \cup H_k(a_{k,i} \cap a_{k,j})\Big) \\ &\leq c_g\big(b_{k,i} \cap b_{k,j} \triangle H_k(\emptyset)\big) + c_g\big(H_k(\emptyset)\big) \\ &\leq \frac{\beta}{2} + \frac{\beta}{2}. \end{aligned}$$

By 15(b), we may define for $k \in \omega$ and $i < t_k$: $p_{k,i} = \min\{n \colon ld_g(b_{k,i}, n) \geq 2 \cdot \mu\}$. To keep the notation transparent, we assume that $p_{k,i} \leq p_{k,j}$ for $i < j < t_k$. Let r denote $2\lceil \frac{1}{\mu} \rceil$.

16. CLAIM. *Let $k \geq l$ and $i < t_k - r$. Then there is an index j such that $ld_g(b_{k,j} \setminus p_{k,i}, p_{k,j}) \geq \mu$.*

Proof. Fix $k > l$ and $i < t_k - r$. Instead of $b_{k,i}, p_{k,i}$ we write b_i, p_i. By 7(a), the function $\nu(\cdot) = ld_g(\cdot, p_i)$ is a probability measure on $P(\omega)$. If $\nu(b_j) \geq \mu$ for all $j \in [i, i + r]$, then we obtain a contradiction, as

$$\begin{aligned} 1 &\geq \nu\Big(\bigcup_{j=1}^{i+r} b_j\Big) \\ &\geq \sum_{j=1}^{i+r} \nu(b_j) - \nu\Big(\bigcup_{i \leq j < j' \leq i+r} b_j \cap b_{j'}\Big) \\ &\geq \sum_{j=1}^{i+r} \mu - \sum_{i \leq j < j' \leq i+r} \nu(b_j \cap bj') \\ &\geq (r + 1)\mu - \binom{r+1}{2}\beta \qquad (1) \\ &= \mu(r + 1) - \frac{(r + 1)r}{2}\frac{\mu}{(r + 1)^2} \qquad (2) \\ &> \mu r \\ &= \mu \cdot 2\Big\lceil \frac{1}{\mu} \Big\rceil \\ &\geq 2. \end{aligned}$$

Inequalitiy (1) follows from 15(c), and equality (2) from the definitions of r and β.

Fix now $j \in [i, i+r]$ such that $\nu(b_j) < \mu$. Since $\nu(b_i) \geq 2\mu$, by our choice of p_i, we must have $j > i$ and hence, $p_j \geq p_i$. Therefore, $\mu > ld_g(b_j, p_i) \geq ld_g(b_j \cap p_i, p_j)$. On the other hand, $ld_g(b_j \setminus p_i, p_j) \geq 2\mu$. Since $ld_g(\cdot, p_j)$ is a measure, the desired inequality $ld_g(b_j \setminus p_i, p_j) \geqq \mu$ follows.

17. COROLLARY. *If $k \geq l$ and $i < t_k - r$, then $p_{k,i+r} \geq \mathrm{acc}_g(p_{k,i}, \mu, 1)$.*

18. COROLLARY. *If $k \geq l$ and $i < t_k - r$, then $p_{k,t_k-r} \geq \mathrm{acc}_g(p_{k,0}, \mu, \lfloor \frac{t_k}{r} \rfloor - 1)$.*

Since $p_{k,0} > t$ by the choice of l, we can replace the right hand side of 17 by $\mathrm{acc}_g(t, \mu, \lfloor \frac{t_k}{r} \rfloor - 1)$.

For $k \geq l$ let $j \in [t_k - r, t_k)$ be such that

$$ld_g(b_{k,j} \setminus p_{k,t_k-r-1}, p_{k,t_k-r-1}) \geq \mu$$

and put $w_k = b_{k,j}$. Then w_k is as required in Lemma 14.

From now on we shall consider specifically the ideals I_1 and $I_{\log}$.

19. DEFINITION . The following statement will be abbreviated by ST in the sequel. "$\exists \gamma, \mu > 0 \, \forall \epsilon > 0 \, \forall l \in \omega \, \exists n > l \, \exists w \in \mathrm{Fin} \, \exists W \subset P(w)$ *(i), (ii), (iii),* and *(iv)*", where:

(i) $\min w \geq \exp(n^\gamma)$.

