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ON THE UNIFORM CONVERGENCE OF INTERPOLATORY POLYNOMIALS

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Abstract

Here we consider a problem on weighted (0, 2) interpolation. We choose the interpolatory conditions in such a way that we get the polynomial of degree $\leqslant 2n-1$, satisfying those conditions. Moreover we prove that the sequence of these interpolatory polynomials under certain conditions converges uniformly to a function belonging to the Zygmund class.

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Let $x_1, x_2, \ldots, x_n$ be any n points such that $-1 \leqslant x_n < x_{n-1} < \ldots < x_2 < x_1 \leqslant 1$ and let $\{\alpha_\nu\}_1^n$ and $\{\beta_\nu\}_1^n$ be arbitrary real numbers. Balázs (1961) considered an interesting problem on weighted $(0,2)$ interpolation by choosing $\{x_\nu\}_1^n$ as the roots of $P_n^{(\alpha)}(x)\,(\alpha > -1)$, the ultraspherical polynomial of degree n. By weighted $(0,2)$ interpolation we mean the construction of the polynomial $S_n(x)$ of degree $2n$ such that

$$S_n(x_\nu) = \alpha_\nu, \quad \nu = 1, 2, \ldots, n, \tag{1.1}$$

$$\{\rho(x)\, S_n(x)\}''_{x=x_\nu} = \beta_\nu, \quad \nu = 1, 2, \ldots, n, \tag{1.2}$$

and

$$S_n(0) = \sum_{\nu=1}^{n} \alpha_\nu\, l_\nu^2(0), \tag{1.3}$$

where $\rho(x) = (1-x^2)^{(1+\alpha)/2}$ is the weight function and $l_\nu(x)$ is the fundamental polynomial of Lagrange interpolation. Balázs proved that there exists a unique polynomial $S_n(x)$ of degree $2n$ satisfying (1.1), (1.2) and (1.3) if n is even but for

n odd there does not exist a unique polynomial $S_n(x)$ of degree $2n$ satisfying the above conditions. Moreover he showed that if the condition (1.3) is dropped there does not exist a unique polynomial $S_n^*(x)$ of degree $2n-1$ for both n even and odd satisfying the above conditions (1.1) and (1.2).

For the uniform convergence Balázs (1961) obtains the following:

THEOREM 1. *Let $f(x)$ be a continuous function in $[-1,1]$ and let $f'(x) \in \mathrm{Lip}\,\mu$, $\frac{1}{2} < \mu \leqslant 1$. Further, let $\alpha_\nu = f(x_\nu)$ and $\beta_\nu = o(n^{\frac{1}{2}})(1-x_\nu^2)^{\frac{1}{2}(\alpha-3)}$. Then the sequence of polynomials $S_n(x)$ converges uniformly to $f(x)$ in $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$, $0<\varepsilon<1$ (ε being an arbitrary fixed positive number).*

Prasad (1970) extended the results of Balázs to the case when the nodes of interpolation are unsymmetrical by choosing $\{x_\nu\}_1^n$ as the zeros of $P_n^{(\alpha,-\alpha)}(x)$ $(\alpha > -1)$, the Jacobi polynomial of degree n with $(\alpha, \beta) = (\alpha, -\alpha)$. Similar results were obtained by Prasad and Eckert (1973) for different nodes of interpolation. Here our aim is to choose the interpolatory conditions in such a way so that we get a polynomial of degree $\leqslant 2n-1$ and that the corresponding uniform convergence theorem is valid for a wider class of functions. In order to accomplish this we shall prove the following:

THEOREM 2. *If n is a positive integer, $\{x_\nu\}_1^n$ are the roots of $P_n(x)$, the Legendre polynomial of degree n and $\{\alpha_\nu\}_1^n$ and $\{\beta_\nu\}_1^n$ are arbitrary real numbers then there is, in general, no polynomial $R_n(x)$ of degree $\leqslant 2n-1$ such that*

$$R_n(x_\nu) = \alpha_\nu, \quad \nu = 1, 2, \ldots, n, \tag{1.4}$$

and

$$[(1-x^2)^\lambda R_n(x)]''_{x=x_\nu} = \beta_\nu, \quad \nu = 1, 2, \ldots, n, \tag{1.5}$$

for $\lambda = \frac{3}{2}$ or $\lambda = \frac{1}{2}$ and if there exists such a polynomial then there is an infinity of them.

