4

Lifting Geometry to Mapping Spaces II: (Weak) Riemannian Metrics

In this chapter, we will discuss Riemannian metrics on infinite-dimensional spaces. Particular emphasis will be placed on the new challenges which arise on infinite-dimensional spaces.

General Assumption for This Chapter To ensure the existence of integrals we shall always assume that the manifolds in this chapter are modelled on (Mackey) complete locally convex spaces.

4.1 Weak and Strong Riemannian Metrics

Riemannian metrics come in several flavours on infinite-dimensional spaces which are not present in the finite-dimensional setting. The strongest flavour (as we shall see) is the notion of a strong Riemannian metric which is treated in classical monographs such as Lang (1999) and Klingenberg (1995).

4.1 Definition Let M be a manifold modelled on a locally convex space E. A *weak Riemannian metric* g on M is a smooth map

$$g\colon TM \oplus TM \to \mathbb{R}, \qquad (v_x, w_x) \mapsto g_x(v_x, w_x)$$

(where $TM \oplus TM$ is the Whitney sum, see §C.9) satisfying

(a) $g_x := g|_{T_xM \times T_xM}$ is symmetric and bilinear for all $x \in M$,
(b) $g_x(v,v) \geq 0$ for all $v \in T_xM$ with $g_x(v,v) = 0$ if and only if $v = 0$.

If a weak Riemannian metric satisfies in addition, for all $x \in M$,

(c) the topology of the inner product space (T_xM, g_x) coincides with the topology of T_xM as a subspace of TM,

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we say that g is a *strong Riemannian metric*. A manifold with a weak (/strong) Riemannian metric will be called a *weak (/strong) Riemannian manifold*.

4.2 Example Every Hilbert space $(H, \langle\cdot,\cdot\rangle)$ becomes a strong Riemannian manifold using the identifications $TH = H \times H$, $TH \oplus TH = H^3$ and setting $g_c(v,w) := \langle v,w\rangle$.

4.3 Example Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. The locally convex space $C^\infty([0,1],H)$ with the compact-open C^∞-topology is a Fréchet space (but not a Banach or Hilbert space!). Consider the L^2-inner product on this space as

$$\langle f,g\rangle_{L^2} := \int_0^1 \langle f(t),g(t)\rangle \mathrm{d}t.$$

This is a bilinear map on $C^\infty([0,1],H)$. By construction of the compact open C^∞-topology, the inclusion of $C^\infty([0,1],H) \to C([0,1],H)_{\text{c.o.}}$ is continuous linear and it is not hard to prove that the mapping $\beta\colon C([0,1],H)^2 \to H$, $(f,g) \mapsto \int_0^1 \langle f(t),g(t)\rangle \mathrm{d}t$ is continuous bilinear. In conclusion, the L^2-inner product is a continuous bilinear form on $C^\infty([0,1],H)$. Interpreting the locally convex space as a manifold, we have

$$\begin{aligned} TC^\infty([0,1],H) &= C^\infty([0,1],H) \times C^\infty([0,1],H) \cong C^\infty([0,1],H \times H) \\ &= C^\infty([0,1],TH). \end{aligned}$$

We obtain an isomorphism $TC^\infty([0,1],H) \oplus TC^\infty([0,1],H) \cong C^\infty([0,1],H^3)$ (cf. also §C.13) which transforms $f_c, g_c \in T_c C^\infty([0,1],H)$ into a triple (c,f,g) of H-valued functions. Thus

$$g_{L^2}\colon TC^\infty([0,1],H) \oplus TC^\infty([0,1],H) \cong C^\infty([0,1],H^3) \to \mathbb{R}, \quad (c,f,g) \mapsto \langle f,g\rangle_{L^2}$$

is a weak Riemannian metric, called the *L^2-metric*. Note that the L^2-metric is not a strong Riemannian metric as the topology of the inner product space is the one induced by the inclusion $C^\infty([0,1],H) \subseteq L^2([0,1],H)$ where the space on the right-hand side is the space of all square-integrable functions with the L^2-topology.

4.4 If (M,g) is a weak Riemannian manifold, there is an injective linear map

$$\begin{aligned} \flat\colon TM &\to T^*M = \bigcup_{x\in M} L(T_xM,\mathbb{R}), \\ T_xM \ni v &\mapsto g_x(v,\cdot), \end{aligned}$$

where T^*M is the dual tangent bundle; see Remark 1.45. If M is a finite-dimensional manifold, then it follows by counting dimensions that $\flat$ is (fibre-wise) an isomorphism and thus $\flat$ is an isomorphism of vector bundles;

Similarly, if g is a strong Riemannian metric, $\flat$ is an isomorphism of vector bundles; see Proposition 4.5. The inverse of $\flat$ in these situations is often denoted by $\sharp$ and the pair of isomorphisms is known as the musical isomorphisms of a Riemannian manifold. For infinite-dimensional manifolds, the map $\flat$ will, in general, not be surjective.

The next result yields a useful characterisation of strong Riemannian metrics (which is the definition of a strong Riemannian metric in the classical texts such as Lang (1999) and Klingenberg (1995)). We just mention that for the proof some tools from functional analysis are required (which we cite but we do not provide a detailed review here).

4.5 Proposition *Let (M, g) be a weak Riemannian manifold. The following conditions are equivalent:*

(a) *g is a strong Riemannian metric;*
(b) *M is a Hilbert manifold and $\flat\colon TM \to T^*M$ is surjective;*
(c) *M is a Hilbert manifold and $\flat\colon TM \to T^*M$ is a vector bundle isomorphism.*

In particular, on every finite-dimensional manifold M, a weak Riemannian metric is automatically strong.

Proof *Step 1: Strong Riemannian metric implies Hilbert and surjective $\flat$.* If g is a strong Riemannian metric, every tangent space $T_x M$ is a Hilbert space (since it is Mackey-complete; see Remark 1.13). As the tangent spaces are isomorphic to the modelling spaces of M, we see that M is a Hilbert manifold. Now the surjectivity of $\flat$ is a consequence of the Riesz representation theorem (Meise and Vogt, 1997, 11.9).

Step 2: $\flat$ surjective on a Hilbert manifold implies that $\flat$ is a bundle isomorphism. Note first that since M is a Hilbert manifold, there is a canonical smooth structure on the dual bundle (see Remark 1.45), whence it makes sense to consider $\flat$ as a smooth map between TM and T^*M. By construction, we know that $\flat$ induces continuous vector space isomorphisms $T_x M \to (T_x M)^*$ for every tangent space. Hence the open mapping theorem (Rudin, 1991, I. 2.11) shows that $\flat$ is fibre-wise an isomorphism, which together with Lang (1999, III. Proposition 1.3) shows that $\flat$ is a vector bundle isomorphism.

Step 3: Hilbert manifold and $\flat$ being a bundle isomorphism imply that g is strong. Consider $x \in M$ and note that as g is continuous on $TM \oplus TM$, g_x is continuous with respect to the Hilbert space topology of the fibre. Let us denote the Hilbert space norm on $T_x M$ by $\|\cdot\|$. Then the ball $B_1^{g_x}(0) := \{y \in T_x M \mid g_x(y, y) < 1\}$ is by construction $B_1^{g_x}(0)$ a disc 0-neighbourhood

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(see Definition A.6) and the norm $\|\cdot\|_{g_x}$ induced by g_x (aka the Minkowski functional of the ball, Lemma A.16) is continuous. As the unit ball in $(T_xM, \|\cdot\|)$ is bounded, this shows already that there is a constant $K > 0$ with $\|\cdot\| \leq K\|\cdot\|_{g_x}$. To see that the norms are equivalent, we have to prove that $B^1_{g_x}(0)$ is bounded in the Hilbert space topology. As $\flat$ is surjective, every bounded linear functional on T_xM is of the form $\flat(v)$ for some $v \in T_xM$. Applying the Cauchy–Schwarz inequality, we derive

$$\sup_{y \in B_1^{g_x}(0)} |\flat(v)(y)| = \sup_{y \in B_1^{g_x}(0)} |g_x(v,y)| \leq \|v\|_{g_x}.$$

We conclude that every bounded linear functional is bounded on $B_1^{g_x}(0)$. This property is called weakly bounded and it is known that on a Hilbert space, every weakly bounded set in a locally convex space is bounded (Rudin, 1991, Theorem 3.13). Summing up, g_x induces the Hilbert space topology of T_xM and since x was arbitrary, this shows that g is a strong Riemannian metric. □

4.6 Example The space of immersions $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$ is an open subset of $C^\infty(\mathbb{S}^1, \mathbb{R}^d)$ by Lemma 2.6. We endow it with a weak Riemannian metric (an invariant version of the L^2-metric, §4.2)

$$g_c(h,k) := \int_{\mathbb{S}^1} \langle h(\theta), k(\theta)\rangle \|\dot{c}\| \mathrm{d}\theta.$$

Recall that since $\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) \Subset C^\infty(\mathbb{S}^1, \mathbb{R}^d)$, we have an identification

$$T_c \,\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d) = C^\infty(\mathbb{S}^1, \mathbb{R}^d);$$

see C.12. Working in the identification we immediately see that the musical morphism reduces to the map

$$\flat_c(h) = \|\dot{c}\| \cdot h, h \in C^\infty(\mathbb{S}^1, \mathbb{R}^d),$$

where the dot signifies pointwise multiplication. So the image of $T_c \,\mathrm{Imm}(\mathbb{S}^1, \mathbb{R}^d)$ under $\flat_c$ can be identified as $C^\infty(\mathbb{S}^1, \mathbb{R}^d)$. However, the (topological) dual space of $C^\infty(\mathbb{S}^1, \mathbb{R}^d)$ is the larger space $\mathcal{D}'(\mathbb{S}^1)^d$ of $\mathbb{R}^d$-valued distributions (see Taylor, 2011, Section 3).