(ii) $c_{\log}(w) \geq \mu$.

(iii) $\#(W) \leq 2^n$.

(iv) $\forall z \subset w \, \exists y \in W \, c_{\log}(z \triangle y) < \epsilon$.

20. LEMMA. *Suppose CSPD holds and the algebras $P(\omega)/I_1$ and $P(\omega)/I_{\log}$ are isomorphic. Then ST holds.*

Proof. Let $\underline{F}: P(\omega)/I_1 \longrightarrow P(\omega)/I_{\log}$ be an isomorphism. By *CSPD*, there exists an $A \subset \omega$ which contains infinitely many sets of the form $[t_1, t_1 \cdot 2^{t_1})$ (i. e., $\left[t_1, \mathrm{acc}_1(t_1, \frac{1}{2}, t_1)\right)$; see 13(a)), and a continuous lifting $F: P(A) \longrightarrow P(\omega)$ of $\underline{F}|P(A)/I_1$. Apply Lemma 8 to $\underline{F}$, F and $y_1 = [t_1, t_1 \cdot 2^{t_1})$ to get sequences $(u_k)_{k\in\omega}$, $(v_k)_{k\in\omega}$, $(H_k)_{k\in\omega}$, a set B and a function F^* that satisfy (1)–(4). Then apply Lemma 14 to these objects and $t = 3$.

Let μ be the constant given by Lemma 14. By 13(b), $\exp\left((\frac{2}{2-\mu})^{\delta_k}\right)$ is a safe estimate from below of $\min w_k$. It is easily seen that $\delta_k \geq t_k/\nu$ for a certain positive constant ν and all k.

Now $(\frac{2}{2-\mu})^{\delta_k} \geq (\frac{2}{2-\mu})^{t_k/\nu} = \exp\left(t_k/\nu \cdot \ln(\frac{2}{2-\mu})\right) = 4^{t_k \cdot \gamma} \geq n(k)^\gamma$, where $\gamma = \frac{\ln 2 - \ln(2-\mu)}{\nu \cdot \ln 4}$ and $n(k) = t_k \cdot 2^{t_k}$.

We have thus found μ, γ as in ST. For every fixed ϵ, l, the w will be one of the w_k's given by Lemma 14, n will be the corresponding $n(k)$, and W the corresponding $W_k = \{H_k(x) : x \subset [t_k, t_k \cdot 2^{t_k})\}$. Clearly, (i), (ii) and (iii) are satisfied by these choices.

We show that (iv) also holds for some large enough k. Suppose that for some $\epsilon > 0$ and $k_0 \in \omega$ we are unable to choose w_k for $k \geq k_0$ so that (iv) holds. For every $k \geq k_0$ pick $z_k \subset w_k$ such that $c_{\log}(z_k \triangle y) \geq \epsilon$ for all $y \in W_k$.

Let $Z = \bigcup_{k=k_0}^{\infty} z_k$. Since $Z \subset F^*(B)$, and since $\underline{F}$ was supposed to be a Boolean homomorphism mapping $P(\omega)/I_1$ onto $P(\omega)/I_{\log}$, there is an $X \subset B$ such that $F^*(X) \triangle Z \in I_{\log}$. Let $y_k = H_k\big(X \cap [t_k, t_k \cdot 2^{t_k})\big)$. Then $y_k \in W_k$, and $F^*(X) = \bigcup_{k=0}^{\infty} y_k$. By 7(c), $g_{\log}\big(F^*(X) \triangle Z \cap v_k\big) < \epsilon$ for sufficiently large k. But $F^*(X) \triangle Z \cap v_k = y_k \triangle z_k$ for $k \geq k_0$. This contradicts our choice of z_k, and thus concludes the proof of 20.

The next lemma is the last brick needed in the proof of 3.

21. LEMMA. *ST is false.*

Proof. Let $\gamma, \mu > 0$, and choose $\epsilon = \frac{\mu}{4}$. Assume that n, w, W are such that (i)–(iii) hold. It suffices to show that if n exceeds a certain number 1 which depends only on γ, μ and ϵ, then there exists a $z \subset w$ such that $c_{\log}(z \triangle y) \geq \epsilon$ for every $y \in W$.

