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BOUNDED POINTWISE APPROXIMATION OF SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. We characterize open subsets U of $\mathbb{R}^N$ in which the bounded solutions of certain elliptic equations can be approximated pointwise by uniformly bounded solutions that are continuous in $\bar{U}$. This result is established in terms of certain capacities.

For closed subsets X, this characterization allows us to approximate bounded solutions in X^o uniformly on relatively closed subsets of X^o by solutions continuous on certain subsets of the boundary of X.

1. **Introduction.** Vitushkin's theorem [16] on uniform approximation of holomorphic functions on compact subsets of the complex plane gives, in terms of the continuous analytic capacity, the necessary and sufficient conditions on compact subsets K of the complex plane in order that holomorphic functions in the interior of K and continuous on K could be approximated by rational functions with poles off K. Vitushkin's result has been extended to unbounded closed subsets of the complex plane by Hadjiiski [7] making use of a covering lemma by A. M. Davie [4].

In a natural extension this scheme has been considered in the theory of approximation of solutions of elliptic operators. For these spaces we recall the work of Bagby for the L^p-norm [1], Verdera for the C^m-norm [15], O'Farrell with Lipschitz norm [9] and for solutions of elliptic systems with the C^m-norm by Tarkhanov [13]. In [1] Bagby proves a theorem on approximation in the mean of solutions of certain homogeneous elliptic equations. One of the most important ideas in Bagby's paper is the definition of a set of capacities with the objective to construct approximating functions with fixed coefficients in their Laurent expansion up to a fixed order. Bagby's scheme requires to define one capacity for each possible Laurent expansion of the functions to be approximated. Using this idea, Tarkhanov proves in [13] the analogue of Vitushkin's theorem for C^m-approximation of the solutions of homogeneous elliptic systems of differential equations on compact subsets of $\mathbb{R}^N$.

Gamelin and Garnett [6] considered Vitushkin's scheme in order to approximate bounded analytic functions on open subsets U of the complex plane with compact boundary by analytic functions on U and continuous on $\bar{U}$. This choice of approximated and approximating functions required a weak type of convergence, namely the bounded pointwise convergence. In this paper, with Tarkhanov's result as a starting point, we characterize proper open subsets U of $\mathbb{R}^N$ such that the class of continuous functions on $\bar{U}$ that satisfy on U a homogeneous elliptic equation is dense with respect to bounded

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pointwise convergence in the space of bounded solutions on U. This result is in some sense an extension of Gamelin and Garnett's theorem to a larger class of equations and to more general sets U without any restrictions on the boundary of U.

Let X be any compact set of the plane where the functions continuous in X and analytic on X^o are dense with respect to bounded pointwise convergence in $H^\infty(X^o)$. If E is relatively open subset of ∂X and $F \subset X^o$ is closed relative to $X^o \cup E$, then any function in $H^\infty(X^o)$ can be approximated uniformly on F by functions analytic on X^o and continuous on E. This result, due to A. Stray [12, Theorem 2.1], makes a decisive use of Gamelin and Garnett's theorem. Thus, our characterization of bounded pointwise density gives an extension of Stray's theorem to proper closed subsets of $\mathbb{R}^N$ and for solutions of the elliptic equations considered.

2. **Preliminaries.** We begin by giving some notation which will remain in effect for the rest of the paper. For any subset A of $\mathbb{R}^N$ we use the standard notation of $A^o, \bar{A}, \partial A$ to describe the interior, the closure and the boundary of A respectively. The characteristic function of A is denoted by χ_A. We will use the letters C, M, R to denote constants that can change its value from line to line. We write $C = C(P, N, \ldots)$ to indicate that C depends on $P, N, \ldots$ and so on. For any function f we denote the supremum of f on A by $\|f\|_A$. In case that $A = \mathbb{R}^N$ we only write $\|f\|$.

Let $\mathcal{D}(\mathbb{R}^N)$ be the complex vector space of all C^∞ functions in $\mathbb{R}^N$ with compact support. Its dual, the space of Schwartz distributions, is denoted by $\mathcal{D}'(\mathbb{R}^N)$. We use the notation $\langle T, \varphi \rangle$ to denote action of the distribution $T \in \mathcal{D}'(\mathbb{R}^N)$ on a function $\varphi \in \mathcal{D}(\mathbb{R}^N)$. The class of functions defined on a locally compact Hausdorff space X vanishing at infinity is denoted by $C_o(X)$.

If $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$, we let $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$. If the polynomial Q is given by $Q(\xi) = \sum a_\alpha \xi^\alpha$, we let $\bar{Q}(\xi) = \sum \bar{a}_\alpha \xi^\alpha$ and $Q(D) = \sum a_\alpha D^\alpha$.

Consider in $\mathbb{R}^N$ the equation $P(D)f = 0$ generated by a homogeneous polynomial $P(\xi) = \sum_{|\alpha|=p} P_\alpha \xi^\alpha$ of degree p with complex coefficients such that

1. P satisfies the ellipticity condition: $P(\xi) \neq 0$ if $\xi \in \mathbb{R}^N \setminus \{0\}$.
2. $p < N$.

Under these assumptions, the differential operator $P(D)$ considered admits a fundamental solution $V \in \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ whose restriction to $\mathbb{R}^N \setminus \{0\}$ is a real-analytic homogeneous function of degree $p - N$ [8, Chapter III].

Let $\mathcal{P}_k$ denote the space of all homogeneous polynomials of degree k in n variables with complex coefficients. This space has the Hilbert space structure with the inner product [10, Chapter III, Section 3.1]

$$\{R_1, R_2\} = R_1(D)\bar{R}_2 = \bar{R}_2(D)R_1 = \sum_{|\alpha|=k} \alpha!\, c_\alpha^{(1)} \overline{c_\alpha^{(2)}},$$

where R_1 and R_2 are elements of $\mathcal{P}_k$ with

$$R_i(\xi) = \sum_{|\alpha|=k} c_\alpha^{(i)} \xi^\alpha \quad \text{for } \xi \in \mathbb{R}^N;\ i = 1, 2.$$

Let $\mathcal{H}_k(P)$ denote the orthogonal complement of the vector subspace $P(\xi)[\mathcal{P}_{k-p}]$ of $\mathcal{P}_k$. Then we can write

$$\mathcal{P}_k = P(\xi)[\mathcal{P}_{k-p}] \oplus \mathcal{H}_k(P).$$

For each multi-index α with $|\alpha| = k$ we introduce the polynomial $\mathcal{N}_\alpha(\xi) = \sum_{|\beta|=k} n_{\alpha\beta}\xi^\beta$ that represents the projection of ξ^α on $\mathcal{H}_k(P)$. These polynomials characterize $\mathcal{H}_k(P)$ in the following way

PROPOSITION 2.1 ([14, LEMMA 2.6]). *The polynomial* $h(\xi) = \sum_{|\alpha|=k} c_\alpha \xi^\alpha$ *belongs to* $\mathcal{H}_k(P)$ *if, and only if, for every multi-index* α *with* $|\alpha| = k$ *we have* $c_\alpha = \sum_{|\beta|=k} n_{\beta\alpha} c_\beta$.