4.7 Example Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space with $S_H = \{x \in H \mid \|x\| = 1\}$ (Example 1.34). The Hilbert sphere is a strong Riemannian manifold with the induced metric $g_x(v,w) := \langle v, w\rangle$ (where $T_xS_H = \{v \in H \mid \langle v, x\rangle = 0\}$).

The distinction between strong and weak Riemannian metrics has far-reaching consequences (see e.g. §4.2 on geodesic distance).

Exercises

4.1.1 Verify Example 4.2: Every Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is a strong Riemannian manifold.

4.1.2 Show that the mapping $\mu\colon C([0,1],H)^2 \to \mathbb{R}$, $(f,g) \mapsto \langle f,g\rangle_{L^2}$ is continuous bilinear (hence smooth), see Example 4.3.
Hint: $\int_0^1\colon C([0,1],\mathbb{R}) \to \mathbb{R}$, $f \mapsto \int_0^1 f(t)\mathrm{d}t$ is linear. Exploit that $C([0,1],\mathbb{R})$ is a Banach space to prove continuity via usual integral estimates.

4.1.3 Verify that the Hilbert sphere is a strong Riemannian manifold (Example 4.7).

4.1.4 We shall treat Riemannian metrics on spaces of smooth functions on the sphere.
Hint: If you are not familiar with integration on manifolds: Use $\theta\colon [0,2\pi] \to \mathbb{S}^1$, $t \mapsto (\cos(t),\sin(t))$ to reduce the integral to a usual integral; see note at the beginning of Chapter 5 and compare Lee (2013, Chapter 16).

(a) Let (M,g) be a strong Riemannian metric. Use C.13 to show that the L^2-metric

$$g_c^{L^2}(h,k) := \int_{\mathbb{S}^1} g_{c(\theta)}(h(\theta),k(\theta))\mathrm{d}\theta$$

defines a weak Riemannian metric on $C^\infty(\mathbb{S}^1,M)$.

(b) Verify that the metric in Example 4.6 is a weak Riemannian metric.

4.2 The Geodesic Distance on a Riemannian Manifold

General Assumption In this section (M,g) denote a *strong* Riemannian manifold if nothing else is said. For convenience, we shall always assume that M is connected.

Having a Riemannian metric at our disposal, we can define the length of curves.

4.8 Definition Let $c\colon [a,b] \to M$ be a piecewise C^1-curve.[1] Then we define the *length* and the *energy* of c as

[1] That is, there exists a partition of $[a,b]$ into subintervals such that on each of them the curve restricts to a C^1-curve.

$$\mathrm{Len}(c) := \int_a^b \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\mathrm{d}t,$$
$$\mathrm{En}(c) := \frac{1}{2}\int_a^b g_{c(t)}(\dot{c}(t), \dot{c}(t))\mathrm{d}t.$$

For $x, y \in M$ then define

$$\Gamma(x,y) := \{c\colon [0,1] \to M \mid c(0) = x, c(1) = y, \text{ and } c \text{ is piecewise } C^1\}.$$

Finally, we define the *geodesic distance* between points $x, y \in M$ as

$$\mathrm{dist}(x,y) := \inf_{c\in\Gamma(x,y)} \mathrm{Len}(c) = \inf_{c\in\Gamma(x,y)} \int_0^1 \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))}\mathrm{d}t.$$

Due to the Cauchy–Schwarz inequality, for curves $c\colon [a,b] \to M$ we find (see Klingenberg, 1995, Proposition 1.8.7) that

$$\mathrm{Len}(c)^2 \leq 2\mathrm{En}(c)(b-a) \qquad \text{(with equality if and only if } \dot{c} \text{ is constant).} \tag{4.1}$$

4.9 Remark Note that since every interval $[a,b]$ is diffeomorphic to $[0,1]$, the chain rule implies that the definition of $\Gamma(x,y)$ and of the geodesic distance does not depend on $[0,1]$. It is only a convenient choice for us and we shall ignore this choice in the construction of paths to avoid cumbersome reparametrisation arguments.

4.10 Lemma *The geodesic distance is a pseudo-distance, that is,*

(a) $\mathrm{dist}(x,y) \geq 0$ *for* $x, y \in M$,
(b) $\mathrm{dist}(x,y) = \mathrm{dist}(y,x)$ *for all* $x, y \in M$,
(c) $\mathrm{dist}(x,z) \leq \mathrm{dist}(x,y) + \mathrm{dist}(y,z)$ *for all* $x, y, z \in M$.

The proof of this lemma is left as an exercise.

On strong Riemannian manifolds, the geodesic distance is also *point separating*, that is, in addition to the properties from Lemma 4.10, it satisfies

(d) $\mathrm{dist}(x,y) \neq 0$ for all $x, y \in M$ with $x \neq y$.

Moreover, one can prove the following result for strong Riemannian metrics.

4.11 Theorem (Klingenberg, 1995, Theorem 1.9.5) *Let* (M,g) *be a strong Riemannian metric. The function* $\mathrm{dist}\colon M \times M \to \mathbb{R}$ *defines a metric on* M. *The topology derived from* dist *coincides with the given topology of* M.

However, in general, *on infinite-dimensional manifolds* the geodesic distance might not be a metric. Indeed it might be non-point separating and even stronger, the geodesic distance might be vanishing.[2] Note that if c is a path connecting $x \neq y$, then $\mathrm{Len}(c) > 0$ (or in other words $\mathrm{Len}(c) = 0$ implies that the path is constant). Thus the vanishing of $\mathrm{dist}(x, y)$ means that we can find arbitrarily short paths connecting the two points. We showcase this in the following example.

4.12 Example (Magnani and Tiberio, 2020) Consider the space $(\ell^2, \langle \cdot, \cdot \rangle)$ of all square-summable real sequences; see Example 3.11. The map $A\colon \ell^2 \to \ell^2$, $(x_n)_{n\in\mathbb{N}} \mapsto (\frac{1}{n^3} x_n)_{n\in\mathbb{N}}$ is continuous linear and induces a bilinear symmetric map $B\colon \ell^2 \times \ell^2 \to \mathbb{R}, B(\mathbf{x}, \mathbf{y}) := \langle \mathbf{x}, A\mathbf{y} \rangle$. Identifying the tangent spaces of ℓ^2, we obtain a weak Riemannian metric via

$$g\colon T\ell^2 \oplus T\ell^2 = \bigcup_{\mathbf{p}\in\ell^2} \ell^2 \times \ell^2 \to \mathbb{R}, \quad (T_{\mathbf{p}}\ell^2)^2 \ni (\mathbf{x}, \mathbf{y}) \mapsto e^{-\|\mathbf{p}\|^2} B(\mathbf{x}, \mathbf{y}).$$

We will now prove that the weak Riemannian manifold (ℓ^2, g) has vanishing geodesic distance, that is, that for every $\mathbf{p} \neq \mathbf{q}$ in ℓ^2, we have $\mathrm{dist}(\mathbf{p}, \mathbf{q}) = 0$. To this end, let $\mathbf{e}_n$ be the sequence with 1 in the nth place and zeroes everywhere else. We construct a path from $\mathbf{p}$ to $\mathbf{q}$ via

$$c_n\colon [0,1] \to \ell^2, \quad t \mapsto \begin{cases} \mathbf{p} + 3tn\mathbf{e}_n, & t \in [0, 1/3], \\ \mathbf{p} + n\mathbf{e}_n + (3t-1)(\mathbf{q} - \mathbf{p}), & t \in [1/3, 2/3], \\ \mathbf{q} + (3 - 3t)n\mathbf{e}_n, & t \in [2/3, 1]. \end{cases}$$

By construction, c_n is a piecewise linear curve connecting $\mathbf{p}$ to $\mathbf{q}$ and passing through $\mathbf{p} + n\mathbf{e}_n$ and $\mathbf{q} + n\mathbf{e}_n$ on the way. We claim that $\mathrm{Len}(c_n) \to 0$ as $n \to \infty$ and thus $\mathrm{dist}(\mathbf{p}, \mathbf{q}) = 0$. To see this, we observe that

$$c_n'(t) = 3n\mathbf{e}_n, t \in [0, 1/3[, \quad c_n'(t) = 3(\mathbf{q} - \mathbf{p}), t \in]1/3, 2/3[,$$
$$c_n'(t) = -3n\mathbf{e}_n, \ t \in]2/3, 1].$$

Moreover, since $\mathbf{p}, \mathbf{q} \in \ell^2$ there is $N > 0$ such that every component of $\mathbf{p}$ and $\mathbf{q}$ with $n > N$ satisfies $|\pi_n(\mathbf{p})|, |\pi_n(\mathbf{q})| < 1/3$ (here π_n is the projection on the nth component). Hence we see that for every such n we have $\pi_n(\mathbf{p} + t(\mathbf{q} - \mathbf{p})) > -1$.

