J. Aust. Math. Soc. **80** (2006), 229–262

SOME NEW BESOV AND TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH PARA-ACCRETIVE FUNCTIONS ON SPACES OF HOMOGENEOUS TYPE

DONGGUO DENG and DACHUN YANG✉

(Received 8 April 2004; revised 27 December 2004)

Communicated by T. Dooley

Abstract

Let $(X, \rho, \mu)_{d,\theta}$ be a space of homogeneous type with $d > 0$ and $\theta \in (0, 1]$, b be a para-accretive function, $\epsilon \in (0, \theta]$, $|s| < \epsilon$, and $a_0 \in (0, 1)$ be some constant depending on d, ϵ and s. The authors introduce the Besov space $b\dot{B}^s_{pq}(X)$ with $a_0 < p \leq \infty$ and $0 < q \leq \infty$, and the Triebel-Lizorkin space $b\dot{F}^s_{pq}(X)$ with $a_0 < p < \infty$ and $a_0 < q \leq \infty$ by first establishing a Plancherel-Pôlya-type inequality. Moreover, the authors establish the frame and the Littlewood-Paley function characterizations of these spaces. Furthermore, the authors introduce the new Besov space $b^{-1}\dot{B}^s_{pq}(X)$ and the Triebel-Lizorkin space $b^{-1}\dot{F}^s_{pq}(X)$. The relations among these spaces and the known Hardy-type spaces are presented. As applications, the authors also establish some real interpolation theorems, embedding theorems, Tb theorems, and the lifting property by introducing some new Riesz operators of these spaces.

2000 *Mathematics subject classification*: primary primary 42B35; secondary 46E35, 42B25, 43A99, 47B06, 42B20, 47A30, 47B38.

Keywords and phrases: space of homogeneous type, para-accretive function, Plancherel-Pôlya inequality, Besov space, Triebel-Lizorkin space, Calderón reproducing formula, interpolation, embedding theorem, Tb theorem, Riesz potential, lifting property.

1. Introduction

It is well known that the remarkable $T1$ theorem given by David and Journé provides a general criterion for the $L^2(\mathbb{R}^n)$-boundedness of generalized Calderón-Zygmund singular integral operators; see [5, 35]. The $T1$ theorem, however, cannot be directly

The first author was supported by NNSF (No. 10171111) of China and the second (corresponding) author was supported by NNSF (No. 10271015) and RFDP (No. 20020027004) of China. The authors also want to thank the referee for their several helpful comments which made this article more readable.

applied to the Cauchy integral on Lipschitz curves. Meyer in [30] (see also [31]) observed that if the function 1 in the $T1$ theorem is allowed to be replaced by a bounded complex-valued function b satisfying $0 < \delta \le \operatorname{Re} b(x)$ almost everywhere, then this result would imply the $L^2(\mathbb{R}^n)$ boundedness of the Cauchy integral on all Lipschitz curves. Replacing the function 1 by an accretive function, McIntosh and Meyer in [30] proved the Tb theorem, where b is an accretive function. David, Journé, and Semmes in [6] introduced a more general class of $L^\infty(\mathbb{R}^n)$ functions b, namely, the so-called para-accretive functions. They proved that the function 1 in the $T1$ theorem can be replaced by para-accretive functions, which is why it is now called the Tb theorem. Moreover, they showed that the para-accretivity is also necessary in the sense that if the Tb theorem holds for a bounded function b, then b is para-accretive. Moreover, Meyer in [31] observed that if $b(x)$ is a bounded function and $1 \le \operatorname{Re} b(x)$, one can then define the modified Hardy space $H^1_b(\mathbb{R}^n)$ simply via the classical Hardy space $H^1(\mathbb{R}^n)$, that is, the space $H^1_b(\mathbb{R}^n)$ is defined by the collection of all functions f such that bf is in the classical Hardy space $H^1(\mathbb{R}^n)$. This space has the advantage of the cancellation adapted to the complex measure $b(x)\,dx$ and is closely related to the Tb theorem, where b is an accretive function. In fact, Han, Lee and Lin recently proved in [17] that if $T^*(b) = 0$, then the Calderón-Zygmund operator T is bounded from the classical $H^p(\mathbb{R}^n)$ to a new Hardy space $H^p_b(\mathbb{R}^n)$ for $n/(n+\epsilon) < p \le 1$, where $\epsilon \in (0, 1]$ is some positive constant which depends on the regularity of the kernel of the considered Calderón-Zygmund operators. When $p, q > 1$, the Besov spaces, $b\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{B}^s_{pq}(X)$, the Triebel-Lizorkin spaces, $b\dot{F}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$, were considered by Han in [14], and the related Tb theorem was also established in that paper.