THEOREM 3. *If n is even and $\{x_\nu\}_1^n$ are the roots of $P_n(x)$, the Legendre polynomial of degree n, then to prescribed values α_ν and β_ν ($\nu = 1, 2, \ldots, n$) there is a uniquely determined polynomial $R_n(x)$ of degree $\leqslant 2n-1$ such that*

$$R_n(x_\nu) = \alpha_\nu, \quad \nu = 1, 2, \ldots, n, \tag{1.6}$$

and

$$[(1-x^2)^\lambda R_n(x)]''_{x=x_\nu} = \beta_\nu, \quad \nu = 1, 2, \ldots, n, \tag{1.7}$$

for $0 \leqslant \lambda < \frac{3}{2}$, $\lambda \neq \frac{1}{2}$. *If n is odd there is, in general, no unique polynomial* $R_n(x)$ *of degree* $\leqslant 2n-1$ *which satisfies* (1.6) *and* (1.7).

Obviously,

$$R_n(x) = \sum_{\nu=1}^{n} \alpha_\nu \mu_\nu(x) + \sum_{\nu=1}^{n} \beta_\nu \sigma_\nu(x), \tag{1.8}$$

where $\mu_\nu(x)$ and $\sigma_\nu(x)$ are as in Theorem 4.

THEOREM 4. *If n is even,* $1 \leqslant \nu \leqslant n$ *and* $\frac{1}{2} < \lambda < \frac{3}{2}$ *then*

$$\sigma_\nu(x) = \frac{P_n(x)(1-x^2)^{\frac{1}{2}-\lambda}}{2(1-x_\nu^2)^{\lambda-1} P_n'(x_\nu)} \left[A_\nu \int_{-1}^{x} P_n(t)\, v(t)\, dt + \int_{-1}^{x} l_\nu(t)\, v(t)\, dt \right], \tag{1.9}$$

where

$$v(t) = (1-t^2)^{\lambda-3/2} \tag{1.10}$$

and

$$A_\nu \int_{-1}^{1} P_n(t)\, v(t)\, dt + \int_{-1}^{1} l_\nu(t)\, v(t)\, dt = 0 \tag{1.11}$$

and

$$\mu_\nu(x) = l_\nu^2(x) + \frac{P_n(x)\, l_\nu'(x)}{2P_n'(x_\nu)} + C_\nu \sigma_\nu(x) + P_n(x) q_{n-1}(x), \tag{1.12}$$

where

$$(1-x^2)^{\lambda-\frac{1}{2}} q_{n-1}(x) = \frac{\lambda-\frac{3}{2}}{P_n'(x_\nu)} \left[B_\nu \int_{-1}^{x} P_n(t)\, v(t)\, dt + \int_{-1}^{x} t l_\nu'(t)\, v(t)\, dt \right], \tag{1.13}$$

$$B_\nu \int_{-1}^{1} P_n(t)\, v(t)\, dt + \int_{-1}^{1} t l_\nu'(t)\, v(t)\, dt = 0 \tag{1.14}$$

and

$$C_\nu = \frac{n(n+1)+2(\lambda-1)}{(1-x_\nu^2)^{1-\lambda}} - \frac{(4\lambda^2-10\lambda+11)\, x_\nu^2}{(1-x_\nu^2)^{2-\lambda}}. \tag{1.15}$$

REMARK 1. It is worth while to point out that if we take $\lambda = 0$ in (1.7) then we get a theorem for pure (0, 2) interpolation which is analogous to Theorem 2 of Surányi and Turán (1955). Further, for the case $\lambda = 0$ we can obtain the explicit representation of these polynomials and prove a uniform convergence theorem but we shall not do so as it is already done by Prasad (1972–73). Now we are in a position to state the following:

THEOREM 5. *Let* $f(x)$ *be a continuous function in* $[-1, 1]$ *and let it satisfy the Zygmund condition*

$$|f(x+h) - 2f(x) + f(x-h)| = o(h) \tag{1.16}$$

in $(-1,1)$. *Further, let* $\alpha_\nu = f(x_\nu)$ *and* $\beta_\nu = o(n)(1-x_\nu^2)^{-3/4}$, $\nu = 1,2,\ldots,n$ *then for* $\lambda = 1$ *the sequence of polynomials* $R_n(x,f)$ *given by* (1.8) *converges uniformly to* $f(x)$ *in every closed interval* $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$, ε *being fixed,* $0<\varepsilon<1$.

REMARK 2. We shall give here the proof of Theorem 2 and Theorem 5 only. The proof of Theorem 3 can be given along the same lines as Theorem 2 and the proof of Theorem 4 could be obtained along the same lines as given by Balázs and Turán (1958).

2. Proof of Theorem 2

We give here the proof of Theorem 2 for $\lambda = \frac{3}{2}$. For $\lambda = \frac{1}{2}$ the proof can be carried out exactly in the same manner. Our aim is to show that if $\alpha_\nu = 0$ for $\nu = 1,2,\ldots,n$ and $\beta_\nu = 0$ for $\nu = 1,2,\ldots,n$, then there exists a polynomial $R_n(x)$ of degree $\leqslant 2n-1$ which is not identically zero but satisfies conditions (1.4) and (1.5). Thus by invoking a well-known theorem from the theory of equations we get the desired result.

Due to the conditions of the theorem it is evident that

$$R_n(x) = P_n(x)\,g_{n-1}(x), \tag{2.1}$$

where $g_{n-1}(x)$ is a polynomial of degree $\leqslant n-1$. Since $[(1-x^2)^{3/2} R_n(x)]''_{x=x_\nu} = 0$, for $\nu = 1,2,\ldots,n$, it follows on using the differential equation for $P_n(x)$ that

$$(1-x_\nu^2)\,g'_{n-1}(x_\nu) - 2x_\nu\, g_{n-1}(x_\nu) = 0, \quad \nu = 1,2,\ldots,n,$$

from which we conclude that

$$(1-x^2)\,g'_{n-1}(x) - 2x g_{n-1}(x) = cP_n(x), \tag{2.2}$$

where c is a numerical constant. Now we set

$$g_{n-1}(x) = \sum_{k=0}^{n-1} a_k P_k(x). \tag{2.3}$$

From Sansone (1959), Chapter 3, p. 179, we know that

$$(1-x^2)\,P'_n(x) = nP_{n-1}(x) - nxP_n(x). \tag{2.4}$$

Hence from (2.2), (2.3) and (2.4) we obtain

$$\sum_{k=1}^{n-1} ka_k P_{k-1}(x) - \sum_{k=0}^{n-1} (k+2)\, a_k x P_k(x) = cP_n(x). \tag{2.5}$$

Further, from Sansone (1959), Chapter 3, p. 178, we have

$$(n+1)\,P_{n+1}(x) + nP_{n-1}(x) = (2n+1)\,xP_n(x). \tag{2.6}$$

So from (2.6) and (2.5) it follows that

$$cP_n(x) = \sum_{k=1}^{n-2}\left[\frac{k(k+1)}{2k+3}a_{k+1} - \frac{k(k+1)}{2k-1}a_{k-1}\right]P_k(x) - \frac{n(n-1)}{2n-3}a_{n-2}P_{n-1}(x)$$
$$-\frac{n(n+1)}{2n-1}a_{n-1}P_n(x).$$

Now equating the coefficients of $P_k(x)$ on both sides, we get

$$\text{(2.7)}\qquad \frac{k(k+1)}{2k+3}a_{k+1} - \frac{k(k+1)}{2k-1}a_{k-1} = 0, \quad k = 1, 2, \ldots, n-2,$$
$$-\frac{n(n-1)}{2n-3}a_{n-2} = 0$$
$$-\frac{n(n+1)}{2n-1}a_{n-1} = c.$$