[2] We say the geodesic distance is *non-vanishing* if there exist $x, y \in M$ such that $\mathrm{dist}(x, y) \neq 0$. So every point separating geodesic distance is non-vanishing but not vice versa.

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We now estimate the length $\mathrm{Len}(c_n)$:

$$\begin{aligned}\int_0^1 c_n(t)\mathrm{d}t &= 3\left(\int_0^{1/3} e^{-\|c_n(t)\|^2}\sqrt{B(n\mathbf{e}_n, n\mathbf{e}_n)}\mathrm{d}t\right.\\ &\quad + \int_{1/3}^{2/3} e^{-\|c_n(t)\|^2}\underbrace{\sqrt{B(\mathbf{q}-\mathbf{p},\mathbf{q}-\mathbf{p})}}_{=:\,K>0}\,\mathrm{d}t\\ &\quad \left.+ \int_{2/3}^{1} e^{-\|c_n(t)\|^2}\sqrt{B(-n\mathbf{e}_n, -n\mathbf{e}_n)}\mathrm{d}t\right)\\ &\leq 3\left(\int_0^{1/3}\sqrt{\langle n\mathbf{e}_n, \frac{1}{n^2}\mathbf{e}_n\rangle}\mathrm{d}t + K\int_{1/3}^{2/3} e^{-(n^2+\|\mathbf{p}+(3t-1)(\mathbf{q}-\mathbf{p})\|^2+2n\langle \mathbf{e}_n, \mathbf{p}+(3t-1)(\mathbf{q}-\mathbf{p})\rangle)}\mathrm{d}t\right.\\ &\quad \left.+ \int_{2/3}^{1}\sqrt{B(-n\mathbf{e}_n, -n\mathbf{e}_n)}\mathrm{d}t\right)\\ &\leq \frac{1}{\sqrt{n}} + 3K\int_{1/3}^{2/3} e^{-n^2-2n\pi_n(\mathbf{p}+t(\mathbf{q}-\mathbf{p}))}\mathrm{d}t + \frac{1}{\sqrt{n}} \overset{n>N}{\leq} \frac{2}{\sqrt{n}} + Ke^{-n^2+2n}.\end{aligned}$$

We conclude that the length of the curve c_n converges to zero as $n \to \infty$ whence the geodesic distance vanishes.

Another interesting example in this regard is the invariant L^2-metric on the group of circle diffeomorphisms. Before we state this example, let us exhibit a general construction principle for (weak) Riemannian metrics on Lie groups.

4.13 (Invariant metrics on a Lie group) Let G be a (perhaps infinite-dimensional) Lie group with Lie algebra $\mathbf{L}(G)$. Assume that $\langle \cdot, \cdot \rangle \colon \mathbf{L}(G) \times \mathbf{L}(G) \to \mathbb{R}$ is a continuous inner product on the Lie algebra. Then we define a *right invariant (weak) Riemannian metric* via the following formula:

$$\langle V, W\rangle_g := \langle T\rho_g^{-1}(V), T\rho_g^{-1}(W)\rangle, \quad \text{for all } V, W \in T_gG. \tag{4.2}$$

Here ρ_g is the right translation by g and we remark that due to the smoothness of the group operations the resulting metric is indeed a (weak) Riemannian metric. By construction, the right invariant metric is invariant under the right action of the Lie group G on TG via right multiplication.

Note that by replacing every ρ_g in (4.2) by the left translation λ_g, we can obtain a *left-invariant (weak) Riemannian metric* associated to the given inner product.

4.14 Example (Right-invariant L^2-metric on $\mathrm{Diff}(\mathbb{S}^1)$) We consider $\mathrm{Diff}(\mathbb{S}^1)$ again as an open subset of $C^\infty(\mathbb{S}^1, \mathbb{S}^1)$. Recall from Example 3.5 that this

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manifold structure turns the diffeomorphism group into a Lie group. Moreover, $T_\varphi \operatorname{Diff}(\mathbb{S}^1) = C^\infty_\varphi(\mathbb{S}^1, T\mathbb{S}^1) \cong C^\infty_\varphi(\mathbb{S}^1, \mathbb{S}^1 \times \mathbb{R}) \cong C^\infty(\mathbb{S}^1, \mathbb{R})$.[3]

On $C^\infty(\mathbb{S}^1, \mathbb{R})$ we have the inner product (where we refer to the discussion in the beginning of Chapter 5 for the meaning of the integral):

$$\langle u, v\rangle_{L^2} := \int_{\mathbb{S}^1} f(\theta) g(\theta) \mathrm{d}\theta.$$

Plugging this into (4.2), we obtain the *(right)-invariant L^2-metric*

$$g_\varphi^{L^2,\mathrm{inv}}(u,v) := \langle u \circ \varphi^{-1}, v \circ \varphi^{-1}\rangle_{L^2}.$$

Thanks to a theorem of Michor and Mumford, the geodesic distance with respect to this metric vanishes; see, for example, Kolev (2008, Theorem 4).

Exercises

4.2.1 Establish the estimate (4.1).

4.2.2 Prove Lemma 4.10 and verify that the pseudodistance does not depend on the choice of interval (Remark 4.9).

4.2.3 The following provide the details for Example 4.12:

- (a) Show that the map A makes sense and is linear and continuous and thus the map B is bilinear and symmetric.
- (b) Prove that g is a weak Riemannian metric on ℓ^2.

4.2.4 Prove that the construction of right- (or left-)invariant metrics on a Lie group from 4.13 yields a weak Riemannian metric.
Hint: To check smoothness of the metric, use Lemma 3.12 to identify the Whitney sum $TG \oplus TG$.

Geodesics on Infinite-dimensional Manifolds (Informal Discussion)

To get a better understanding of the distance on weak Riemannian metrics, it seems useful to study curves ‘of shortest length’ between two points the so-called *geodesics*. Before we study geodesics in the next section, we discuss some aspects in an informal way.

If the manifold is a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ viewed as a strong Riemannian manifold (Example 4.2), the curve of shortest length between $a, b \in H$ is the straight line

$$\mathbb{R} \ni t \mapsto \gamma(t) := (b-a)t + a \in H. \tag{4.3}$$

[3] Here we exploit that $\mathbb{S}^1$ is a Lie group, whence the tangent bundle $T\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$ is trivial.

Note that this curve also satisfies $\operatorname{dist}(\gamma(t),\gamma(s)) = \|b-a\||t-s|$, and so it realises the shortest distance between any two given points on the line. On a manifold, we would like to compute curves which satisfy the same property at least locally, that is, in a neighbourhood of each point the curve realises the shortest distance for every pair of points on the curve.[4] Thanks to (4.1), one can equivalently describe geodesics between p and q as curves of minimal energy, that is, extrema of the energy En restricted to $\Gamma(p,q)$. Hence to find geodesics we consider the derivative of the energy. Working locally in a chart (U,φ) of M (and suppressing most identifications in the notation; see Lemma C.17), this yields, for the derivative $d\mathrm{En}(c;h)$, the formula

$$\int_0^1 \frac{1}{2} d_1 g_U(c, c'(t), c'(t); h)) - d_1 g(c(t), h(t), c'(t); c'(t))) \\ - g_U(c(t), h(t), c''(t))\mathrm{d}t.$$

To find the geodesics, one would now have to isolate h in the expression to rewrite the differential in the form $\int_0^1 g_c(\ldots,h)\mathrm{d}t$ and extract the geodesic equation. In general, this is not possible, as the $\flat$-map is not an isomorphism, whence the existence of geodesics for weak Riemannian metrics is a priori unclear (see also Proposition 4.5).

We remark that the geodesics of weak Riemannian metrics are also of independent interest in the context of Euler–Arnold theory (see Chapter 7). There, certain partial differential equations can be interpreted as geodesic equations on infinite-dimensional manifolds.

4.15 Example (Inviscid Burgers equation) One can show (Kolev, 2008, 3.2) that geodesics with respect to the invariant L^2-metric from Example 4.14 correspond to solutions of the inviscid Burgers equation (also known as the Hopf equation):

$$u_t + 3uu_x = 0 \text{ (subscripts denoting partial derivatives).}$$

We shall investigate a similar situation later in Example 7.4.

The observant reader should have noted that we are talking about geodesics (which are the solutions of the inviscid Burgers equation) for a weak Riemannian metric which was described in Example 4.14 as having vanishing geodesic distance. It might be tempting to think that this implies that all geodesics must be constant (since only these curves have length 0 and geodesics were

[4] The sphere $\mathbb{S}^1$ is a Riemannian manifold by Example 4.7. For $x \in \mathbb{S}^1$ consider a closed curve running in a great circle from x around the sphere. Then this curve realises the shortest distance from x to another point y until it passes the point antipodal to x. However, as long as we restrict to an open neighbourhood which does not contain points antipodal to each other, the curve realises the shortest distance from one point to the other.