The main purpose of this paper is to study the Besov spaces, $b\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{B}^s_{pq}(X)$, the Triebel-Lizorkin spaces, $b\dot{F}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$ when $p \le 1$ or $q \le 1$, and the related Tb Theorem. We will do this in the setting of metric spaces, or more generally, spaces of homogeneous type.

Analysis on metric spaces has recently obtained an increasing interest; see [13, 25, 27, 34]. In particular, the theory of function spaces on metric spaces, or more generally, the spaces of homogeneous type in the sense of Coifman and Weiss in [3, 4] has been well developed; see [16, 19–24, 28, 29, 42, 48]. It is well known that the spaces of homogeneous type in Definition 1.1 below include the Euclidean space, the C^∞ Riemannian manifolds, the boundaries of Lipschitz domains, and, in particular, the Lipschitz manifolds introduced recently by Triebel in [41] and the isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel in [39] that the d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set, the generalized Sierpinski carpet and so forth. We point out that the spaces of homogeneous type in Definition 1.1 also include the post critically finite self-similar fractals studied by Kigami in [26] and by Strichartz in [36], and the metric spaces with heat kernel studied by Grigor'yan, Hu and Lau in [12]. More examples of spaces of homogeneous type can be found in [3, 4, 34].

We now state some necessary definitions and notation of spaces of homogeneous type. A *quasi-metric* ρ on a set X is a function $\rho : X \times X \to [0, \infty)$ satisfying that

(i) $\rho(x, y) = 0$ if and only if $x = y$;
(ii) $\rho(x, y) = \rho(y, x)$ for all $x,\ y \in X$;
(iii) there exists a constant $A \in [1, \infty)$ such that for all x, y and $z \in X$,

$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in X : \rho(y, x) < r\}$ for all $x \in X$ and $r > 0$ form a basis.

In what follows, we set $\operatorname{diam} X = \sup\{\rho(x, y) : x, y \in X\}$. We also assume the following conventions. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper, C will denote a positive constant which is independent of the main parameters, however it may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For any $q \in [1, \infty]$, we denote by q' its conjugate index, that is, $1/q + 1/q' = 1$. Let A be a set and we will denote by χ_A the characteristic function of A.

DEFINITION 1.1 ([23]). Let $d > 0$ and $\theta \in (0, 1]$. A *space of homogeneous type*, $(X, \rho, \mu)_{d,\theta}$, is a set X, together with a quasi-metric ρ, a nonnegative Borel regular measure μ on X, and, in addition, there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam} X$ and $x, x', y \in X$,

$$\mu(B(x, r)) \sim r^d \tag{1.1}$$

and

$$|\rho(x, y) - \rho(x', y)| \leq C_0 \rho(x, x')^{\theta}[\rho(x, y) + \rho(x', y)]^{1-\theta}. \tag{1.2}$$

The space of homogeneous type defined above is an variant of the space of homogeneous type introduced by Coifman and Weiss in [3]. In [29], Macias and Segovia have proved that one can replace the quasi-metric ρ of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar{\rho}$ which yields the same topology on X as ρ such that $(X, \bar{\rho}, \mu)$ is the space defined by Definition 1.1 with $d = 1$.

Throughout this paper, we will always assume that $\mu(X) = \infty$.

Let us now recall the definitions of para-accretive functions and the space of test functions.