Consider now the following polynomials which will play a fundamental role. For any multi-index α, define $\mathcal{M}_\alpha(y)$ by

$$\mathcal{M}_\alpha(y) = (-1)^k \sum_{|\beta|=k} \frac{n_{\beta\alpha}}{\beta!} y^\beta.$$

Given $F \subset \mathbb{R}^n$, let $O(F)$ denote the vector space of solutions of $P(D)f = 0$ in a neighborhood of F. For the rest of the paper K will denote a compact subset of $\mathbb{R}^N$. We now establish the theorem of representation for the functions in $O(\mathbb{R}^N \setminus K)$ that vanish at infinity.

THEOREM 2.2 ([14, LEMMA 4.6]). *Let* $f \in O(\mathbb{R}^N \setminus K)$ *vanish at infinity. If* x_0 *is a point of* K *and* $r_0 = \sup_{y\in\partial K}\{|y - x_0|\}$, *then, in the complement of the closed ball* $\overline{B(x_0, r_0 a N^2)}$,

$$f(x) = \sum_\alpha D^\alpha V(x - x_0) c_\alpha, \tag{2.1}$$

where the series converges absolutely and uniformly on closed subsets of the complement of $\overline{B(x_0, r_0 a N^2)}$. *Moreover, the coefficients* c_α *with* $|\alpha| = k$ *are defined by*

$$c_\alpha = c_\alpha(f, x_0) = \int \mathcal{M}_\alpha(y - x_0)\big(P(D)f\big)(y)\, dy$$

and satisfy the condition $c_\alpha = \sum_{|\beta|=k} n_{\beta\alpha} c_\beta$.

We have an equivalent representation of the coefficients c_α that will be very useful in the next section. Consider the Green Operator $G(g,f)$ of the differential operator $P(D)$

$$G(g,f) = \sum_{|\beta+\gamma+1_j|=p} (-1)^{|\beta|+(j-1)} p_{\beta+\gamma+1_j} \partial^\beta g \partial^\gamma f \, dy[j]$$

[3, Section 14.3.4] where $dy[j] = \bigwedge_{\substack{i=1,\dots,n \\ i\neq j}} dy_i$, and 1_j represents the unit vector on the x_j-axis. Then any function $f \in O(\mathbb{R}^N \setminus K)$ vanishing at infinity can be represented in the neighborhood of the infinity given in the theorem by

$$f(x) = -\int_\Gamma G\big(V(x - y), f(y)\big),$$

where Γ is its boundary. This representation gives the equivalent definition of the coefficients c_α as

$$c_\alpha = c_\alpha(f, x_0) = -\int_\Gamma G\big(\mathcal{M}_\alpha(y - x_0), f(y)\big).$$

We point out that this representation of the coefficients c_α does not depend on the choice of the surface Γ provided that Γ is contained in $\mathbb{R}^N \setminus K$, surrounds K once and is oriented so that the outwards normal vector points toward infinity.

The relevance of Vitushkin's result for holomorphic functions in the plane is the sufficiency of just one capacity to fix three coefficients of the Laurent expansion of the admissible functions [5, Chapter VIII]. Here, we follow the scheme established by Bagby [1] and for each possible Laurent expansion of the function to be approximated we define one capacity.

Let Ω be an open subset of $\mathbb{R}^N$, K a compact subset of Ω and $H(\xi) = \sum_{|\alpha|=k} c_\alpha \xi^\alpha$ a fixed element of $\mathcal{H}_k(P)$ with $k \in \mathbb{N}$. Denote by $\mathcal{B}_H(K, \Omega)$ the set of functions $f \in O(\mathbb{R}^N \setminus K)$ vanishing at infinity, with Laurent expansion

$$f(x) = \lambda \sum_{|\alpha|=k} D^\alpha V(x) c_\alpha + o(|x|^{p-N-k}) \quad \text{as } x \longrightarrow \infty$$

for some scalar $\lambda \in \mathbb{C}$ and such that $\|f\|_\Omega = \sup_{x\in\Omega} |f(x)| < \infty$.

Let $C(\Omega)$ be the set of all continuous functions on Ω, and denote $C_H(K, \Omega) = \mathcal{B}_H(K, \Omega) \cap C(\Omega)$.

DEFINITION 2.3. We define the following capacities:

$$\gamma_H(K, \Omega) = \sup_{\substack{f\in\mathcal{B}_H(K,\Omega)\\ \|f\|_\Omega\leq 1}} |\langle P(D)f, \bar{H}\rangle| = \sup_{\substack{f\in\mathcal{B}_H(K,\Omega)\\ \|f\|_\Omega\leq 1}} \Big|\int_\Gamma G\big(\bar{H}(y), f(y)\big)\Big|, \tag{2.2}$$

$$\alpha_H(K, \Omega) = \sup_{\substack{f\in C_H(K,\Omega)\\ \|f\|_\Omega\leq 1}} |\langle P(D)f, \bar{H}\rangle| = \sup_{\substack{f\in C_H(K,\Omega)\\ \|f\|_\Omega\leq 1}} \Big|\int_\Gamma G\big(\bar{H}(y), f(y)\big)\Big|. \tag{2.3}$$

If F is an arbitrary precompact subset of Ω, then we set $\gamma_H(F, \Omega) = \sup \gamma_H(K, \Omega)$ and $\alpha_H(F, \Omega) = \sup \alpha_H(K, \Omega)$ where the supremum is taken over all compact subsets K of F.

These capacities are similar to the capacities defined by Tarkhanov in [13] for the C^m-approximation. For the main properties of this capacities see [13]. Here we mention two equivalent definitions of those capacities that will be useful later.

LEMMA 2.4. *The identity*

$$\gamma_H(K, \Omega) = \frac{\{H, H\}}{\inf \|f\|_\Omega}$$

holds, where the infimum is taken over all functions $f \in O(\mathbb{R}^N \setminus K)$ that vanish at infinity with

$$f(x) = \sum_{|\alpha|=k} D^\alpha V(x) c_\alpha + o(|x|^{p-N-k}) \quad \textit{as } x \longrightarrow \infty.$$

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If we take the infimum over the functions on $C_H(K,\Omega)$ then we get

$$\alpha_H(K,\Omega) = \frac{\{H,H\}}{\inf \|f\|_\Omega}.$$

From now on we take $\Omega = \mathbb{R}^N$ so we write $\gamma_H(F) = \gamma_H(F,\mathbb{R}^N)$ and $\alpha_H(F) = \alpha_H(F,\mathbb{R}^N)$ for simplicity.

Another useful tool for the resolution of our problem is the Localization Operator that we define below.

DEFINITION 2.5. For a fixed $g \in \mathcal{D}(\mathbb{R}^N)$ we define the Localization Operator V_g: $\mathcal{D}'(\mathbb{R}^N) \longrightarrow \mathcal{D}'(\mathbb{R}^N)$ by

$$V_g(f) = V * [gP(D)f] \quad \text{if } f \in \mathcal{D}'(\mathbb{R}^N).$$

The main properties of the Localization Operator are incorporated in the following lemmas. For a complete study see [1, Section 5] or [13, Section 3].