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described as smooth curves (locally) minimising the length between their endpoints). This, however, is wrong (the reader might either consult the partial differential equations (PDE) literature, or observe – see Example 7.4 – that the related equation $u_t + uu_x = 0$ is the geodesic equation of a weak Riemannian metric with non-vanishing geodesic distance). This equation is the Burgers equation and it admits non-constant solutions. So what's wrong here? For strong Riemannian metrics one can show that every geodesic is (locally) length minimising. However, there are also other characterisations of geodesics (which we will discuss in the next section) which coincide with our informal definition for strong Riemannian metrics. In the weak setting, the situation is (as always) more complicated.

Exercises

4.2.5 We verify some details concerning geodesics in Hilbert space.

(a) Show that the path $c(t) = (b-a)t + a, t \in [0,1]$, (4.3), realises the shortest length with respect to all C^1-curves connecting a to b in the Hilbert space H.
Hint: Prove first for a path consisting of two line segments meeting at a point p that its length is longer than the length of c if p does not lie on c. Then use that C^1-paths are rectifiable.

(b) Prove the distance property claimed after (4.3): For all $s,t \in \mathbb{R}$, $\operatorname{dist}(\gamma(s),\gamma(t)) = \|b-a\||t-s|$. Note that this is (up to reparametrisation to unit speed) the geodesic property for curves in metric spaces; see Bridson and Haefliger (1999, Definition 1.3).

4.3 Geodesics, Sprays and Covariant Derivatives

In this section, we consider three objects associated to a Riemannian metric (see e.g. Lang, 1999; Abraham et al., 1988). This will enable us to study geodesics on Riemannian manifolds. The idea is that every (strong) Riemannian metric induces a covariant derivative, a metric spray and a connector. These can be used to conveniently describe geodesics. The main point is to introduce the concepts of spray and covariant derivative while we relegate many details of the constructions to the literature, for example, Lang (1999, Chapter VIII).

General Assumption In this section, (M,g) will denote a *strong* Riemannian manifold if nothing else is said. Thus there will be no need to worry about the existence of solutions to certain differential equations. Note that for weak

Riemannian metrics similar computations are in principle possible, but require the careful checking of several technical details.[5]

We first define certain vector fields on the tangent bundle $\pi_M : TM \to M$ which are called *sprays*. For any spray, geodesics can be defined and if the spray is the metric spray associated to a Riemannian metric, these geodesics will turn out to be the geodesics of the Riemannian metric.

4.16 Definition (Spray) A vector field $S \in \mathcal{V}(TM)$ on the tangent manifold TM is said to be of *second order* if

$$T\pi_M(S(v)) = v \text{ for all } v \in TM.$$

For each $t \in \mathbb{R}$ we denote by $t_{TM} : TM \to TM$ the vector bundle morphism which is given in each fibre by multiplication with t. Now a second-order vector field S is a *spray* if

$$S(tv) = T(t_{TM})(t \cdot S(v)) \text{ for all } t \in \mathbb{R}, v \in TM. \tag{4.4}$$

To understand the meaning of (4.4), let us localise in a chart. (Warning: The next identities hold up only to identifications in charts, which we will suppress in the notation!)

4.17 Choose $U \Subset M$ such that $TU \cong U \times E$ (where E is the model space of M). Then $TTU \cong (U\times E)\times(E\times E)$. Now if S is a second-order vector field, its restriction to U is given for $(u,V) \in U \times E$ by $S(u,V) = ((u,V),V,S_2(u,V))$. If S is a spray, the equation (4.4) reads on U as follows:

$$S(u,tV) = (u,tV,tV,S_{U,2}(u,tV)) = (u,tV,tV,t^2 S_{U,2}(u,V)).$$

The map $S_{U,2}$ is thus quadratic with respect to scalar multiplication in the fibre. Furthermore, this implies together with Exercise 4.3.1 that $S_{U,2}(x,v) = \frac{1}{2} d_2^2 S_2((x,0);(v,v))$.

We define $B\colon M \to C^\infty(TM \oplus TM, TM)$ as the map which is locally on a chart domain (U,φ) given by the symmetric bilinear map $B_U(x;v,w) := d_2^2 S_{U,2}((x,0);(v,w))$. Associated to this bilinear map is the quadratic form $Q_U(x,v) := B_U(x;v,v)$.[6] Note that in finite-dimensional Riemannian geometry $-B$ is represented by the so-called *Christoffel symbols* (Lang, 1999, pp. 213–214).

4.18 (Integral curves of second-order vector fields) Let S be a smooth second-order vector field on the manifold M and assume that S admits integral curves

[5] We shall do this in Chapter 5 for the L^2-metric on $C^\infty(\mathbb{S}^1, M)$.

[6] We recall that due to the identity $B_U(x; v, w) = \frac{1}{2}(Q_U(x, v + w) - Q_U(x, v) - Q_U(x, w))$, the quadratic form carries exactly the same information as the bilinear form.

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(i.e. C^1-curves $\beta\colon J \to TM$ with $S(\beta) = \dot{\beta}$[7]). Note that an integral curve $\beta\colon J \to TM$ of a second-order vector field S satisfies $(\pi_M \circ \beta)^{\cdot}(t) = \beta(t)$.

4.19 Definition A C^2-curve $\alpha\colon J \to M$ is a *geodesic* of the spray S if $\dot{\alpha}\colon J \to TM$ is an integral curve of S, or equivalently, if α satisfies the *geodesic equation*

$$\ddot{\alpha}(t) = S\left(\dot{\alpha}(t)\right), \quad \text{for all } t \in J.$$

Note that $\dot{\alpha}(t) = T_t\alpha(1) \in TM$. Equivalently, the geodesic equation becomes (in local coordinates) the equation $\ddot{\alpha}(t) = B_{\alpha(t)}(\dot{\alpha}(t), \dot{\alpha}(t))$.

4.20 Example A strong Riemannian metric g induces a spray $S^g\colon TM \to T^2M$ which we describe locally on a chart domain $U \Subset M, TU = U \times E, T^2U = (U \times E) \times E \times E$ (again suppressing the identifications!). Namely, we think of the local representative $g_U\colon U \times E \times E \to \mathbb{R}$, $(x,v,w) \mapsto g_x(v,w)$ of the metric as a mapping with three components. Then we define the associated spray via $S_U^g(x,v) := ((x,v),(v,\Gamma_U(x,v)))$ for $v \in T_xM$, where the quadratic form Γ is the unique map which satisfies for all $v,w \in T_xU$.

$$g_U(x,\Gamma_U(x,v),w) = \frac{1}{2} d_1g_U(x,v,v;w) - d_1g_U(x,v,w;v), \qquad (4.5)$$

Here d_1g_U denotes the partial derivative with respect to the first component (see Proposition 1.20). We leave the verification that the local mappings S_U^g yield a spray S^g on TM to the reader (Exercise 4.3.2). Note that the formula (4.5) makes sense for weak Riemannian metrics, but since $\flat$ is, in general, not surjective for a weak Riemannian metric (see Proposition 4.5), it is a priori not clear whether the condition defines a unique map Γ_U.

4.21 Definition (Metric spray) Let (M,g) be a weak Riemannian manifold. A spray S is called *metric spray* if its associated quadratic form Q satisfies for each chart (U,φ) the formula (4.5), that is, for all $v,w \in T_xU$.

$$g_U(x,Q_U(x,v),w) = \frac{1}{2} d_1g_U(x,v,v;w) - d_1g_U(x,v,w;v), \qquad (4.6)$$

Let us point out that the metric spray S^g associated to a Riemannian metric g can be interpreted in the sense of Hamiltonian mechanics: It can be shown (see Abraham et al., 1988, p. 493) that the spray S^g is the Hamiltonian vector field on TM associated with the kinetic energy function

[7] If the manifold M is modelled on a Banach space, such curves always exist due to the standard ODE solution theory. Beyond Banach spaces, the existence of such curves needs to be established for the special case at hand.

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$e\colon TM \to \mathbb{R}$, $v_x \mapsto \frac{1}{2}g_x(v_x, v_x)$. This has several interesting consequences, such as conservation of energy by the geodesics (again we refer to Abraham et al., 1988, Supplement 8.1.B, for more information). We shall return to the relation between energy and geodesics in Chapter 7.

The next example shows that there is not necessarily a metric spray. It was pointed out to me by C. Maor and we urge the reader to compare it with Theorem 4.30.