DEFINITION 1.2. A bounded complex-valued function b on X, a space of homogeneous type, is said to be *para-accretive* if there exist constants $C_1 > 0$ and $\kappa \in (0, 1]$ such that for all balls $B \subset X$, there is a ball $B' \subset B$ with $\kappa\mu(B) \leq \mu(B')$ satisfying

$$\frac{1}{\mu(B)}\left|\int_{B'} b(x)\,d\mu(x)\right| \geq C_1 > 0.$$

DEFINITION 1.3 ([14]). Let b be a para-accretive function. Fix $\gamma > 0$ and $\theta \geq \beta > 0$. A function f defined on X is said to be a *test function* of type (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \leq C\dfrac{r^\gamma}{(r+\rho(x, x_0))^{d+\gamma}}$;

(ii) $|f(x) - f(y)| \leq C\left(\dfrac{\rho(x,y)}{r+\rho(x,x_0)}\right)^\beta \dfrac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$ for $\rho(x, y) \leq [r + \rho(x, x_0)]/2A$;

(iii) $\displaystyle\int_X f(x)b(x)\,d\mu(x) = 0.$

If f is a test function of type (x_0, r, β, γ) related to a para-accretive function b, we write $f \in \mathscr{G}_b(x_0, r, \beta, \gamma)$ and the norm of f in $\mathscr{G}_b(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathscr{G}_b(x_0,r,\beta,\gamma)} = \inf\{C : \text{(i) and (ii) hold}\}.$$

Now fix $x_0 \in X$ and let $\mathscr{G}_b(\beta, \gamma) = \mathscr{G}_b(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathscr{G}_b(x_1, r, \beta, \gamma) = \mathscr{G}_b(\beta, \gamma)$$

with an equivalent norm for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathscr{G}_b(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathscr{G}_b(\beta, \gamma)$. Also, let the dual space $(\mathscr{G}_b(\beta, \gamma))'$ be all linear functionals $\mathscr{L}$ from $\mathscr{G}_b(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists a $C \geq 0$ such that for all $f \in \mathscr{G}_b(\beta, \gamma)$,

$$|\mathscr{L}(f)| \leq C\|f\|_{\mathscr{G}_b(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathscr{G}_b(\beta, \gamma))'$ and $f \in \mathscr{G}_b(\beta, \gamma)$. Clearly, for all $h \in (\mathscr{G}_b(\beta, \gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathscr{G}_b(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well known that even when $X = \mathbb{R}^n$, $\mathscr{G}_b(\beta_1, \gamma)$ is not dense in $\mathscr{G}_b(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows, for a given $\epsilon \in (0, \theta]$, we let $\mathring{\mathscr{G}}_b(\beta, \gamma)$ be the completion of the space $\mathscr{G}_b(\epsilon, \epsilon)$ in $\mathscr{G}_b(\beta, \gamma)$ when $0 < \beta, \gamma < \epsilon$.

Let b be a para-accretive function. As usual, we write

$$b\mathscr{G}_b(\beta, \gamma) = \{f : f = bg \text{ for some } g \in \mathscr{G}_b(\beta, \gamma)\}.$$

https://doi.org/10.1017/S1446788700013094 Published online by Cambridge University Press

If $f \in b\mathcal{G}_b(\beta, \gamma)$ and $f = bg$ for some $g \in \mathcal{G}_b(\beta, \gamma)$, then the norm of f is defined by $\|f\|_{b\mathcal{G}_b(\beta,\gamma)} = \|g\|_{\mathcal{G}_b(\beta,\gamma)}$. By this definition, it is easy to see that

$$f \in \left(b\mathring{\mathcal{G}}_b(\beta, \gamma)\right)' \quad \text{if and only if} \quad bf \in \left(\mathring{\mathcal{G}}_b(\beta, \gamma)\right)', \tag{1.3}$$

where we define $bf \in \left(\mathring{\mathcal{G}}_b(\beta, \gamma)\right)'$ by $\langle bf, g\rangle = \langle f, bg\rangle$ for all $g \in \mathring{\mathcal{G}}_b(\beta, \gamma)$.