LEMMA 2.6 ([13, SECTION 3]). *If $g \in \mathcal{D}(\mathbb{R}^N)$ and $f \in \mathcal{D}'(\mathbb{R}^N)$ then*

1. $V_g(f) = gf + \sum_{\substack{|\alpha|\neq 0 \\ |\alpha+\beta|=p}} a_{\alpha\beta}[D^\beta V * fD^\alpha g]$.

2. *$V_g(f)$ is continuous at every point of continuity of f and vanishes at infinity.*

3. *$P(D)V_g(f) = gP(D)f$. Hence, $V_g(f)$ is a solution of the operator $P(D)$ off the support of g and wherever f is, and $f - V_g(f)$ is a solution in the interior of the set on which g equals to* 1.

LEMMA 2.7. *Let $g \in \mathcal{D}(\mathbb{R}^N)$ satisfy* $\operatorname{supp} g \subset B(0,\delta)$ *and $\|D^\alpha g\| \leq R_{|\alpha|}\delta^{-|\alpha|}$ for any multi-index α. Then $\|V_g(f)\| \leq C\|f\|$ for any bounded measurable function f.*

3. **Bounded pointwise approximation.** Let U be an open subset of $\mathbb{R}^N$. We denote by $O^\infty(U)$ the subspace of $O(U)$ of bounded functions on U, and by $\mathcal{A}(U)$ the subspace of $O(U)$ of continuous functions on $\bar{U}$. Clearly, $O^\infty(U)$ is a Banach space with the supremum norm in U.

DEFINITION 3.1. We say that $\mathcal{A}(U)$ is bounded pointwise dense in $O^\infty(U)$ if each $f \in O^\infty(U)$ is the pointwise limit on U of a sequence $\{f_n\}$ in $\mathcal{A}(U)$ satisfying $\|f_n\|_U \leq M\|f\|_U$.

The characterization in terms of capacities of the open sets where this kind of approximation holds was established by Gamelin and Garnett in [6] for $P(D) = \bar{\partial}$ in the complex plane. The independence of the constant M with respect to the function f can be easily established by a category argument.

We begin with two lemmas. The first lemma is a higher dimensional version of a result of A. Davie [4], but for our proof we follow the construction given in [2]. Davie's lemma allowed Hadjiiski [7] to generalize the Vitushkin theorem to unbounded sets.

LEMMA 3.2. *Let Ω be a proper open subset of $\mathbb{R}^N$. Then, for any positive function $\rho(x)$ in $C_0(\Omega)$, there exists a sequence $\{B_j\}_{j=1}^\infty = \{B_j(x_j,\tau_j)\}_{j=1}^\infty$ of open balls whose closures lie in Ω such that:*

1. $\Omega = \bigcup_{j=1}^{\infty} B_j$.

2. No point of Ω *lies in more than* $Z = Z(N)$ *balls* B_j.

3. If $x \in B_j$ *then* $\tau_j < \rho(x)$.

4. There is a C^∞*-partition of unity* $\{\phi_j\}_{j=1}^{\infty}$ *subordinate to* $\{B_j\}_{j=1}^{\infty}$ *with* $\|D^\alpha \phi_j\| \leq R_{|\alpha|}\tau_j^{-|\alpha|}$ *for any multi-index* α.

5. For all $x \in \mathbb{R}^N$,

$$\sum_j \min\left(1, \frac{\tau_j^{N+1}}{|x - x_j|^{N+1}}\right) \leq C, \tag{3.1}$$

C being an absolute constant.

PROOF. Take any positive function $\rho \in C_0(\Omega)$, and consider the increasing sequence of compact subsets of Ω defined by

$$V_k = \{x \in \Omega : \rho(x) \geq 2^{-k}\}, \quad k = 1, 2, \ldots$$

that cover Ω. For each integer k define $\alpha_k = \min\bigl(1, \operatorname{dist}(V_k, \mathbb{R}^N \setminus \Omega)\bigr)$ and $\beta_k = \min\bigl(1, \operatorname{dist}(V_k, \mathbb{R}^N \setminus V_{k+1})\bigr)$.

Now, choose recursively a sequence $\{m_k\}_{k=1}^{\infty}$ of integers satisfying $m_{k-1} < m_k$ and $\beta_k > 2^{k-m_k+1}$. For $k = 1, 2, \ldots$ let $\{B_{k,q}\}_{q=1}^{\infty}$ be an enumeration of the balls of radius $\frac{3}{4}2^{-m_k}$ for which the coordinates of their centers are integral multiples of 2^{-m_k}.

Let k_0 be the smallest k for which V_{k_0} is nonempty. We construct a covering starting with the balls $B_{k_0,q}$ that intersect V_{k_0}. Recursively, for $k > k_0$ we select those balls $B_{k,q}$ which intersect V_k and are not contained in the union of the balls that have already been selected. The covering of Ω considered consists of the coecentric balls with a double radii; we renumber this sequence $\{2B_{k,q}\}$ as $\{B_j\}_{j=1}^{\infty}$.

Note that, by the choice of m_k, any ball of the k-generation meets only the balls of the $(k-1)$ and $(k+1)$-generation. Moreover, this latter property, along with the choice of centers and radii, guarantees that the maximum number of balls that contains any point does not exceed an absolute constant Z. This covering $\{B_j\}$ of Ω satisfies 1), 2) and 3).

It is easy to get a partition of unity satisfying 4). We choose $\phi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \phi \leq 1$ such that $\phi(x) \equiv 1$ in $|x| \leq \frac{\sqrt{2}}{2}$, $\phi(x) \equiv 0$ if $|x| > \frac{23}{32}$ and $\|D^\alpha \phi\| \leq R_{|\alpha|}$. Then, for $j \in \mathbb{N}$, we define $\phi_j(x) = \phi(\frac{3(x-x_j)}{4\tau_j})$, $\varphi_1(x) = \phi_1(x)$, and $\varphi_j(x) = \phi_j(x)\prod_{i=1}^{j-1}\bigl(1 - \phi_i(x)\bigr)$.

Since $\{\frac{1}{2}B_j\}$ covers Ω and each function ϕ_j equals one for $|x| \leq \sqrt{2} \cdot 2^{-m_k}$, we conclude that $\{\phi_j\}$ satisfies 4).

To get 5) we first prove that for any k

$$\sum_q \min\left(1, \frac{(2^{-m_k})^{N+1}}{|x - x_{k,q}|^{N+1}}\right) \leq C \min\left(1, \frac{2^{-m_k}}{\operatorname{dist}(x, \bigcup_q B_{k,q})}\right) \tag{3.2}$$

for any k and any point $x \in \mathbb{R}^N$, with $C = C(N)$ an absolute constant, and where the sum ranges over the indices q such that $2B_{k,q}$ is included in the covering $\{B_j\}_{j=1}^{\infty}$.

Indeed, fix $x \in \mathbb{R}^N$ and consider the annulus $\triangle\bigl(x, l\frac{3}{2}2^{-m_k}, (l+1)\frac{3}{2}2^{-m_k}\bigr)$. Then, the number of centers $x_{k,q}$ contained in this annulus is less than $N4^N Z l^{N-1}$ [1, Lemma 7.1].