4.22 Example (A weak Riemannian metric without metric spray) We return to the weak Riemannian metric on the Hilbert space $(\ell^2.\langle\cdot,\cdot\rangle)$ from Example 4.12:

$$g\colon T\ell^2 \oplus T\ell^2 = \bigcup_{\mathbf{p}\in\ell^2} \ell^2 \times \ell^2 \to \mathbb{R}, \quad T_{\mathbf{p}}\ell^2 \ni (\mathbf{x}, \quad \mathbf{y}) \mapsto e^{-\|\mathbf{p}\|^2} \sum_{n=1}^{\infty} \frac{\mathbf{x}_n \mathbf{y}_n}{n^3},$$

where the subscripts denote elements of a sequence. A quick computation shows that

$$d_1 g(\mathbf{p}, \mathbf{x}, \mathbf{y}; \mathbf{w}) = -2\langle \mathbf{p}, \mathbf{w}\rangle g(\mathbf{p}, \mathbf{x}, \mathbf{y}) = -2g(\mathbf{p}, (n^3 \mathbf{p}_n)_{n\in\mathbb{N}}, \mathbf{w}) \sum_{n=1}^{\infty} \frac{\mathbf{x}_n \mathbf{y}_n}{n^3},$$

where the last equality only makes sense if the sequence $(n^3\mathbf{p}_n)_{n\in\mathbb{N}}$ is contained in ℓ^2. Plugging this identity into the right-hand side of (4.6), we see that

$$\begin{aligned}
&\frac{1}{2} d_1 g(\mathbf{p}, \mathbf{x}, \mathbf{x}; \mathbf{w}) - d_1 g(\mathbf{p}, \mathbf{x}, \mathbf{w}; \mathbf{x}) \\
&\quad = -g(\mathbf{p}, (n^3 \mathbf{p}_n)_{n\in\mathbb{N}}, \mathbf{w}) \sum_{n=1}^{\infty} \frac{\mathbf{x}_n^2}{n^2} + 2\langle \mathbf{p}, \mathbf{x}\rangle g(\mathbf{p}, \mathbf{x}, \mathbf{w}) \\
&\quad = g\left(\mathbf{p}, 2\langle \mathbf{p}, \mathbf{x}\rangle \mathbf{x} - \left(\sum_{n=1}^{\infty} \mathbf{x}_n / n^3\right) (n^3 \mathbf{p}_n)_{n\in\mathbb{N}}, \mathbf{w}\right).
\end{aligned}$$

We conclude that if a spray existed, its quadratic form needs to be given by

$$Q(\mathbf{p}, \mathbf{x}) = 2\langle \mathbf{p}, \mathbf{x}\rangle \mathbf{x} - \left(\sum_{n=1}^{\infty} \mathbf{x}_n / n^3\right) (n^3 \mathbf{p}_n)_{n\in\mathbb{N}}$$

and this expression is ill defined if $(n^3\mathbf{p}_n)_{n\in\mathbb{N}}$ is not contained in ℓ^2. Hence g does not admit a metric spray.

Exercises

4.3.1 Let $U \Subset E$, where E is a locally convex space, $0 \in U$ and $f\colon U \to E$ is smooth. Prove that if f is *locally homogeneous of order p*,

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that is, $f(tx) = t^p f(x)$ for all $(t,x) \in \mathbb{R} \times U$ such that $tx \in U$, then $f(x) = \frac{1}{p!} d^p f(0; x, \ldots, x)$.

4.3.2 Show that the local formula (4.5) defines a spray on a chart domain U. Then show that for any other chart V, the change of charts relates the local formulae obtained in this way to each other, that is, the sprays can be combined to define a spray on TM.

4.3.3 Let S be a spray on a manifold M, (U,φ) and (V,ψ) are two charts of M with change of charts $\tau := \psi \circ \varphi^{-1}$. Prove the following change of charts formulae for the local expression of the spray and the associated bilinear form (see 4.17):

$$S_{V,2}(\tau(x), d\tau(x;v)) = d^2\tau(x;v,v) + d\tau(x; S_{U,2}(x,v)), \tag{4.7}$$

$$B_V(\tau(x); d\tau(x;v), d\tau(x;w)) = d^2\tau(x;v,w) + d\tau(x; B_U(x;v,w)). \tag{4.8}$$

4.3.4 Define the subspace $S := \{\mathbf{x} \in \ell^2 \mid (n^3 \mathbf{x}_n)_{n\in\mathbb{N}} \in \ell^2\}$ of ℓ^2. Endow S with the restriction of the weak Riemannian metric $g_{\mathbf{p}}(\mathbf{x},\mathbf{y}) = e^{-\|\mathbf{p}\|^2} \sum_{n=1}^{\infty} \frac{\mathbf{x}_n \mathbf{y}_n}{n^3}$ from Example 4.12. Show that the resulting weak Riemannian metric:

(a) has vanishing geodesic distance;

(b) admits a metric spray and deduce that the (non-)existence of a (smooth) metric spray does not imply the non-degeneracy of the geodesic distance.

Remark: Again I am indebted to C. Maor for pointing this example out to me.

Covariant Derivatives

On a locally convex space E, we can identify vector fields with functions, whence for two vector fields $X,Y \in C^\infty(E,E)$, we can differentiate Y in the direction of X via $X.Y(v) = dY(v, X(v))$ (see Appendix D). This yields again a smooth function from E to E. On a manifold M, the corresponding construction for vector fields would be $TY \circ X: M \to T^2M$ which is obviously *not* a vector field. Hence to define a derivative taking two vector fields to vector fields, an additional structure is needed.

4.23 Definition A *covariant derivative* $\nabla: \mathcal{V}(M)\times\mathcal{V}(M) \to \mathcal{V}(M)$, $(X,Y) \mapsto \nabla_X Y$ is a bilinear map satisfying the properties

(a) for $f \in C^\infty(M,\mathbb{R}), X,Y \in \mathcal{V}(M)$ we have

1. $\nabla_{fX} Y = f \nabla_X Y$,

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2. $\nabla_X(fY) = \mathcal{L}_X(f)Y + f\nabla_X Y$ (where $\mathcal{L}_X(f)$ is the Lie derivative (D.3)).

(b) $\nabla_X Y - \nabla_Y X = [X,Y]$ for all $X,Y \in \mathcal{V}(M)$.

We shall show now that every spray induces a covariant derivative. Recall from Appendix D that for a chart (φ,U) we denote by X_φ the local representative of a vector field $X \in \mathcal{V}(M)$.

4.24 Proposition *Given a spray S on M, there exists a unique covariant derivative ∇ such that in a chart (φ,U), the derivative is given by the local formula*

$$(\nabla_X Y)_\varphi(x) = X_\varphi.Y_\varphi(x) - B_U(x;X_\varphi(x),Y_\varphi(x)). \tag{4.9}$$

We call ∇ the covariant derivative associated to the spray S. *Suppressing the indices, the above formula reads $\nabla_X Y = X.Y - B(X,Y)$.*

Proof Define $\nabla_X Y$ locally over U via the formula (4.9). In Exercise 4.3.6 we shall see that this formula has all properties of a covariant derivative for vector fields on U (and all of the defining properties of a covariant derivative localise on U!). Obviously we can repeat the construction for every chart in an atlas $\mathcal{A}$. It suffices now to prove that the family $((\nabla_X Y)_\varphi)_{\varphi\in\mathcal{A}}$ induces a vector field, that is, in view of Lemma D.7, it suffices to prove that the local representatives are related by the change of charts. We will check this for $\tau := \psi\circ\varphi^{-1}$, that is, we prove $d\tau\circ(\nabla_X Y)_\varphi = (\nabla_X Y)_\psi\circ\tau$. Note first that by construction $Y_\psi\circ\tau = d\tau\circ Y_\varphi$ and thus (1.7) yields

$$d(Y_\psi\circ\tau)(x;v) = d(d\tau(x;Y_\varphi)(x;v) = d^2\tau(x;Y_\varphi(x),v) + d\tau(x,dY_\varphi(x;vV)). \tag{4.10}$$

We now apply the change of charts formulae for the spray and associated bilinear form from Exercise 4.3.3:

$$\begin{aligned}
(\nabla_X Y)_\psi(\tau(z)) &= dY_\psi(\tau(z);d\tau(z;X_\varphi(z)) \\
&\quad - B_V(\tau(z);d\tau(z;X_\varphi(z)),d\tau(z;Y_\varphi(z))) \\
&\overset{(4.8)}{=} d(Y_\psi\circ\tau)(z;X_\varphi(z)) - d^2\tau(z;Y_\varphi(z),X_\varphi(z)) - d\tau(z;B_U(z;X_\varphi(z),Y_\varphi(z))) \\
&\overset{(4.10)}{=} d^2\tau(z;Y_\varphi(z),X_\varphi(z)) + d\tau(x,dY_\varphi(x;X_\varphi(z))) \\
&\quad - d^2\tau(z;Y_\varphi(z),X_\varphi(z)) - d\tau(z;B_U(z;X_\varphi(z),Y_\varphi(z))) \\
&= d\tau(z;(\nabla_X Y)_\varphi(z)),
\end{aligned}$$

where we have exploited that the second derivative $d^2\tau(z;\cdot)$ is symmetric by Schwarz' theorem, Exercise 1.3.3. □

4.25 Remark For later use, we observe that the covariant derivative depends only on the values of the field in whose direction we derivate: Let X,Y be vector fields and ∇ be a covariant derivative associated to a spray S. Then the formula (4.9) shows that $\nabla_X Y(p)$ depends only on $X(p)$ but not on the vector field X in a neighbourhood of p.

As soon as we have a covariant derivative associated to a spray, we can define an associated curvature tensor. We recall the definition here for later use.