If n is even then from (2.7) we get that

$$\text{(2.8)}\qquad a_0 = a_2 = a_4 = \ldots = a_{n-2} = 0$$

and

$$\text{(2.9)}\qquad a_{2k+1} = -\frac{(4k+3)}{n(n+1)}c, \quad k = 0, 1, 2, \ldots, \tfrac{1}{2}n-1.$$

Similarly, if n is odd then from (2.7) we obtain

$$\text{(2.10)}\qquad a_1 = a_3 = \ldots = a_{n-2} = 0$$

and

$$\text{(2.11)}\qquad a_{2k} = -\frac{(4k+1)}{n(n+1)}c, \quad k = 0, 1, \ldots, \tfrac{1}{2}(n-1).$$

Consequently from (2.8), (2.9), (2.10), (2.11), (2.1) and (2.3) it follows that

$$\text{(2.12)}\qquad R_n(x) = -\frac{cP_n(x)}{n(n+1)}\sum_{k=0}^{\frac{1}{2}n-1}(4k+3)P_{2k+1}(x) \quad \text{for } n \text{ even}$$

and

$$\text{(2.13)}\qquad R_n(x) = -\frac{cP_n(x)}{n(n+1)}\sum_{k=0}^{\frac{1}{2}(n-1)}(4k+1)P_{2k}(x) \quad \text{for } n \text{ odd.}$$

Thus it is evident that there does not exist in general a unique polynomial $R_n(x)$ of degree $\leqslant 2n-1$ satisfying conditions (1.4) and (1.5) regardless whether n is even or odd and if there exists such a polynomial there are infinity of them.

3. Some lemmas

In this section we state and prove a few lemmas which will help us in arriving at the proof of Theorem 5.

LEMMA 3.1. *Let $f(x)$ be a continuous function satisfying the Zygmund condition* (1.16) *in* $[-1,1]$ *then there exists a sequence of polynomials $\Phi_n(x)$ of degree $\leqslant n$ with the following properties*:

$$|f(x)-\Phi_n(x)| = o(n^{-1})\,[(1-x^2)^{\frac{1}{2}}+n^{-1}], \tag{3.1}$$

$$|\Phi_n'(x)| = o(\log n) \tag{3.2}$$

and

$$|\Phi_n''(x)| = o(n)\min\,[(1-x^2)^{-\frac{1}{2}}, n] \tag{3.3}$$

which holds uniformly in $[-1,1]$.

For the proof of (3.1) and (3.3) see Freud (1958). The proof of (3.2) can be given along the same lines as (3.3).

If $Q_n(x)$ is any polynomial of degree $\leqslant 2n+1$ and n is an even positive integer then due to Prasad and Varma (1969) we know that

$$Q_n(x) = \sum_{i=0}^{n+1} Q_n(x_i)\,r_i(x) + \sum_{i=1}^{n} Q_n''(x_i)\,\rho_i(x), \tag{3.4}$$

where

$$1 = x_0 > x_1 > \dots > x_n > x_{n+1} = -1 \tag{3.5}$$

are the zeros of $(1-x^2)P_n(x)$ and

$$r_i(x_j) = \begin{matrix} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j, \end{matrix} \quad j = 0,1,\dots,n+1,$$

$$r_i''(x_j) = 0, \quad j = 1,2,\dots,n,$$

$$\rho_i(x_j) = 0, \quad j = 0,1,\dots,n+1,$$

$$\rho_i''(x_j) = \begin{matrix} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j, \end{matrix} \quad j = 1,2,\dots,n,$$

and that $r_i(x)$ and $\rho_i(x)$ are given by Theorem 2.1 of Prasad and Varma (1969), p. 55. Now if $\lambda = 1$ then from (1.9) to (1.15) one can easily see that

$$\sigma_\nu(x) = \frac{\rho_\nu(x)}{1-x^2}, \quad \nu = 1,2,\dots,n, \tag{3.6}$$

and

(3.7) $$\mu_\nu(x) = \frac{(1-x_\nu^2)\, r_\nu(x)}{1-x^2}, \quad \nu = 1,2,\ldots,n.$$

Now we shall have the following:

LEMMA 3.2. *For* $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$ *and* n *even, we have*