Now, taking into account that it is possible to meet centers in each annulus only for l greater than

$$l_0 = \frac{1}{2}\frac{\operatorname{dist}(x, \bigcup_q B_{k,q})}{\frac{3}{2}2^{-m_k}},$$

we prove (3.2)

$$\begin{aligned}\sum_q \frac{(2^{-m_k})}{|x - x_{k,q}|^{N+1}} &\leq \left(\frac{2}{3}\right)^{N+1} N4^N Z \sum_{l=l_0}^{\infty} \frac{1}{l^2} \\ &\leq C\sum_{l=l_0}^{\infty}\frac{1}{2^l} \leq C\frac{2^{-m_k}}{\operatorname{dist}(x, \bigcup_q B_{k,q})}.\end{aligned}$$

We can now prove 5). Fix $x \notin \Omega$. Then, $\operatorname{dist}(x, \bigcup_q B_{k,q}) > \alpha_k - 3\cdot 2^{m_k} > \beta_k - 3\cdot 2^{-m_k} > 2^{k-1-m_k}$. By (3.2), this yields

$$\sum_j \min\left(1, \frac{r_j^{N+1}}{|x - x_j|^{N+1}}\right) \leq C\sum_k \min\left(1, \frac{2^{-m_k}}{\operatorname{dist}(x, \bigcup_q B_{k,q})}\right) \leq C\sum_k \frac{2^{-m_k}}{2^{k-1-m_k}} \leq C.$$

On the other hand, if $x \in \Omega$, we can assume that $x \in V_i \setminus V_{i-1}$. Again, (3.2) allows us to divide (3.1) into three sums:

$$\sum_j \min\left(1, \frac{r_j^{N+1}}{|x - x_j|^{N+1}}\right) \leq C\left(\sum_{k=k_0}^{i-2} + \sum_{k=i-1}^{i+2} + \sum_{k=i+3}^{\infty}\right) \min\left(1, \frac{2^{-m_k}}{\operatorname{dist}(x, \bigcup_q B_{k,q})}\right).$$

Let us estimate each part. For the first one, take into account that $\operatorname{dist}(x, \bigcup_q B_{k,q}) > \beta_k - 3\cdot 2^{-m_k} > 2^{k-1-m_k}$, since x is not in V_{i-2}. Hence,

$$\sum_{k=k_0}^{i-2} \min\left(1, \frac{2^{-m_k}}{\operatorname{dist}(x, \bigcup_q B_{k,q})}\right) \leq \sum_{k=k_0}^{i-2} \frac{2^{-m_k}}{\beta_k} \leq \sum_{k=k_0}^{i-2} \frac{1}{2^{k-1}}.$$

The second one is trivially bounded by 4. Thus, it only remains to estimate the third sum. Since $\operatorname{dist}(x, \bigcup_q B_{k,q}) > \beta_{k-3}$ we have

$$\sum_{k=i+3}^{\infty} \min\left(1, \frac{2^{-m_k}}{\operatorname{dist}(x, \bigcup_q B_{k,q})}\right) \leq \sum_{k=i+3}^{\infty} \frac{2^{-m_k}}{\beta_{k-3}} \leq \sum_{k=i+3}^{\infty} \frac{2^{-m_{k-3}}}{\beta_{k-3}} \leq \sum_{k=i+3}^{\infty} \frac{1}{2^{k-3}}.$$

In the last two inequalities we have made use of the recursive properties of the sequence $\{m_k\}_{k=1}^{\infty}$.

With these estimates, we finish the proof of the lemma by taking C to be the maximum of the constants calculated above.

The next lemma is an extension to solutions of more general differential equations of a localization result for analytic functions in the plane due to Gamelin and Garnett [6].

LEMMA 3.3. *Let U be an open subset of $\mathbb{R}^N$ with a compact boundary and $f \in O^\infty(U)$ vanish at infinity. Let $\{U_j\}_{j=1,\ldots,m}$ be a finite open covering of ∂U such that in each $U \cap U_j$, f is a pointwise limit of a sequence $\{h_{j,n}\}_{n=1}^{\infty}$ in $\mathcal{A}(U \cap U_j)$ satisfying $\|h_{j,n}\|_{U\cap U_j} \leq M\|f\|_{U\cap U_j}$ with M independent of f.*

Then f can be approximated pointwise on U by a sequence $\{f_n\} \subset \mathcal{A}(U)$ satisfying

$$\|f_n\|_U \leq \lambda \|f\|_U,$$

where λ depends only on the covering $\{U_j\}$.

PROOF. Take $f \equiv 0$ off U. Let $\{\varphi_j\}_{j=1}^m \subset \mathcal{D}(\mathbb{R}^N)$ satisfying $\operatorname{supp} \varphi_j \subset U_j$, $0 \leq \varphi_j \leq 1$ and $\sum \varphi_j \equiv 1$ near ∂U. Then, $f = \sum_{j=1}^m V_{\varphi_j}(f)$.

For every $j \in \{1, \ldots, m\}$, we choose a sequence $\{h_{j,n}\}_{n=1}^\infty$ in $A(U \cap U_j)$ such that $\|h_{j,n}\|_{U \cap U_j} \leq M \|f\|_{U \cap U_j}$ and $h_{j,n}(x) \longrightarrow f(x)$ for any $x \in U \cap U_j$. We can assume that $h_{j,n} \in C(\mathbb{R}^N)$ and $h_{j,n} \longrightarrow 0$ off $\bar{U}$ [10, Chapter VI]. Hence, there exists a bounded function h_j such that $\{h_{j,n}\}$ converges to h_j in the weak-star topology of $L^\infty(\mathbb{R}^N)$.

Consider $g_{j,n} = V_{\varphi_j}(h_{j,n})$, and define $G_n = \sum_j g_{j,n}$. Then $G_n \in \mathcal{A}(U)$ and $\|G_n\| \leq \lambda_1 M \|f\|$, where λ_1 depends only on $\{\varphi_j\}_{j=1,\ldots,m}$.

Since $\operatorname{supp} \varphi_j \subset U_j$, the pointwise convergence of $\{h_{j,n}\}$ to f on $U_j \cap U$ implies the same convergence of $\varphi_j h_{j,n}$ to $\varphi_j f$ on U. Hence, for every x in U,

$$\begin{aligned}
\lim_{n\to\infty} g_{j,n}(x) &= \lim_{n\to\infty} \Big(\varphi_j(x) h_{j,n}(x) + \sum_{\substack{|\alpha| \neq 0 \\ |\alpha+\beta| = p}} a_{\alpha\beta} [D^\beta V * h_{j,n} D^\alpha \varphi_j](x) \Big) \\
&= \varphi_j(x) f(x) + \sum_{\substack{|\alpha| \neq 0 \\ |\alpha+\beta| = p}} a_{\alpha\beta} [D^\beta V * f D^\alpha \varphi_j](x) \\
&\quad + \sum_{\substack{|\alpha| \neq 0 \\ |\alpha+\beta| = p}} a_{\alpha\beta} \int_{\partial U} D^\beta V(x-y) h_j(y) D^\alpha \varphi_j(y)\, dy \\
&= V_{\varphi_j}(f)(x) + \sum_{\substack{|\alpha| \neq 0 \\ |\alpha+\beta| = p}} a_{\alpha\beta} H_{j,\alpha,\beta}(x),
\end{aligned}$$

where $H_{j,\alpha,\beta} = [D^\beta V * \chi_{\partial U} h_j D^\alpha \varphi_j] \in \mathcal{A}(U)$ and $\|H_{j,\alpha,\beta}\| \leq \lambda_2 M \|f\|$, with λ_2 depending only on $\{\varphi_j\}_{j=1,\ldots,m}$. Denote by $H_j = \sum_{\substack{|\alpha| \neq 0 \\ |\alpha+\beta| = p}} a_{\alpha\beta} H_{j,\alpha,\beta}$.