4.26 Definition Let M be a manifold with a covariant derivative ∇. For vector fields $X, Y, Z \in \mathcal{V}(M)$, we define the linear map

$$R(X,Y)\colon \mathcal{V}(M) \to \mathcal{V}(M) \text{ given by the formula}$$
$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then one can show that R is a trilinear map in the variables X, Y, Z called the *curvature* associated to the covariant derivative; see Exercise 4.3.10. If S is a spray inducing ∇ one also says that R is the curvature of S. Similarly if ∇ is the metric derivative of a (weak) Riemannian metric g, we say that R is the curvature of the metric g.

Curvature (associated to the metric spray) is a fundamental invariant of a Riemannian manifold (Klingenberg, 1995; Lang, 1999; Gallot et al., 2004). Here we mention only that the curvature of certain infinite-dimensional (weak and strong) Riemannian manifolds plays a crucial role in important applications of infinite-dimensional geometry such as Arnold's result (Arnold, 1966) on the practical impossibility of long-term weather forecasts. Also there is an interesting divide between the curvature of strong and weak Riemannian metrics: If (M,g) is a strong Riemannian manifold, the curvature is always (locally) bounded (this follows from the fact that it can be represented as a smooth section into a suitable tensor bundle). However, there are examples of weak Riemannian manifolds with covariant derivative such that the curvature is unbounded (with respect to the norm induced by the metric) locally as well as a multilinear operator on the tangent space at a single point (see Exercise 4.3.11).

4.27 Similar to the construction of a covariant derivative, every spray induces a bundle morphism $K\colon T^2M \to TM$ which is locally (on a chart domain (U,φ)) given by

$$K(x,y,v,w) := (x, w - B_U(x; y, v)). \tag{4.11}$$

This bundle morphism is a linear connection[8] called the *connector* of the spray. By definition of the connector, the associated covariant derivative is $\nabla_X Y = K \circ TY \circ X$.

[8] A linear connection for a bundle $p\colon E \to M$ is given by a bundle map $k\colon TE \to E$ over the bundle projection which is given in bundle trivialisations by bilinear maps as in (4.11). See, for example, Klingenberg (1995, 1.5).

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4.28 Remark If the manifold admits smooth cutoff functions (e.g. if M is finite dimensional; see also §A.4) one can show (Lang, 1999, §VII.2 Theorem 2.4) that every covariant derivative is associated to a smooth spray as in Proposition 4.24.

4.29 Definition A covariant derivative ∇ on a Riemannian manifold (M,g) is called *metric derivative* if for all $X,Y,Z \in \mathcal{V}(M)$, the following equation holds

$$X.g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Here $X.f = df \circ X$ is the derivative of $f\colon M \to \mathbb{R}$ in the direction of the field X.

4.30 Theorem (Lang, 1999, §VIII.4 Theorem 4.1 and 4.2) *Let (M,g) be a strong Riemannian manifold. Then g admits a metric derivative and there exists a unique spray $S\colon TM \to T^2M$, the* metric spray*, whose associated covariant derivative is the metric derivative.*

4.31 Remark To every (strong) Riemannian metric, there is an associated metric spray and a metric derivative. The metric spray turns out to be the one we computed in 4.17 (whence it coincides with the metric spray in Definition 4.21). For a weak Riemannian metric we have no guarantee that a metric spray exists as Example 4.22 shows, but if a metric spray exists, its associated covariant derivative is the metric derivative.

We investigate some examples of Riemannian manifolds for which these objects can be computed explicitly.

4.32 Example (The trivial example) Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space considered as a strong Riemannian manifold. Due to Exercise 4.3.5, the covariant derivative is $\nabla_X Y = X.Y$. This implies that the bilinear form B of the associated spray (aka the Christoffel symbols) needs to vanish, and we see that the connector and the metric spray associated to the metric are

$$K\colon T^2H \cong H^4 \to TH \cong H^2, \qquad (x,u,v,w) \mapsto (x,w),$$
$$S\colon TH \cong H^2 \to T^2H \cong H^4, \qquad (x,u) \mapsto (x,u,u,0).$$

Finally, from the formula for the covariant derivative one sees that the curvature R identically vanishes on H (one also says that H is flat).

4.33 Example (The submanifold example) Recall from Example 4.7 that the Hilbert sphere S_H of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a submanifold of H. This structure turns S_H into strong Riemannian manifold with respect to the pullback metric

$$g_x(V,W) := \langle V, W \rangle \text{ for } W \in T_x S_H = \{y \in H \mid \langle x, y \rangle = 0\}.$$

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We shall now describe the covariant derivative of this Riemannian metric. Define the smooth map $\mathrm{pr}\colon S_H \times H \to H$, $(p,v) \mapsto v - \langle p,v\rangle p$, and note that for fixed $p \in S_H$, $\mathrm{pr}_p := \mathrm{pr}(p,\cdot)$ is just the orthogonal projector onto the tangent space $T_p S_H$. Since the tangent space $T_p S_H$ has been identified with the orthogonal complement of p in H, we may identify vector fields on S_H with smooth maps $X\colon S_H \to H$ which satisfy $\langle X(p),p\rangle = 0$, for all $p \in S_H$. Using these identifications, we can then define a map

$$\nabla^{S_H} : \mathcal{V}(S_H)^2 \to \mathcal{V}(S_H), \nabla_X^{S_H} Y(p) := \mathrm{pr}_p(dY(p,X(p))). \tag{4.12}$$

It is a straightforward computation (Exercise 4.3.7) that ∇^{S_H} defines a covariant derivative on S_H. Let us show now that it is the metric derivative of the pullback metric g. Pick vector fields $X,Y,Z \in \mathcal{V}(S_H)$. We can now compute as follows:

$$\begin{aligned} X.g(Y,Z)(p) &= X.\langle Y,Z\rangle(p) = d\langle Y,Z\rangle(p;X(p)) \\ &= \langle dY(p;X(p));Z(p)\rangle + \langle Y(p),dZ(p;X(p))\rangle \\ &= g(\nabla_X^{S_H} Y,Z)(p) + g(Y,\nabla_X^{S_H} Z)(p), \end{aligned}$$

where the last equality follows from the fact that vector fields on S_H take their image in the orthogonal complement of the base point. Thus ∇^{S_H} is the metric derivative of the pullback metric. The connector and the metric spray are now simply the restrictions of the ones from Example 4.32. There is also an associated formula (called the Gauss equations) relating the curvature of the sphere to the one of the flat ambient space (see Klingenberg, 1995, Example 1.11.6).

The formulae for covariant derivative, spray and connector in Example 4.33 might seem ad hoc and indeed there is not much to see due to the simplicity of the data of the ambient space. However, the whole procedure turns out to be a special case of a formula which relates the covariant derivative of an isometrically immersed submanifold to the covariant derivative of the ambient manifold (see Klingenberg, 1995, Theorem 1.10.3). For the general notion, one needs to define the covariant derivative of vector fields along smooth maps. For now, we shall define this concept only for the special case of a covariant derivative for vector fields along curves (but see Definition 5.4).

4.34 For a C^1-curve $c\colon [a,b] \to M$, we say a C^1 curve $\alpha\colon [a,b] \to TM$ *lifts* c if $\pi_M \circ \alpha = c$. Denote by $\mathrm{Lift}(c)$ the *set of all lifts* of c. Note that the pointwise operations turn $\mathrm{Lift}(c)$ into a vector space on which the smooth functions $C^\infty([a,b],\mathbb{R})$ act by pointwise multiplication.

If X is a (smooth) vector field on M, then for every C^1-curve $c\colon [a,b] \to M$, the curve $X \circ c$ is a lift of c. Note, however, that not every lift of a curve needs to arise as the composition of a vector field and a curve (e.g. if the curve intersects itself, there can be lifts taking different values for the different time parameters associated to the intersection).

4.35 Consider a C^1-curve $c\colon [a,b] \to M$ and a chart (U,φ) of M. We define a local representative $c_U := \varphi \circ c|_{c^{-1}(U)}$ of c in the chart φ. For curves with $\pi_M \circ \alpha = c$ of M, we also define the *principal part* with respect to the chart (U,φ). Namely, we set $T\varphi \circ \alpha|_{\alpha^{-1}(TU)} = (\varphi \circ c|_{\alpha^{-1}(TU)}, \alpha_U) = (c_U(t), \alpha_U(t))$ for some C^1-map α_U.

Furthermore, define for c a curve $\dot{c}\colon [a,b] \to TM$ with the property $\pi_{TM} \circ \dot{c} = c$ as follows: In any chart (U,φ) the principal part of $\dot{c}$ is $(\dot{c})_U(t) := (\varphi \circ c)'(t) = (c_U)'(t)$. Obviously the definition does not depend on the choice of charts and we note that if c is a C^2-curve, then $\dot{c} \in \operatorname{Lift}(c)$. We will later use the same notation for mappings from the circle $\mathbb{S}^1$ with values in a manifold. Note that (up to a harmless identification) the new definition will coincide with the one here; see §5.1.