DEFINITION 1.4 ([16]). Let b be a para-accretive function. A sequence $\{S_k\}_{k\in\mathbb{Z}}^b$ of linear operators is said to be an *approximation to the identity* of order $\epsilon \in (0, \theta]$ associated to b if there exists $C_2 > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in X$, the kernel of S_k, denoted by $S_k(x, y)$, is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $|S_k(x, y)| \le C_2 \dfrac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}$;

(ii) $|S_k(x, y) - S_k(x', y)| \le C_2 \left(\dfrac{\rho(x, x')}{2^{-k} + \rho(x, y)}\right)^{\epsilon} \dfrac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}$ for $\rho(x, x') \le (2^{-k} + \rho(x, y))/2A$;

(iii) $|S_k(x, y) - S_k(x, y')| \le C_2 \left(\dfrac{\rho(y, y')}{2^{-k} + \rho(x, y)}\right)^{\epsilon} \dfrac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}$ for $\rho(y, y') \le (2^{-k} + \rho(x, y))/2A$;

(iv) $$\left|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]\right| \le C_2 \left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)}\right)^{\epsilon} \left(\frac{\rho(y, y')}{2^{-k} + \rho(x, y)}\right)^{\epsilon} \frac{2^{-k\epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}$$

for $\rho(x, x') \le (2^{-k} + \rho(x, y))/2A$ and $\rho(y, y') \le (2^{-k} + \rho(x, y))/2A$;

(v) $\displaystyle\int_X S_k(x, y) b(y)\, d\mu(y) = 1$;

(vi) $\displaystyle\int_X S_k(x, y) b(x)\, d\mu(x) = 1$.

REMARK 1.5. By Coifman's construction in [6], if b is a given para-accretive function, one can construct an approximation to the identity of order θ such that $S_k(x, y)$ has compact support when one variable is fixed, that is, there is a constant $C_3 > 0$ such that for all $k \in \mathbb{Z}$, $S_k(x, y) = 0$ if $\rho(x, y) \ge C_3 2^{-k}$.

REMARK 1.6. We also remark that in the sequel, if the approximation to the identity as in Definition 1.2 exists, then all the results still hold when b and b^{-1} are bounded. It seems that we do not need to assume that b is a para-accretive function. However, in [6], it was proved that the existence of the approximation to the identity as in Definition 1.2 is equivalent to the para-accretivity of b.

In the next section, we introduce the norms $\|\cdot\|_{b\dot{B}^s_{pq}(X)}$ and $\|\cdot\|_{b\dot{F}^s_{pq}(X)}$ on some distribution spaces via approximations to the identity. Then we establish a Plancherel-

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Pôlya-type inequality related to these norms by using the discrete Calderón reproducing formula in [17]. From this inequality, we deduce that the definitions of these norms are independent of the choice of approximations to the identity. By the discrete Calderón reproducing formula again, we further verify that the definition of these norms are also independent of the choice of the distribution spaces under some restrictions. After these preparations, we introduce the Besov space $b\dot{B}^s_{pq}(X)$ and the Triebel-Lizorkin space $b\dot{F}^s_{pq}(X)$. Using the discrete Calderón reproducing formula, we then establish the frame and the Littlewood-Paley function characterizations of these spaces. By the results in [6] and [17], it is easy to see that $b\dot{F}^0_{p2}(X) = L^p(X)$ with an equivalent norm if $1 < p < \infty$, and $b\dot{F}^0_{p2}(X) = H^p(X)$ with an equivalent norm if $d/(d+\epsilon) < p \le 1$. It is still unknown if this is true for any other s, p and q.

In Section 3 of this paper, we introduce the new Besov space $b^{-1}\dot{B}^s_{pq}(X)$ and the new Triebel-Lizorkin space $b^{-1}\dot{F}^s_{pq}(X)$. The frame and the Littlewood-Paley function characterization of these spaces are also given. Moreover, the relations among the spaces $b\dot{B}^s_{pq}(X)$, $b\dot{F}^s_{pq}(X)$, $b^{-1}\dot{B}^s_{pq}(X)$, $b^{-1}\dot{F}^s_{pq}(X)$, and the known Hardy spaces are presented.

Section 4 is devoted to some applications of the theory of these spaces. Using the frame characterization of these spaces, we first establish some real interpolation theorems. Some embedding theorems on these spaces are also presented. Using the interpolation theorem, we further establish the Tb theorems on these spaces and consider the boundedness of new Riesz potentials in these spaces. As a corollary, we obtain the boundedness of the new Riesz potential operator I_α for $0 \le \alpha < \epsilon$ with $\epsilon \in (0, \theta]$ from the Hardy spaces $H^p(X)$ to the Hardy spaces $H^q_b(X)$, where $1/q = 1/p - \alpha/d$ and $d/(d+\epsilon-\alpha) < p < \infty$. Moreover, using the Tb Theorem, we further establish the lifting property via the Riesz potential operators of these spaces.