The sequence $f_n = G_n - \sum_{j=1}^m H_j$ satisfies $f_n \in \mathcal{A}(U)$, $\|f_n\| \leq (\lambda_1 + \lambda_2) M \|f\|$ and $f_n(x) \longrightarrow \sum_{j=1}^m V_{\varphi_j}(f)(x) = f(x)$ as $n \longrightarrow \infty$ for every $x \in U$. This completes the proof.

Now we establish the main result of this section. As said before, Tarkhanov's scheme of approximation in C^m—norm solutions of elliptic systems is the starting point for our result, as it is Vitushkin's scheme for Gamelin and Garnett's characterization of bounded pointwise approximation given in [6]. The characterization of those proper open subsets U of $\mathbb{R}^N$ where $\mathcal{A}(U)$ is bounded pointwise dense in $\mathcal{O}^\infty(U)$ is given in terms of capacities γ and α.

THEOREM 3.4. *Let U be a proper open subset of $\mathbb{R}^N$. Then, the following assertions are equivalent:*

i) $\mathcal{A}(U)$ is bounded pointwise dense in $\mathcal{O}^\infty(U)$.

ii) For each $r > 1$ there exists $\eta > 0$ such that

$$\gamma_H\big(B(x,\delta)\setminus U\big) \le \eta\alpha_H\big(B(x,r\delta)\setminus U\big) \tag{3.3}$$

for any $H \in \mathcal{H}_k$ $(k \in \mathbb{N})$, all $x \in \mathbb{R}^N$ and all $\delta > 0$.

iii) There exist constants $\delta_0 > 0$, $r > 1$ and $\eta > 0$ such that if $H \in \bigcup_{k=0}^{p} \mathcal{H}_k$ then, for every point $x \in \mathbb{R}^N$ and for all $\delta \le \delta_0$,

$$\gamma_H\big(B(x,\delta)\setminus U\big) \le \eta\alpha_H\big(B(x,r\delta)\setminus U\big). \tag{3.4}$$

PROOF. First we prove that iii) implies i). Fix $\delta \le \delta_0$ and $f \in O^\infty(U)$. For this δ we consider a δ-neighborhood Ω_δ of ∂U, Lemma 3.2 then yields a sequence of balls $\{B_j^\delta(x_j,\tau_j)\}_{j=1}^\infty$ and a partition of unity $\{\phi_j^\delta\}_{j=1}^\infty$ for Ω_δ satisfying 1)–5). Without loss of generality, we can assume $\sigma_j < \frac{\delta}{r^p}$ for any j.

Set $f \equiv 0$ off U and write $f_j = V_{\phi_j}(f)$. By property 3) of the chosen partition of unity and Lemma 2.7 we can write $f = \sum f_j + G$, where $\|f_j\| \le M\|f\|$ for each j, with $M = M(P,N)$ and $G \in O(\mathbb{R}^N)$.

To prove the approximation statement for f it is suffices to find functions $\Phi_j \in C(\mathbb{R}^N)$ satisfying

a) $\operatorname{supp} P(D)\Phi_j \subset B(x_j, r^p\sigma_j)\setminus U$;

b) $\|\Phi_j\| \le M'\|f\|$ $\big(M' = M'(P,N,\eta)\big)$;

c) f_j and Φ_j have the same coefficients in their Laurent expansions at infinity with indices $|\alpha| \le p$.

Assume for the moment the existence of these functions and define $\Phi_\delta = \sum_j \Phi_j + G$. Then we have a sequence $\{\Phi_\delta\}_{\delta>0}$ in $\mathcal{A}(U)$, that converges pointwise to f in U as $\delta \longrightarrow 0$, and is uniformly bounded by $M''\|f\|$, with $M'' = M''(P,N,\eta,r)$.

Indeed, by c) the Laurent expansion of $f_j - \Phi_j$ has the expression

$$f_j(x) - \Phi_j(x) = \sum_{|\alpha|\ge p+1} D^\alpha V(x-x_j)c_\alpha$$

for any x outside some ball centered at x_j. The uniform boundedness of f_j and Φ_j allows us to estimate the coefficients c_α in the following way. Take $\varphi \in C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\varphi \subset B\big(x_j,(r^p+1)\tau_j\big)$, $\varphi \equiv 1$ in $B(x_j, r^p\sigma_j)$ and $\|D^\alpha\varphi\| \le R_{|\alpha|}\big((r^p+1)\tau_j\big)^{-|\alpha|}$. Then,

$$\begin{aligned} |c_\alpha| &= \left|(-1)^p \int_{\mathbb{R}^N} P(D)[\varphi(y)\mathcal{M}_\alpha(y-x_j)]\big(f_j(y)-\Phi_j(y)\big)\,dy\right| \\ &\le \Big(\sum|P_\beta|\Big)\|f_j-\Phi_j\|\sup_{|\beta|=p}\int\Big|D^\beta[\varphi(y)\mathcal{M}_\alpha(y-x_j)]\Big|\,dy \\ &\le M_1\|f\|\Big(\sup_{\substack{|\beta|=p\\|\gamma|\le p}} C_{\beta-\gamma}\Big)\big((r^p+1)\tau_j\big)^{-p}\frac{N^{|\alpha|}}{(|\alpha|-p)!}\frac{\big((r^p+1)\tau_j\big)^{N+|\alpha|}}{N+|\alpha|-p} \\ &\le M_1\|f\|\frac{(r^p+1)^N}{N+1}\tau_j^{N-p}\frac{\big(N(r^p+1)\tau_j\big)^{|\alpha|}}{(|\alpha|-p)!}, \end{aligned} \tag{3.5}$$

where we have mainly made use of the Leibniz formula, the properties of φ and the estimates of the derivatives of the polynomials $\mathcal{M}_\alpha$ [14, Lemma 2.2].