4.36 Proposition (Lang, 1999, §VIII.3 Theorem 3.1) *Let S be a spray on M with associate bilinear form B and $c \in C^2([a,b],M)$. Then there exists a unique linear map*

$$\nabla_{\dot{c}}\colon \operatorname{Lift}(c) \to \{\gamma \in C([a,b],TM) \mid \pi_M \circ \gamma = c\}$$

which in a chart (U,φ) is given by the formula

$$(\nabla_{\dot{c}}\alpha)_U(t) = \alpha_U'(t) - B_U(c_U(t); c_U'(t), \alpha_U(t)). \tag{4.13}$$

Furthermore, $\nabla_{\dot{c}}$ acts as a derivation on multiplication with C^1-functions φ, that is,

$$\nabla_{\dot{c}}(\varphi\alpha) = \varphi'(t)\nabla_{\dot{c}}\alpha(t) + \varphi(t)\nabla_{\dot{c}(t)}\alpha(t).$$

Proof The proof is similar to Proposition 4.24 and left as Exercise 4.3.9. □

4.37 For a C^2-curve c, a lift γ is *c-parallel* if $\nabla_{\dot{c}}\alpha = 0$. In a chart (U,φ) this is equivalent to $\alpha_U'(t) = B_U(c_U(t); c_U'(t), \alpha_U(t))$. Thanks to Definition 4.19, c is a geodesic for the spray S if and only if $\dot{c}$ is c-parallel, that is, if and only if it satisfies the *geodesic equation*

$$\nabla_{\dot{c}}\dot{c} = 0.$$

4.38 Example From Exercise 4.2.5 we see that a lift $\alpha\colon [a,b] \to H \times H$ of a curve c in a Hilbert space H is c-parallel if and only if its principal part $\mathrm{pr}_2 \circ \alpha$ is constant, that is, the lift corresponds to a curve which is parallel to c in the usual sense of the word.

From the informal discussion at the end of the last section, we know that a geodesic for a Riemannian manifold should (at least locally) be the shortest path between points which are not far apart. From the presentation in this section this is not apparent. A full proof requires more techniques,[9] whence we just state the following.

4.39 Theorem (Klingenberg, 1995, Theorem 1.9.3) *Let (M, g) be a strong Riemannian manifold, $p \in M$. Then there is an open p-neighbourhood U_p and a constant $\eta > 0$ such that every geodesic starting in U_p of length $< \eta$ is a curve of minimal length between its endpoints.*

Thus a geodesic is always at least locally a curve of minimal length among all curves connecting two (close enough) points on the geodesic.

Exercises

4.3.5 Show that for a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ considered as a strong Riemannian manifold, the metric derivative is given by the usual derivative, that is, $\nabla_X Y = X.Y = dY \circ (\mathrm{id}, X)$. Deduce that geodesics in this case are of the form $at + b$, $a, b \in H$ (see Exercise 4.2.5).
Hint: The metric derivative is unique. If it is $X.Y$, then the bilinear map B of the associated metric spray needs to vanish.

4.3.6 Let (U, φ) be a chart for the manifold M. Show that the local formula (4.9) induces a covariant derivative on $\mathcal{V}(U)$.
Hint: Review Appendix D to prove the statement on the Lie bracket.

4.3.7 Show that (4.12) defines a covariant derivative on S_H.

4.3.8 Consider a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ in the canonical way as a strong Riemannian manifold. Show that for a curve $c \in C^2([a,b], H)$ the covariant derivative $\nabla_{\dot{c}} f = \dot{f}_H$ for all $f = (c, f_H) \in \mathrm{Lift}(c)$. Deduce that geodesics in $(H, \langle\cdot,\cdot\rangle)$ are lines in H.

4.3.9 Use Exercise 4.3.3 to work out a proof for Proposition 4.36.

4.3.10 Let M be a manifold with spray S and associated covariant derivative ∇. Work locally in a chart (U, φ) of M, where we write $X_\varphi, Y_\varphi, Z_\varphi$ for the local representatives of vector fields and B_U for the bilinear form associated to the spray.

(a) Establish the following local formula for the curvature:

$$\begin{aligned}(R(X,Y)Z)_\varphi = B_U(B_U(Y_\varphi, Z_\varphi), X_\varphi)) - B_U(B_U(X_\varphi, Z_\varphi), Y_\varphi)) \\ + d_1 B_U(X_\varphi, Z_\varphi; Y_\varphi) - d_1 B_U(Y_\varphi, Z_\varphi; X_\varphi).\end{aligned}$$

[9] In Proposition 7.2 we shall see that a curve is geodesic if and only if it extremises energy. As energy bounds length, a geodesics extremises the length (i.e. it is locally of minimal length).

Hint: Recall that B_U takes three arguments $B_U(x; X_\varphi(x), Y_\varphi(x))$ and use Proposition 4.24.

(b) Deduce that $R\colon \mathcal{V}(M)^3 \to \mathcal{V}(M)$ is a trilinear map which satisfies

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \text{ (Bianchi identity).}$$

4.3.11 Consider again the subspace $S \subseteq \ell^2$ with the weak Riemannian metric from Exercise 4.3.4. Derive an explicit formula for the curvature $R(X,Y)Z$ and show that the curvature at the point $s = 0$ is unbounded. *Hint:* Everything is local. Since $S \subseteq \ell^2$ is a subspace of a Hilbert space, S admits smooth bump functions. Thus every vector can be continued to a smooth vector field and we write the curvature now for vectors (understanding that they can be continued to vector fields). Let e_i be the vector in ℓ^2 whose only non-zero entry is 1 in the ith place. Set $u_i = \frac{1}{\sqrt{i^3}} e_i$ and work out a formula for $g_0(R(u_i,u_j)u_j|_{s=0}, u_i)$ by exploiting that the e_i are orthonormal with respect to g_0. What happens for $i, j \to \infty$?

Weak Riemannian Metrics with and without Metric Derivative

For a weak Riemannian metric one cannot expect, in general, that there exists a metric derivative associated to the metric. An example of a weak Riemannian metric without an associated covariant derivative can be found in Bauer et al. (2014, p. 12): For a Sobolev-type right invariant metric on a certain subgroup of the diffeomorphism group $\mathrm{Diff}(\mathbb{R})$, the geodesic equation and the covariant derivative do not exist on the subgroup.[10]

4.40 Remark To remedy the problem that the categorisation of weak and strong Riemannian metrics is not sharp enough to capture the existence of covariant derivatives and related important structures for weak Riemannian metrics (see Chapter 5 for an example of a weak Riemannian metric which admits a covariant derivative, connector and metric spray), several authors have proposed a finer classification of infinite-dimensional Riemannian metrics.

We mention here the following concepts and refer interested readers to the original sources for more information.

- Micheli et al. (2013) define a *robust Riemannian metric* as a weak Riemannian metric g on M such that

[10] We will not discuss any details here as this would require at least a discussion of manifold structures on function spaces with non-compact domain. It is worth remarking, though, that the geodesic equation of this metric is related to the non-periodic Hunter–Saxton equation; see Bauer et al. (2014, p. 12 Theorem).

(a) the associated metric derivative exists, and
(b) the Hilbert space completions $\overline{T_x M}^{g_x}$ form a smooth vector bundle $\bigcup_{x \in M} \overline{T_x M}^{g_x}$ whose trivialisations extend the trivialisations of the bundle TM.

In particular, this setting is strong enough to enable certain calculations of curvature for the weak Riemannian manifold.

- Stacey (2008) strengthens the notion of a weak Riemannian structure via an additional structure: As observed in 4.4, one of the main differences between the strong and the weak setting is the failure of the mapping $\flat \colon TM \to T^*M$ to be an isomorphism. While this cannot be directly remedied, requiring a so-called *co-orthogonal structure* allows one to obtain a map replacing the inverse $\sharp$ (which does not exist) of $\flat$ by a mapping with dense image. Exploiting the additional structure it is possible to define a Dirac operator on loop spaces.

We refer to §5.1 for computations of metric derivatives for the (weak) L^2- and H^1-metrics.

4.4 Geodesic Completeness and the Hopf–Rinow Theorem

In this section, we investigate geodesic completeness of infinite-dimensional Riemannian manifolds and the Hopf–Rinow theorem. We will see that this theorem fails in infinite-dimensional geometry as it is built on top of (local) compactness of the underlying manifold. Let us first recall some definitions.

4.41 Definition Let (M, g) be a (weak) Riemannian manifold. Then M

(a) is *metrically complete*, if the geodesic distance turns M into a complete metric space (i.e. Cauchy sequences with respect to the geodesic distance converge);
(b) is *geodesically complete* if every geodesic can be continued for all time;
(c) *has minimising geodesics* if for every two points in the same connected component of M, there exists as length minimising geodesic c connecting the points (i.e. for a, b there is a geodesic c with $\operatorname{Len}(c) = \operatorname{dist}(a, b)$).

In finite dimensions the Hopf–Rinow theorem holds (see Klingenberg, 1995, Theorem 2.1.3).

4.42 Theorem (Hopf–Rinow) *Let (M, g) be a* finite-dimensional *Riemannian manifold. Then M is metrically complete if and only if it is geodesically complete. Moreover, if M is metrically complete, it has minimising geodesics.*

In infinite-dimensional settings Theorem 4.42 does not hold. What remains true, however, is that metrical completeness of a strong Riemannian manifold implies geodesic completeness.