Finally, we mention that the theory of the spaces $b\dot{F}^s_{\infty q}(X)$ and $b^{-1}\dot{F}^s_{\infty q}(X)$ with $|s| < \epsilon \in (0, \theta]$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$ has been established in [48].

2. Spaces $b\dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X)$

Let b be a para-accretive function as in Definition 1.2. We now introduce the norms $\|\cdot\|_{b\dot{B}^s_{pq}(X)}$ and $\|\cdot\|_{b\dot{F}^s_{pq}(X)}$ on some distribution spaces via the approximations to the identity. In what follows, we denote $\max(0, x)$ for any $x \in \mathbb{R}$ simply by x_+.

DEFINITION 2.1. Let b be a para-accretive function as in Definition 1.2. Let $\epsilon \in (0, \theta]$, $\{S_k\}^b_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ as in Definition 1.4, and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Let $|s| < \epsilon$.

(i) If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ and $0 < q \le \infty$, for all

$f \in \left(\mathring{\mathscr{G}}_b(\beta,\gamma)\right)'$ with

$$\max\{0, -s + d(1/p - 1)_+\} < \beta < \epsilon \quad \text{and} \quad \max\{0, s - d/p\} < \gamma < \epsilon, \tag{2.1}$$

we define the norm $\|f\|_{b\dot{B}^s_{pq}(X)}$ by

$$\|f\|_{b\dot{B}^s_{pq}(X)} = \left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k(f)\|^q_{L^p(X)}\right\}^{1/q}$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

(ii) If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, for all $f \in \left(\mathring{\mathscr{G}}_b(\beta,\gamma)\right)'$ with β, γ as in (2.1), we define the norm $\|f\|_{b\dot{F}^s_{pq}(X)}$ by

$$\|f\|_{b\dot{F}^s_{pq}(X)} = \left\|\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(X)}$$

with the usual modification made when $q = \infty$.

The following theorem indicates that the definitions of the norms $\|\cdot\|_{b\dot{B}^s_{pq}(X)}$ and $\|\cdot\|_{b\dot{F}^s_{pq}(X)}$ are independent of the choice of approximations to the identity.

THEOREM 2.2. *Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0,\theta]$ and $|s| < \epsilon$. Let $\{S_k\}^b_{k\in\mathbb{Z}}$ and $\{G_k\}^b_{k\in\mathbb{Z}}$ be two approximations to the identity of order ϵ as in Definition 1.4, $D_k = S_k - S_{k-1}$ and $E_k = G_k - G_{k-1}$ for $k \in \mathbb{Z}$.*

(i) *If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ and $0 < q \le \infty$, for all $f \in \left(\mathring{\mathscr{G}}_b(\beta,\gamma)\right)'$ with $0 < \beta,\ \gamma < \epsilon$, we have*

$$\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k(f)\|^q_{L^p(X)}\right\}^{1/q} \sim \left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|E_k(f)\|^q_{L^p(X)}\right\}^{1/q}.$$

(ii) *If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, for all $f \in \left(\mathring{\mathscr{G}}_b(\beta,\gamma)\right)'$ with $0 < \beta, \gamma < \epsilon$, we have*

$$\left\|\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(X)} \sim \left\|\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}|E_k(f)|^q\right\}^{1/q}\right\|_{L^p(X)}.$$

To prove Theorem 2.2, we first recall the following construction given by Christ in [2], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type.

LEMMA 2.3. *Let X be a space of homogeneous type. Then there exists a collection $\{Q_\alpha^k \subset X : k \in \mathbb{Z},\ \alpha \in I_k\}$ of open subsets, where I_k is some index set, $\delta \in (0,1)$ and $C_4,\ C_5 > 0$, such that*

(i) $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*
(ii) *for any $\alpha,\ \beta,\ k,\ l$ with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;*
(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*
(iv) $\operatorname{diam}(Q_\alpha^k) \leq C_4 \delta^k$;
(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_5\delta^k)$, where $z_\alpha^k \in X$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter rough δ^k and centered at z_α^k. In what follows, we always suppose $\delta = 1/2$. See [22] for how to remove this restriction. In addition, for $k \in \mathbb{Z}$ and $\tau \in I_k$, we will denote by $Q_\tau^{k,\nu}$, $\nu = 1, 2, \ldots, N(k,\tau)$, the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$, where j is a fixed large positive integer. Denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$.