Estimate (3.5) makes it possible to estimate the evaluation of $f_j - \Phi_j$ at any point, as follows:

$$
\begin{aligned}
|f_j(x) - \Phi_j(x)| &\leq \sum_{|\alpha|\geq p+1} |D^\alpha V(x-x_j)|\,|c_\alpha| \\
&\leq M_1 M' \|f\| \frac{(r^p+1)^N}{N+1} \tau_j^{N-p} \sum_{|\alpha|\geq p+1} \frac{\big(N(r^p+1)\tau_j\big)^{|\alpha|}}{|x-x_j|^{N-p+|\alpha|}} \frac{c_1 a^{|\alpha|} |\alpha|!}{(|\alpha|-p)!} \\
&\leq M_2 \|f\| \left(\sum_{l=0}^{\infty} \left(\frac{aN^2(r^p+1)\tau_j}{|x-x_j|} \right)^l \frac{(p+1+l)!}{(l+1)!} \right) \frac{\tau_j^{N-p}\big(aN^2(r^p+1)\tau_j\big)^{p+1}}{|x-x_j|^{N+1}}.
\end{aligned}
$$

So, if $|x - x_j| > 2aN^2(r^p+1)\tau_j$, we have

$$
|f_j(x) - \Phi_j(x)| \leq M_3 \|f\| \frac{\tau_j^{N+1}}{|x-x_j|^{N+1}}. \tag{3.6}
$$

On the other hand, for $|x - x_j| \leq 2aN^2(r^p+1)\tau_j$ we can write

$$
|f_j(x) - \Phi_j(x)| \leq \|f_j\| + \|\Phi_j\| \leq 2M\|f\| \frac{\big(2aN^2(r^p+1)\tau_j\big)^{N+1}}{|x-x_j|^{N+1}}. \tag{3.7}
$$

Formulas (3.6) and (3.7) give

$$
|f_j(x) - \Phi_j(x)| \leq C\|f\| \min\left(1, \frac{\sigma_j^{N+1}}{|x-x_j|^{N+1}}\right) \tag{3.8}
$$

for any $x \in \mathbb{R}^N$,where $C = C(N, P, r, \eta)$.

The estimate (3.8) and the property 5) of the covering $\{B_j\}$ allow us to bound uniformly the function Φ_δ as a consequence of the following estimate

$$
|f(x) - \Phi_\delta(x)| \leq \sum_j |f_j(x) - \Phi_j(x)| \leq C\|f\| \sum_j \min\left(1, \frac{\sigma_j^{N+1}}{|x-x_j|^{N+1}}\right) \leq C\|f\|,
$$

valid for any point $x \in \mathbb{R}^n$, with $C = C(P, N, r, \eta)$.

Now we prove the pointwise convergence to f on U. For a fixed $x \in U$, denote by d the distance of x to ∂U. Then, recalling the construction of the covering $\{B_j\}$ from the proof of Lemma 3.2 above, we have

$$
\begin{aligned}
|f(x) - \Phi_\delta(x)| &\leq \sum_{k=1}^{\infty} \frac{2^{-m_k^\delta}}{\operatorname{dist}(x, \bigcup_p B_{k,p})} \\
&\leq \frac{C\|f\|}{d-\delta} \sum_{k=1}^{\infty} 2^{-m_k^\delta} \leq \frac{C\|f\|}{d-\delta} \sum_{k=m_1^\delta}^{\infty} 2^{-k}.
\end{aligned}
$$

Since $m_1^\delta \longrightarrow \infty$ when $\delta \longrightarrow 0$, we obtain $\Phi_\delta(x) \longrightarrow f(x)$ for $\delta \longrightarrow 0$.

It remains to prove the existence of functions Φ_j satisfying a), b) and c). If we consider the Laurent expansion of each f_j at x_j, $f_j(x) = \sum_\alpha D^\alpha V(x-x_j)b_\alpha$, then the choice of the function φ_j, along with Lemmas 2.6 and 2.7, yields $f_j \in \mathcal{B}_{b_0}\big(B(x_j,\sigma_j)\setminus U\big)$. Now Lemma 2.4 and the estimate (3.4) imply

$$\{b_0, b_0\} \le M\|f\|\gamma_{b_0}\big(B(x_j,\sigma_j)\setminus U\big) \le M\eta\|f\|\alpha_{b_0}\big(B(x_j, r\sigma_j)\setminus U\big). \tag{3.9}$$

From (3.9), the definition of α_{b_0} guarantees the existence of a function $\Phi_j^0 \in C(\mathbb{R}^N)$ with $\operatorname{supp} P(D)\Phi_j^0 \subset B(x_j, r\sigma_j)\setminus U$, bounded by $2M\eta\|f\|$ and with the expansion

$$\Phi_j^0(x) = b_0 V(x-x_j) + o(|x|^{p-N}).$$

Take $F_j^0 = f_j - \Phi_j^0$. Then, $\operatorname{supp} P(D)F_j^0 \subset B(x_j, r\sigma_j)\setminus U$, $\|F_j^0\| \le (M+2M\eta)\|f\| = M_0\|f\|$ and

$$F_j^0(x) = \sum_{|\alpha|\ge 1} b_\alpha^0 D^\alpha V(x-x_j).$$

Since $H_1(\xi) = \sum_{|\alpha|=1} b_\alpha^0 \xi^\alpha \in \mathcal{H}_1(P)$, by Proposition 2.1 and Theorem 2.2, it follows that F_j^0 belongs to $\mathcal{B}_{H_1}\big(B(x_j, r\sigma_j)\setminus U\big)$. Therefore, again Lemma 2.4 and (3.4) give

$$\{H_1, H_1\} \le M_0\|f\|\gamma_{H_1}\big(B(x_j, r\sigma_j)\setminus U\big) \le M_0\eta\|f\|\alpha_{H_1}\big(B(x_j, r^2\sigma_j)\setminus U\big). \tag{3.10}$$

Equation (3.10) provides a function $\Phi_j^1 \in C(\mathbb{R}^N)$ with $\operatorname{supp} P(D)\Phi_j^1 \subset B(x_j, r^2\sigma_j)\setminus U$, $\|\Phi_j^1\| \le 2M_0\eta\|f\| = M_1\|f\|$ and a Laurent expansion of the form

$$\Phi_j^1(x) = \sum_{|\alpha|=1} b_\alpha^0 D^\alpha V(x-x_j) + o(|x|^{p-N-1}).$$

Iterating this procedure, we get at the p-th step a function $\Phi_j^p \in C(\mathbb{R}^N)$ with $\operatorname{supp} P(D)\Phi_j^p \subset B(x_j, r^p\sigma_j)\setminus U$, $\|\Phi_j^p\| \le 2M_{p-1}\eta\|f\| = M_p\|f\|$, and

$$\Phi_j^p(x) = \sum_{|\alpha|=p} b_\alpha^p D^\alpha V(x-x_j) + o(|x|^{-N}).$$

The function $\Phi_j = \sum_{i=0}^p \Phi_j^i$ satisfies a), b) and c) by construction. This completes the proof of this implication.

That ii) implies iii) is trivial, so to conclude the proof of the theorem we shall show that i) implies ii).