(3.8) $$\sum_{\nu=1}^{n} (1-x^2)^{-3/4} |\sigma_\nu(x)| \leqslant k_1 n^{-1},$$

(3.9) $$\sum_{\nu=1}^{n} (1-x_\nu^2)^{-\frac{1}{2}} |\sigma_\nu(x)| \leqslant k_1 n^{-1},$$

(3.10) $$\sum_{\nu=1}^{n} |\sigma_\nu(x)| \leqslant k_1 n^{-1}$$

and

(3.11) $$\sum_{\nu=1}^{n} (1-x_\nu^2)^{-1} |\sigma_\nu(x)| \leqslant k_2 n^{-1} \log n,$$

where k_i, $i = 1,2,\ldots$, *are constants depending on* ε.

PROOF. From (4.9) of Prasad and Varma (1969), p. 58 we know that

(3.12) $$|\rho_\nu(x)| \leqslant \frac{48}{n^{\frac{1}{2}}(1-x_\nu^2)^{7/4}[|P_n'(x_\nu)|]^3} + \frac{3}{n(1-x_\nu^2)[P_n'(x_\nu)]^2} + \frac{(1-x^2)^{1/4}|l_\nu(x)|}{2n^{3/2}|P_n'(x_\nu)|(1-x_\nu^2)^{\frac{1}{2}}}.$$

Since $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$ hence $1/(1-x^2) \leqslant 1/\varepsilon^2$ and on using (3.6) and (3.12) we obtain

(3.13) $$\begin{aligned} &\sum_{\nu=1}^{n} (1-x_\nu^2)^{-3/4} |\sigma_\nu(x)| \\ &\leqslant \frac{48}{n^{\frac{1}{2}}\varepsilon^2} \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)^{5/2}|P_n'(x_\nu)|^3} \\ &\quad + \frac{3}{n\varepsilon^2} \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)^{7/4}[P_\nu'(x_\nu)]^2} \\ &\quad + \frac{1}{2n^{3/2}\varepsilon^2} \sum_{\nu=1}^{n} \frac{(1-x^2)^{1/4}|l_\nu(x)|}{(1-x_\nu^2)^{5/4}|P_n'(x_\nu)|} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

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It is well known from Szegö (1959), Chapter 8, p. 236 that

$$|P_n'(\cos\theta_\nu)| \sim \nu^{-3/2}n^2, \quad \nu = 1,2,\ldots,\tfrac{1}{2}n, \tag{3.14}$$

$$|P_n'(\cos\theta_\nu)| \sim n^2(n-\nu+1)^{-3/2}, \quad \nu = \tfrac{1}{2}n+1,\ldots,n, \tag{3.15}$$

$$(1-x_\nu^2) > \nu^2 n^{-2}, \quad \nu = 1,2,\ldots,\tfrac{1}{2}n, \tag{3.16}$$

and

$$(1-x_\nu^2) > (n-\nu+1)^2 n^{-2}, \quad \nu = \tfrac{1}{2}n+1,\ldots,n. \tag{3.17}$$

Now on using (3.14), (3.15), (3.16) and (3.17) we see that

$$I_1 \leqslant k_3 n^{-1}. \tag{3.18}$$

Similarly, (3.14), (3.15), (3.16) and (3.17) yield

$$I_2 \leqslant k_4 n^{-1}. \tag{3.19}$$

Further, from Prasad and Varma (1969), p. 59 we also have

$$(1-x^2)^{1/4}|l_\nu(x)| \leqslant \frac{3n}{(1-x_\nu^2)^{5/4}[P_n'(x_\nu)]^2}. \tag{3.20}$$

Hence from (3.20), (3.14), (3.15), (3.16) and (3.17) it follows that

$$I_3 \leqslant k_5 n^{-1}. \tag{3.21}$$

Finally from (3.13), (3.18), (3.19) and (3.21) we have (3.8). One can very easily see that (3.9) and (3.10) follow from (3.8).