REMARK 2.4. Since we always assume that $\mu(X) = \infty$ in this paper, for all $k \in \mathbb{Z}$, I_k in Lemma 2.3 is an infinite index set.

Theorem 2.2 is a simple corollary of Lemma 2.3 and the following Plancherel-Pôlya-type inequality.

THEOREM 2.5. *Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$ and $|s| < \epsilon$. Let $\{S_k\}_{k\in\mathbb{Z}}^b$ and $\{G_k\}_{k\in\mathbb{Z}}^b$ be two approximations to the identity of order ϵ as in Definition 1.4, $D_k = S_k - S_{k-1}$ and $E_k = G_k - G_{k-1}$ for $k \in \mathbb{Z}$.*

(i) *If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \leq \infty$ and $0 < q \leq \infty$, there is a constant $C > 0$ such that for all $f \in \left(\mathring{\mathscr{G}}_b(\beta,\gamma)\right)'$ with $0 < \beta,\ \gamma < \epsilon$, we have*

$$\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \sup_{z\in Q_\tau^{k,\nu}} |D_k(f)(z)|^p\right]^{q/p}\right\}^{1/q}$$
$$\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \inf_{z\in Q_\tau^{k,\nu}} |E_k(f)(z)|^p\right]^{q/p}\right\}^{1/q}.$$

(ii) *If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \leq \infty$, there is a constant $C > 0$ such that for all $f \in \left(\mathring{\mathscr{G}}_b(\beta,\gamma)\right)'$ with $0 < \beta,\ \gamma < \epsilon$, we have*

$$\left\|\left\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \sup_{z\in Q_\tau^{k,\nu}} |D_k(f)(z)|^q \chi_{Q_\tau^{k,\nu}}\right\}^{1/q}\right\|_{L^p(X)}$$

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$$\leq C \left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \inf_{z \in Q_\tau^{k,\nu}} |E_k(f)(z)|^q \chi_{Q_\tau^{k,\nu}} \right\}^{1/q} \right\|_{L^p(X)} .$$

To show Theorem 2.5, the following discrete Calderón reproducing formula in [17] will play a crucial role.

LEMMA 2.6. *Let b be a para-accretive function as in Definition* 1.2. *Let $\epsilon \in (0, \theta]$, $\{G_k\}_{k\in\mathbb{Z}}^b$ be an approximation to the identity of order ϵ as in Definition* 1.4, *and $E_k = G_k - G_{k-1}$ for $k \in \mathbb{Z}$. Let $0 < \beta,\ \gamma < \epsilon$. Then there exists a family of functions $\{\tilde{E}_k(x, y)\}_{k\in\mathbb{Z}}$ such that for all fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ and all $f \in \left(\mathring{\mathscr{G}}_b(\beta, \gamma)\right)'$,*

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} E_k(f)(y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} b(x)\tilde{E}_k(y, x) b(y)\, d\mu(y), \tag{2.2}$$

where $\tilde{E}_k(x, y)$ for $k \in \mathbb{Z}$ satisfies (i) *and* (iii) *of Definition* 1.4 *with ϵ replaced by $\epsilon' \in (0, \epsilon)$, and*

$$\int_X \tilde{E}_k(x, y) b(x)\, d\mu(x) = \int_X \tilde{E}_k(x, y) b(y)\, d\mu(y) = 0.$$

Moreover, the series in (2.2) *converges in the sense that for all $g \in \mathring{\mathscr{G}}_b(\beta', \gamma')$ with $\beta < \beta' < \epsilon$ and $\gamma < \gamma' < \epsilon$,*

$$\lim_{M,\, N\to\infty} \left\langle \sum_{|k|\leq M} \sum_{\substack{\tau \in I_k \\ \rho(x_0, z_\tau^k) \leq N}} \sum_{\nu=1}^{N(k,\tau)} E_k(f)(y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} b(\cdot)\tilde{E}_k(y, \cdot) b(y)\, d\mu(y), g \right\rangle = \langle f, g \rangle.$$

Another useful tool in dealing with the Triebel-Lizorkin space is the following lemma which can be found in [8, pages 147–148] for $\mathbb{R}^n$ and [22, page 93] for spaces of homogeneous type.