Metrically complete implies geodesically complete Let $c_X : J \to M$ be a geodesic with $\dot{c}(0) = X \in T_pM$. We argue by contradiction and assume that $\sup J = t^+ < \infty$. We can reparametrise c_X such that $\mathrm{Len}(c_X|_{[r,s]}) = |r - s|$ for all $r, s \in J$. Pick a Cauchy sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \nearrow t^+$. As

$$\mathrm{Len}(c_X(t_n), c_X(t_m)) \leq |t_n - t_m|,$$

we see that the points $(c(t_n))_{n\in\mathbb{N}}$ form a Cauchy sequence with respect to the geodesic distance, whence they converge towards some limit $q \in M$. Pick now $\varepsilon > 0$ so small that $B_\varepsilon^{\mathrm{dist}}(q)$ is the domain of Riemannian normal coordinates (Lang, 1999, §VIII.6. Theorem 6.4). For $t^+ - t_0 < \varepsilon/2$ there is a geodesic γ with $\dot{\gamma}(0) = \dot{c}(t_0)$. Note that by uniqueness of geodesics $\gamma(t) = c(t + t_0)$. Since γ is contained in the normal coordinates around q, the domain of definition of γ contains at least $[-\varepsilon/2, \varepsilon/2]$, whence c can be extended beyond t^+. □

In infinite dimensions the following example shows that metric and geodesic completeness do not imply existence of minimising geodesics. This is mainly a consequence of the lack of local compactness as the generalised version of the Hopf–Rinow theorem (in the context of metric spaces) shows see Bridson and Haefliger (1999, Proposition 3.7).

4.43 Example (Grossman's ellipsoid (McAlpin, 1965)) Let $H = \ell^2$ be the Hilbert space of square summable sequences with the orthonormal basis $(\mathbf{e}_n)_{n\in\mathbb{N}}$ (where $\mathbf{e}_n$ is the sequence with a 1 in the nth place and 0s everywhere else). Recall that the inner product on ℓ^2 is $\langle (x_n)_n, (y_n)_n \rangle = \sum_n x_n y_n$. We define $a_1 = 1$ and $a_n = 1 + 2^{-n}$ for $n \geq 2$ and consider the ellipsoid

$$E := \left\{ (x_n)_{n\in\mathbb{N}} \in \ell^2 \,\middle|\, \sum_{n\in\mathbb{N}} \frac{x_n^2}{a_n^2} = 1 \right\}.$$

Defining the smooth diffeomorphism $F \colon H \to H$, $(x_n)_{n\in\mathbb{N}} \mapsto (a_n x_n)_{n\in\mathbb{N}}$, we see that the ellipsoid is the image of the Hilbert sphere $E = F(S_H)$. As the Hilbert sphere is a submanifold of H by Example 1.34, so is the ellipsoid by Exercise 4.4.2. We endow the ellipsoid with the strong Riemannian metric induced by the embedding $E \subseteq H$ whence it becomes a strong Riemannian manifold. Note also that S_H is a Riemannian manifold with respect to the induced metric. Thanks to Klingenberg (1995, 1.10.13(iii)) geodesics in this Riemannian manifold are given by great circles and this shows that S_H is complete (we shall study such a great circle in the next step). Note that by definition of F we have $F(r\mathbf{e}_1) = r\mathbf{e}_1$ for all $r \in \mathbb{R}$. Moreover, if γ is a path in S_H connecting $\mathbf{e}_1$ to $-\mathbf{e}_1$, then $F \circ \gamma$ is a path connecting $\mathbf{e}_1$ and $-\mathbf{e}_1$ in E. As F is a diffeomorphism every path connecting $\mathbf{e}_1$ and $-\mathbf{e}_1$ in E arises in this way. Now, thanks to Exercise 4.4.2(d), we can apply Theorem 6.9 of Lang (1999, §VIII.6) which also implies that E is complete (and geodesically complete).

We shall now prove that $\mathrm{dist}_E(\mathbf{e}_1, -\mathbf{e}_1) = \pi$, but there exists no path realising the minimal distance. Moreover, we shall prove in Exercise 4.4.2 that the half circle $\gamma(t) = \cos(\pi t)\mathbf{e}_1 + \sin(\pi t)\mathbf{e}_2, t \in [0,1]$ is a geodesic connecting $\mathbf{e}_1$ and $-\mathbf{e}_1$, whence $\mathrm{dist}(\mathbf{e}_1, -\mathbf{e}_1)$ in S_H is π. Consider now an arbitrary $\gamma\colon [0,1] \to H, \gamma(t) = (\gamma_n(t))_{n\in\mathbb{N}}$ in S_H connecting $\mathbf{e}_1$ and $-\mathbf{e}_1$. For the length of the paths in S_H and E we obtain as a special case of Exercise 4.4.3 the following:

$$\pi \leq \mathrm{Len}(\gamma) = \int_0^1 \sqrt{\sum_{n\in\mathbb{N}} \dot{\gamma}_n(t)^2}\mathrm{d}t \leq \int_0^1 \sqrt{\sum_{n\in\mathbb{N}} a_n^2 \dot{\gamma}_n(t)^2} = \mathrm{Len}(F(\gamma)). \tag{4.14}$$

By definition of F, we have equality $\mathrm{Len}(\gamma) = \mathrm{Len}(F\gamma)$ if and only if $\dot{\gamma}_n(t) = 0$ for $n \geq 2$. However, the only curve starting in $\mathbf{e}_1$ satisfying this is the constant curve. Hence we have for all curves joining $\mathbf{e}_1$ and $-\mathbf{e}_1$ the strict inequality $\mathrm{Len}(F \circ \gamma) > \pi$. Considering now the half ellipse $\gamma_n(t) = F(\cos(\pi t)\mathbf{e}_1 + \sin(\pi t)\mathbf{e}_n), t \in [0,1]$ joining $\mathbf{e}_1$ and $-\mathbf{e}_1$ in the $(\mathbf{e}_1, \mathbf{e}_n)$-plane, then

$$\begin{aligned}\mathrm{Len}(\gamma_n) = \int_0^1 \pi \sqrt{\sin^2(\pi t) + (1 + 2^{-n})^2 \cos^2(\pi t)}\mathrm{d}t &\leq \sqrt{1 + 2^{-n}}\pi \to \pi \\ = \mathrm{dist}_E(\mathbf{e}_1, -\mathbf{e}_1).&\end{aligned}$$

This illustrates the failure of the Hopf–Rinow theorem in infinite dimensions. Note that explicit counterexamples are known for the other relations (between completeness, geodesic completeness and existence of minimising geodesics) established in the Hopf–Rinow theorem. See Atkin (1975, 1997) for more information.

4.44 Remark It should be mentioned that parts of the Hopf–Rinow theorem can also be salvaged in the infinite-dimensional setting. However, then certain assumptions on the curvature of the Riemannian manifold are needed. For a manifold with seminegative curvature, geodesic completeness implies completeness; see for example, Lang (1999, §IX.3).

Exercises

4.4.1 Let c be a geodesic in a Riemannian manifold (M, g) with covariant derivative ∇.

(a) Show that if $\varphi\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto at + b$, for $a, b \in \mathbb{R}$, then the reparametrisation $c \circ \varphi$ is also a geodesic.

(b) Let $\mathrm{Len}(c|_{[a,b]}) = L|a - b|$. Show that there is a reparametrisation ψ such that $\mathrm{Len}(c \circ \psi|_{[c,d]}) = |c - d|$ for all c, d in the domain of the reparametrised geodesic.

4.4.2 Check various details for Example 4.43.

(a) Show that the mapping $F\colon H \to H$, $F(\sum x_n e_n) = \sum a_n x_n e_n$ is a smooth map with smooth inverse.
Hint: Observe that F is linear and use the idea that limits and series can be exchanged if the series is dominated by a convergent majorant.

(b) Let $S \subseteq M$ be a split submanifold and $F\colon M \to N$ a diffeomorphism. Show that $F(S)$ is a split submanifold of N.
Hint: See Lemma 1.61.

(c) The covariant derivative on S_H as a submanifold of H with the induced metric is given as $\nabla = \mathrm{pr} \circ \nabla^H$, where ∇^H is the covariant derivative on H and pr the projector onto the tangent space of S_H (see Example 4.33). Prove that the half-circle $\gamma(t) = \cos(\pi t)e_1 + \sin(\pi t)e_2$ is a geodesic in S_H.

(d) Show that for all $x, v \in H$ we have $\|T_x F(v)\| \geq \|v\|$.

4.4.3 Generalise (4.14) in the following way: If $f\colon (M,g) \to (N,h)$ is a map between Riemannian manifolds such that there exists a constant $C > 0$ with

$$\sqrt{h_{f(\pi(v))}(Tf(v), Tf(v))} \geq C\sqrt{g_{\pi(v)}(v,v)} \qquad \text{for all } v \in TM,$$

show that for a piecewise C^1-path $\gamma\colon [a,b] \to M$ we have $\mathrm{Len}(f \circ \gamma) \geq C\,\mathrm{Len}(\gamma)$.