In the same way on using (3.12), (3.20), (3.14), (3.15), (3.16) and (3.17) we arrive at the estimate (3.11). Next we shall prove the following:

LEMMA 3.3. *If* $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$ *and* n *is an even positive integer then*

$$\sum_{\nu=1}^{n} |\mu_\nu(x)| \leqslant k_6 n.$$

PROOF. From (3.7) we see that for $-1+\varepsilon \leqslant x \leqslant 1-\varepsilon$,

$$\sum_{\nu=1}^{n} |\mu_\nu(x)| \leqslant \frac{1}{\varepsilon^2}\sum_{\nu=1}^{n}(1-x_\nu^2)|r_\nu(x)|. \tag{3.22}$$

Further, from Prasad and Varma (1969), p. 60, one can also easily see that

$$\sum_{\nu=1}^{n}(1-x_\nu^2)^{\frac{1}{2}}|r_\nu(x)| \leqslant 894n. \tag{3.23}$$

Consequently Lemma 3.3 follows from (3.22) and (3.23).

4. Proof of Theorem 5

Owing to the uniqueness theorem we get

$$\Phi_n(x) = \sum_{\nu=1}^{n} \Phi_n(x_\nu)\,\mu_\nu(x) + \sum_{\nu=1}^{n} [(1-x^2)\,\Phi_n(x)]''_{x_\nu}\,\sigma_\nu(x), \tag{4.1}$$

where $\Phi_n(x)$ is given by Lemma 3.1. Further, since

$$R_n(x,f) = \sum_{\nu=1}^{n} f(x_\nu)\,\mu_\nu(x) + \sum_{\nu=1}^{n} \beta_\nu\,\sigma_\nu(x) \tag{4.2}$$

it follows that

$$\begin{aligned} |R_n(x,f)-f(x)| &\leqslant |R_n(x,f)-\Phi_n(x)|+|\Phi_n(x)-f(x)| \\ &= J_1+J_2. \end{aligned} \tag{4.3}$$

We shall first determine J_1. From (4.1) and (4.2) we see that

$$\begin{aligned} J_1 &\leqslant \sum_{\nu=1}^{n} |f(x_\nu)-\Phi_n(x_\nu)|\,|\mu_\nu(x)| + \sum_{\nu=1}^{n} |\beta_\nu|\,|\sigma_\nu(x)| + \sum_{\nu=1}^{n} |[(1-x^2)\,\Phi_n(x)]''_{x_\nu}\,\sigma_\nu(x)| \\ &= \theta_1+\theta_2+\theta_3. \end{aligned} \tag{4.4}$$

Now using Lemma 3.1 and Lemma 3.3 we obtain

$$\begin{aligned} \theta_1 &= \sum_{\nu=1}^{n} |f(x_\nu)-\Phi_n(x_\nu)|\,|\mu_\nu(x)| \\ &= o(1). \end{aligned} \tag{4.5}$$

Further, using Lemma 3.2 and the estimate $\beta_\nu = o(n)\,(1-x_\nu^2)^{-3/4}$ we have that

$$\begin{aligned} \theta_2 &= o(n) \sum_{\nu=1}^{n} (1-x_\nu^2)^{-3/4}\,|\sigma_\nu(x)| \\ &= o(1). \end{aligned} \tag{4.6}$$

Finally on using Lemma 3.1 and Lemma 3.2 we get

$$\begin{aligned} \theta_3 &= \sum_{\nu=1}^{n} |[(1-x^2)\,\Phi_n(x)]''_{x_\nu}\,\sigma_\nu(x)| \\ &= o(1)+o(\log n)\,[k_1 n^{-1}]+o(n) \sum_{\nu=1}^{n} (1-x_\nu^2)^{-\frac{1}{2}}\,|\sigma_\nu(x)| \\ &= o(1). \end{aligned} \tag{4.7}$$

Hence from (4.4), (4.5), (4.6) and (4.7) it follows that

$$J_1 = o(1). \tag{4.8}$$

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Similarly, the use of Lemma 3.1 once again yields

$$J_2 = o(1). \tag{4.9}$$

Consequently from (4.3), (4.8) and (4.9) we conclude that

$$|R_n(x, f) - f(x)| = o(1).$$

This completes the proof of the theorem.

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