Fix $H \in \mathcal{H}_k$, $\delta > 0$, $r > 1$ and a point $x_0 \in \mathbb{R}^N$. Take $f \in \mathcal{B}_H\big(B(x_0,\delta)\setminus U\big)$ satisfying $\|f\| \le 1$ and

$$2|\langle P(D)f, \bar{H}\rangle| \ge \gamma_H\big(B(x_0,\delta)\setminus U\big). \tag{3.11}$$

Consider $1 < r' < r$, and define $W = U \cup \{|x-x_0| > r'\delta\}$. Then $W_1 = \{|x-x_0| > \delta\}$ and $W_2 = \{|x-x_0| < r'\delta\}$ cover ∂W. The functions $\{f_n^1(x)\} = \big\{f\big(x+\frac{1}{n}(x-x_o)\big)\big\}$ belong to $\mathcal{A}(W_1)$, are uniformly bounded on W_1, and converge to f pointwise on this set. On the other hand, since $W_2 \cap W \subset U$ and $f \in \mathcal{O}^\infty(U)$, our hypothesis assure the existence of a sequence $\{f_n^2\} \subset \mathcal{A}(W_2 \cap W)$ converging pointwise to f on $W_2 \cap W$ and uniformly bounded on $W_2 \cap W$, by M, say. Applying Lemma 3.3 we get a sequence $\{f_n\}$ in $\mathcal{A}(W)$ that converges pointwise to f on W and is uniformly bounded on W, say by M', with M' independent of f. Since $r' < r$, it follows that $\operatorname{supp} P(D)f_n \subset B(x_0, r\delta) \setminus U$ and, without loss of generality, we can assume that $\{f_n\} \subset C(\mathbb{R}^N)$.

To prove this implication we need for each $n \in \mathbb{N}$ a function $\Phi_n \in C(\mathbb{R}^N)$ satisfying:

1. $\operatorname{supp} P(D)\Phi_n \subset B(x_0, r\delta) \setminus U$.
2. $\|\Phi_n\| \le M_1$ with M_1 independent of f and f_n.
3. $(f_n - f)$ and Φ_n have the same coefficients in their Laurent expansion at infinity with indices $|\alpha| \le k$.

Assume again for the moment the existence of these functions and continue with the proof of (3.3). The cited properties of Φ_n imply the following ones:

a) $\operatorname{supp} P(D)(f_n - \Phi_n) \subset B(x_0, r\delta) \setminus U$.
b) $\|f_n - \Phi_n\| \le M' + M_1 = M_2$.
c) $f_n(x) - \Phi_n(x) = \lambda H(D)V(x - x_0) + o(|x|^{p-N-k})$ as $x \longrightarrow \infty$.
d) $\langle P(D)f, \bar{H}\rangle = \langle P(D)(f_n - \Phi_n), \bar{H}\rangle$.

The latter statements allows us to estimate $\alpha_H\big(B(x_0, r\delta) \setminus U\big)$ as follows:

$$\alpha_H\big(B(x_0, r\delta) \setminus U\big) \ge \frac{|\langle P(D)(f_n - \Phi_n), \bar{H}\rangle|}{M_2} \ge \frac{1}{2M_2}\gamma_H\big(B(x_0, r\delta) \setminus U\big). \tag{3.12}$$

That gives (3.3), with $\eta = 2M_2$.

We now prove the claim concerning the existence of the functions Φ_n. Consider the Laurent expansion of $f_n - f$ at x_0,

$$f_n(x) - f(x) = \sum_\alpha D^\alpha V(x - x_0) b_\alpha,$$

which converges absolutely and uniformly on closed subsets of the complement of a neighborhood of x_0. Since $\|f_n - f\| \le 1 + M'$, (3.5) permits us to estimate b_α by

$$|b_\alpha| = \left|\int \mathcal{M}_\alpha(y - x_0) P(D)(f_n - f)(y)\, dy\right| \le K_{|\alpha|}(1 + M'),$$

with $K_{|\alpha|} = K_{|\alpha|}(P, N, \delta, r, \alpha)$.

Consider now a function $\chi \in \mathcal{D}\big(B(x_0, r\delta) \setminus U\big)$ with $\int \chi = 1$, and define

$$\Phi_n^0(x) = b_0 \int V(x - y)\chi(y)\, dy.$$

Then $\Phi_n^0 \in C^\infty(\mathbb{R}^N)$, $\operatorname{supp} P(D)\Phi_n^0(x) \subset B(x_0, r\delta) \setminus U$, $\|\Phi_n^0\| \le M_0(1 + M')$ with $M_0 = M_0(P, N, \delta, r, U)$, and

$$\Phi_n^0(x) = b_0 V(x - x_o) + o(|x|^{p-N}) \quad \text{as } x \longrightarrow \infty$$

Take $F_n^0 = (f_n - f) - \Phi_n^0$. Then $\|F_n^0\| \leq (1+M_0)(1+M')$, $\operatorname{supp} P(D)F_n^0 \subset B(x_0, r\delta) \setminus U$, and

$$F_n^0(x) = \sum_{|\alpha|\geq 1} b_\alpha^0 D^\alpha V(x - x_0).$$

Using (3.5) again we have

$$|b_\alpha^0| \leq K_{|\alpha|}(1+M_0)(1+M').$$

Similarly, we define

$$\Phi_n^1(x) = \sum_{|\alpha|=1} b_\alpha^0 \int V(x-y)D^\alpha \chi(y)\,dy$$

to produce a function $\Phi_n^1 \in C^\infty(\mathbb{R}^N)$ satisfying $\|\Phi_n\| \leq M_1(1+M')$ with $M_1 = M_1(P, N, \delta, r, U)$, $\operatorname{supp} P(D)\Phi_n^1(x) \subset B(x_0, r\delta) \setminus U$, and

$$\Phi_n^1(x) = \sum_{|\alpha|=1} b_\alpha D^\alpha V(x - x_0) + o(|x|^{p-N-1}) \quad \text{as } x \longrightarrow \infty.$$

Now set $F_n^1 = F_n^0 - \Phi_n^1$, then $\|F_n^1\| \leq (1+M_1)(1+M')$, $\operatorname{supp} P(D)F_n^1 \subset B(x_0, r\delta) \setminus U$ and

$$F_n^1(x) = \sum_{|\alpha|\geq 2} b_\alpha^1 D^\alpha V(x - x_0).$$

Continuing this process by induction, we construct at the k-th step a function $\Phi_n^k \in C^\infty(\mathbb{R}^N)$ given by

$$\Phi_n^k(x) = \sum_{|\alpha|=k} b_\alpha^{k-1} \int V(x-y)D^\alpha \chi(y)\,dy$$

satisfying $\operatorname{supp} P(D)\Phi_n^k(x) \subset B(x_0, r\delta) \setminus U$, $\|\Phi_n\| \leq M_k(1+M')$ with $M_k = M_k(P, N, \delta, r, U)$, and

$$\Phi_n^k(x) = \sum_{|\alpha|=k} b_\alpha D^\alpha V(x - x_0) + o(|x|^{p-N-k}) \quad \text{as } x \longrightarrow \infty.$$

To finish the proof it suffices to take $\Phi_n = \sum_{t=0}^k \Phi_n^t \in C^\infty(\mathbb{R}^N)$ that clearly satisfies 1), 2) and 3). The proof of the theorem is complete.

4. Approximation by functions continuous up the boundary. The characterization of bounded pointwise density established in the previous section allows us to prove Stray's theorem for solutions of the differential operators considered, replacing the compact subsets of the plane with proper closed subsets of $\mathbb{R}^N$.