Let $y \subset w$. The idea is to show that the number of subsets $z \subset w$ which satisfy $c_{\log}(y \triangle z) < \epsilon$ is smaller than $2^{\#(w)} \cdot 2^{-n}$. But we don't know the number of elements of w, so counting these sets, even if possible, would not be of much help. This problem is solved in the following way: We treat $P(w)$ as a probability space, and assign the same probability $2^{-\#(w)}$ to every $z \subset w$. For $m \in \omega$ we define a random variable ξ_m as follows:

$$\xi_m(z) = \begin{cases} \frac{1}{m+1} & \text{if } m \in y \triangle z \\ 0 & \text{if } m \notin y \triangle z. \end{cases}$$

We let $p = \min\{k : ld_{\log}(w, k) \geq \mu\}$ and define a random variable ξ on $P(w)$ by: $\xi(z) = \sum_{m \in w \cap p} \xi_m(z)$.

Note that $c_{\log}(y \triangle z) \geq ld_{\log}(y \triangle z, p) \geq \frac{\xi(z)}{S_{\log}(p)} = \frac{\xi(z)}{\ln p + O(1)}$.

Therefore, $\Pr\big(\{c_{\log}(y \triangle z) < \epsilon\}\big) \leq \Pr\big(\{\xi(z) < (\epsilon + o(1)) \ln p\}\big)$. Hence, we shall be done if we show that for sufficiently large n the following inequality holds:

$$(1) \qquad \Pr\big(\{\xi < (\epsilon + o(1)) \ln p\}\big) < 2^{-n}.$$

Namely, if (1) holds, then $\Pr\big(\{\exists y \in W \; c_{\log}(y \triangle z) < \epsilon\}\big) = \Pr\big(\{\bigcup_{y \in W} c_{\log}(y \triangle z) < \epsilon\}\big) \leq \sum_{y \in W} \Pr\big(\{c_{\log}(y \triangle z) < \epsilon\}\big) \leq |W| \cdot \Pr\big(\{\xi < (\epsilon + o(1)) \ln p\}\big) < 2^n \cdot 2^{-n} = 1$.

The last inequality means that the event which contains all z for which $c_{\log}(y \triangle z) \geq \epsilon$ has a positive probability, and is therefore nonempty. That is precisely what we need.

For the proof of (1), it will be convient to consider centralized random variables. We put: $\tilde{\xi}_m = \xi_m - \frac{1}{2(m+1)}$ for $m \in w \cap p$, and $\tilde{\xi}_m = \sum_{m \in w \cap p} \tilde{\xi}_m$. Then $E[\tilde{\xi}_m] = 0$ for all m, and hence, $E[\tilde{\xi}\,] = 0$ as well.

The $\tilde{\xi}_m$'s are independent. Moreover, $\xi < (\epsilon + o(1)) \ln p$ if and only if $\tilde{\xi}_m < -\sum_{m \in w \cap p} \frac{1}{2(m+1)} + (\epsilon + o(1)) \ln p$. It follows from the choice of p and ϵ that $-\sum_{m \in w \cap p} \frac{1}{2(m+1)} + (\epsilon + o(1)) \ln p \leq \frac{\mu}{2} \ln p + (\frac{\mu}{4} + o(1)) \ln p = -\big(\epsilon + o(1)\big) \ln p$.

Thus (1) can be reformulated in terms of $\tilde{\xi}$ as follows:

$$(2) \qquad \Pr\Big(\big\{\, \tilde{\xi} < -\big(\epsilon + o(1)\big) \ln p \big\}\Big) < 2^{-n}.$$

To be more specific, we show that for sufficiently large n the following strengthening of (2) holds:

$$(3) \qquad \Pr\Big(\big\{\, \tilde{\xi} < -\frac{\epsilon}{2} \ln p \big\}\Big) < 2^{-n}.$$

We need the following.

22. CLAIM. (Folklore) *Let $(\eta_i)_{i=1}^r$ be a sequence of independent random variables such that $E[\eta_i] = 0$ and $|\eta_i| \leq \delta_i$ for every i. Then for arbitrary $\beta > 0$ the following holds:* $\Pr\big(\{\, \sum_{i=1}^r \eta_i \geq \beta \,\}\big) \leq \exp\big(\beta^2 / (4 \sum_{i=1}^r \delta_i^2)\big)$.