For any set X and any $E \subset \partial X$, we denote by $O_E^\infty(X)$ the functions of $O^\infty(X^o)$ that are continuous in E.

THEOREM 4.1. *Let X be a proper closed subset of $\mathbb{R}^N$ where $\mathcal{A}(X^o)$ is dense with respect to bounded pointwise convergence in $O^\infty(X^o)$. Then there exists a constant k such that if $E \subset \partial X$ is relatively open to ∂X, $h \in O^\infty(X^o)$ and $F \subset X^o$ is relatively closed in $X^o \cup E$, then for any $\varepsilon > 0$ we can find a function $f \in O_E^\infty(X)$ such that $\|h - f\|_F < \varepsilon$ and $\|f\| \leq k\|h\|$.*

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PROOF. By Theorem 3.4, the hypothesis provides constants $\eta > 0$ and $r > 1$ such that

$$\gamma_H\big(B(x,\delta)\setminus X^o\big) \le \eta\alpha_H\big(B(x,r\delta)\setminus X^o\big) \tag{4.13}$$

for any $x \in \mathbb{R}^N$, any $\delta > 0$ and any $H \in \bigcup_{l\in\mathbb{N}} \mathcal{H}_l(P)$.

Take sequences $\{V_n\}$ and $\{K_n\}$ of open and compact sets respectively satisfying:

a) $K_n \subset V_n$ for every $n \in \mathbb{N}$;

b) $V_n \cap V_m = \emptyset$ if $|n-m| > 1$;

c) $E = \bigcup_{n=1}^{\infty} K_n$;

d) $V_n \cap F = \emptyset$ for every $n \in \mathbb{N}$;

e) For any compact K in $\mathbb{R}^N \setminus (\partial X \setminus E)$, then $K \cap \bar{V}_n \ne \emptyset$ only for a finite number of open subsets V_n.

Now, for each integer n we choose $\delta_n > 0$ such that $L = d(K_n, \mathbb{R}^N \setminus V_n)/\delta_n$ is sufficiently large. Later on we will specify the precise size of L.

For this δ_n, following a construction due to Bagby [1, p. 779], we can obtain a sequence of points $\{x_{n,k}\}_{k=1}^{\infty}$ and a sequence of functions $\{\varphi_{n,k}\}_{k=1}^{\infty}$ satisfying:

i) $\varphi_{n,k} \in C_0^{\infty}\big(B(x_{n,k},\delta_n)\big)$;

ii) $\sum_{k=1}^{\infty}\varphi_{n,k} \equiv 1$ in $\mathbb{R}^N$;

iii) $\|D^\alpha\varphi_{n,k}\| \le R_{|\alpha|}\delta_n^{-|\alpha|}$;

iv) Each point of $\mathbb{R}^N$ is contained in no more than Z balls $B_{n,k} = B(x_{n,k},\delta_n)$ with Z independent of δ_n;

v) If $I_n = \{k : \bar{B}_{n,k} \cap K_n \ne \emptyset\}$, then $I_n \cap I_{n+2} = \emptyset$ for every n.

Set $h \equiv 0$ off X. Take now $G_{n,k} = V_{\varphi_{n,k}}(h)$. From (4.13), for every k there exists a function $\Phi_{n,k} \in C(\mathbb{R}^N)$ such that:

1) $\operatorname{supp} P(D)\Phi_{n,k} \subset B(x_{n,k}, r^p\delta_n)\setminus X^o$;

2) $\|\Phi_{n,k}\| \le M\|h\|$;

3) $G_{n,k}(x)$ and $\Phi_{n,k}(x)$ have the same coefficients in their Laurent expansions at infinity with indices $|\alpha| \le p$.

For each $n \in \mathbb{N}$, define the function $h_n = \sum_{k\in I_n}(G_{n,k} - \Phi_{n,k})$. From 3) we have that (3.8) holds for any $G_{n,k} - \Phi_{n,k}(k \in I_n)$. Hence, by v), for any point $x \in \mathbb{R}^N$ and any $l \ge 1$ the number of points $x_{n,k}$ contained in the annulus $\Delta\big(x, l\delta_n, (l+1)\delta_n\big)$ is no greater than $N4^NZl^{N-1}$ for [1, Lemma 7.1], and the value of $h_n(x)$ is bounded by

$$|h_n(x)| \le C\|f\|\Big(\sum_{l=1}^{\infty}\frac{\delta_n^{N+1}}{(l\delta_n)^{N+1}}N4^{N+1}Zl^{N-1} + m_0\Big) \le C\|f\|\Big(m_0 + \sum_{l=1}^{\infty}\frac{1}{l^2}\Big), \tag{4.14}$$

where m_0 is the number of centers contained in the ball of center x and radius δ_n. Since m_0 is no greater than $N4^NZ$, from (4.14) we conclude the uniform boundedness of the sequence $\{h_n\}$ by $M'\|f\|$, with $M' = M'(P,N)$.

Fix $0 < \varepsilon < 1$ and choose δ_n such that $B(y, \frac{L\delta_n}{2}) \cap B_{n,k} = \emptyset$ for any $k \in I_n$ and any $y \in \mathbb{R}^N \setminus V_n$. From the choice of δ_n and therefore the value of L, no point $x_{n,k}$ lies in the annulus $\Delta\big(x, \frac{L}{2}\delta_n, (\frac{L}{2}+1)\delta_n\big)$, and by (4.14) we can write

$$|h_n(y)| \le C\|h\|\Big(\sum_{l=L/2}^{\infty}\frac{1}{l^2}\Big) \le \frac{\varepsilon}{2^{n+1}}\|h\|. \tag{4.15}$$

The last inequality holds with an appropriate choice of δ_n.

Define $g = h - \sum_{n=1}^{\infty} h_{2n-1}$. Then $\|g\| \leq M_2\|h\|$, g is in $O^\infty(X^o)$ and is continuous in K_{2n-1} for every $n \in \mathbb{N}$.

We now repeat the process for g instead of h, considering now the pairs (K_{2n}, V_{2n}) to obtain a function $f = g - \sum_{n=1}^{\infty} g_{2n} \in O^\infty(X^o)$, continuous in E that approximate h on F in the following terms:

$$\begin{aligned}\|h-f\|_F \leq \|h-g\|_F + \|g-f\|_F &\leq \Big\|\sum h_{2n-1}\Big\|_F + \Big\|\sum g_{2n}\Big\|_F \\ &\leq \sum \|h_{2n-1}\|_{\mathbb{R}^N\setminus V_{2n-1}} + \sum \|g_{2n}\|_{\mathbb{R}^N\setminus V_{2n}} \leq \Big(\sum \frac{1}{2^n}\Big)\varepsilon\|h\|.\end{aligned}$$

The last inequality follows from (4.15). This concludes the proof of the theorem.

REMARK. In case when $P(D)$ is the laplacian in $\mathbb{R}^N$ with $N > 2$ and X is the closed unit ball $\bar{B}$, the space $O^\infty(X)$ is the well known harmonic Hardy space $h^\infty(B)$, and it is easy to prove the bounded pointwise density of $\mathcal{A}(B)$ without making use of the capacity estimates [11, Chapter IV, §2]. Thus, Theorem 4.1 holds.

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