Proof. Let η_i's be as in the assumption, and let α be an arbitrary positive real.

$$\begin{aligned}\Pr\Big(\big\{\, \sum_{i=1}^r \eta_i \geq \beta \,\big\}\Big) &= \Pr\Big(\big\{\, \alpha \sum_{i=1}^r \eta_i \geq \alpha \cdot \beta \,\big\}\Big) \\ &= \Pr\Big(\big\{\, \exp(\alpha \sum_{i=1}^r \eta_i) \geq \exp(\alpha\beta \,\big\}\Big) \\ &= A.\end{aligned}$$

By Markov's inequality,

$$\begin{aligned}A &\leq \frac{E[\exp(\alpha \cdot \sum_{i=1}^r \eta_i)]}{\exp(\alpha\beta)} \\ &= \frac{E[\prod_{i=1}^r \exp(\alpha \cdot \eta_i)]}{\exp(\alpha\beta)} \\ &= \frac{\prod_{i=1}^r E[\exp(\alpha\eta_i)]}{\exp(\alpha\beta)} \\ &= B.\end{aligned}$$

For every $\gamma \in \Re^+$ the inequality $\exp(\gamma) \leq \gamma + \exp(\gamma^2)$ holds. Therefore,

$$\begin{aligned} B &\leq \frac{\prod_{i=1}^r E[\alpha \cdot \eta_i + \exp(\alpha^2 \eta_i^2)]}{\exp(\alpha\beta)} \\ &= \frac{\prod_{i=1}^r \left(\alpha E[\eta_i] + E[\exp(\alpha^2\eta_i^2)]\right)}{\exp(\alpha\beta)} \\ &= \frac{\prod_{i=1}^r E[\exp(\alpha^2\eta_i^2)]}{\exp(\alpha\beta)} \\ &\leq \frac{\prod_{i=1}^r \exp(r^2\delta_i^2)}{\exp(\alpha\beta)} \\ &= \exp(\alpha^2 \cdot \sum_{i=1}^r \delta_i^2 - \alpha\beta) \\ &= C. \end{aligned}$$

Substituting $\alpha = \beta / (2\sum_{i=1}^r \delta_i^2)$ in C we get: $C = \exp(-\beta^2 / \left(4\sum_{i=1}^r \delta_i^2\right))$, as desired.

Now observe the $\tilde{\xi}$ is a symmetric random variable, and that the sequence $(\tilde{\xi}_m)_{m \in w \cap p}$ satisfies the assumptions of 22 with $\delta_m = \frac{1}{2(m+1)}$. Therefore:

$$\begin{aligned} \Pr\left(\{\tilde{\xi} < -\frac{\epsilon}{2} \ln p\}\right) &= \Pr\left(\{\tilde{\xi} > \frac{\epsilon}{2} \ln p\}\right) \\ &\leq \exp\left(-\frac{\epsilon^2}{4}(\ln p)^2 / \left(4 \sum_{m \in w \cap p} \frac{1}{4(m+1)^2}\right)\right) \\ &= D. \end{aligned}$$

Let l_1 be such that $\frac{\epsilon^2}{4} \cdot l_1^{2\gamma} \geq 1$. If $n > l_1$, then $p \geq \min w \geq \exp(n^\gamma)$, so $\frac{\epsilon^2}{4} \cdot (\ln p)^2 \geq 1$. For $n > l_1$ we can thus estimate:

$$\begin{aligned} D &\leq \exp\left(-1 / \left(\sum_{m \in w \cap p} \frac{1}{(m+1)^2}\right)\right) \\ &= E. \end{aligned}$$

Since $\sum_{m \in w \cap p} \frac{1}{(m+1)^2} \leq \sum_{m = \min w}^{\infty} \frac{1}{(m+1)^2} \leq \int_{\min w}^{\infty} \frac{dx}{x^2} = \frac{1}{\min w}$, and since $\min w \geq \exp(n^\gamma)$, we can further estimate: $E \leq \exp\left(-\exp(n^\gamma)\right)$.

If $l \geq l_1$ is such that $\exp(n^\gamma) \geq n$ for $n > l$, then we get for these n: $\Pr\left(\{\tilde{\xi} < \frac{\epsilon}{2} \ln p\}\right) \leq \exp(-\exp(n^\gamma)) \leq \exp(-n) < 2^{-n}$. This proves (3), and simultaneously concludes the proofs of Lemma 21 and Theorem 3.

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