LEMMA 2.7. *Let $0 < r \leq 1$, $k, \eta \in \mathbb{Z}_+$ with $\eta \leq k$ and for any dyadic cube $Q_\tau^{k,\nu}$ and all $x \in X$, $|f_{Q_\tau^{k,\nu}}(x)| \leq (1 + 2^\eta \rho(x, y_\tau^{k,\nu}))^{-d-\gamma}$, where $y_\tau^{k,\nu}$ is any point in $Q_\tau^{k,\nu}$ and $\gamma > d(1/r - 1)$. Then, for all $x \in X$,*

$$\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}| |f_{Q_\tau^{k,\nu}}(x)| \leq C\, 2^{(k-\eta)d/r} \left[M \left(\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}|^r \chi_{Q_\tau^{k,\nu}} \right)(x) \right]^{1/r},$$

where C is independent of x, k and η, and M is the Hardy-Littlewood maximal operator on X; see [3].

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PROOF OF THEOREM 2.5. We first verify (i). With all the notation as in Theorem 2.5, by (2.2), we have that for all $k \in \mathbb{Z}$ and $z \in X$,

$$D_k(f)(z) = \sum_{k'=-\infty}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} E_{k'}(f)(y_{\tau'}^{k',\nu'}) \int_{Q_{\tau'}^{k',\nu'}} D_k(b\tilde{E}_{k'}(y,\cdot))(z)b(y)\,d\mu(y). \tag{2.3}$$

We recall the estimate in [14, page 66], that there is a constant $C > 0$ such that for all $k,\ k' \in \mathbb{Z}$ and all $y,\ z \in X$,

$$\left|D_k(b\tilde{E}_{k'}(y,\cdot))(z)\right| \le C_{\epsilon'} 2^{-|k-k'|\epsilon'} \frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')} + \rho(y,z))^{d+\epsilon'}}, \tag{2.4}$$

where ϵ' can be any positive number in $(0,\epsilon)$ and $k \wedge k' = \min(k,k')$.

If $y \in Q_{\tau'}^{k',\nu'}$ and $z \in Q_{\tau}^{k,\nu}$, then

$$2^{-(k\wedge k')} + \rho(y,z) \sim 2^{-(k\wedge k')} + \rho(y_{\tau'}^{k',\nu'}, y_{\tau}^{k,\nu}). \tag{2.5}$$

We also recall the well-known inequality that for all $a_j \in \mathbb{C}$ and $q \le 1$,

$$\left(\sum_j |a_j|\right)^q \le \sum_j |a_j|^q. \tag{2.6}$$

Now suppose $p \le 1$. In this case, from (2.3)–(2.6), it follows that

$$\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \sup_{z\in Q_\tau^{k,\nu}} |D_k(f)(z)|^p\right]^{q/p}\right\}^{1/q} \\
&\le C\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu})\left[\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{-|k-k'|\epsilon' p}\mu(Q_{\tau'}^{k',\nu'})^p\right.\right.\right. \\
&\quad \times \left.\left.\left. |E_{k'}(f)(y_{\tau'}^{k',\nu'})|^p \frac{2^{-(k\wedge k')\epsilon' p}}{(2^{-(k\wedge k')} + \rho(y_{\tau'}^{k',\nu'}, y_\tau^{k,\nu}))^{(d+\epsilon')p}}\right]\right)^{q/p}\right\}^{1/q} \\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left[\sum_{k'=-\infty}^{\infty} 2^{(k-k')sp-|k-k'|\epsilon' p+k'd(1-p)} \sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sp}\mu(Q_{\tau'}^{k',\nu'})\right.\right. \\
&\quad \times \left.\left. |E_{k'}(f)(y_{\tau'}^{k',\nu'})|^p \int_X \frac{2^{-(k\wedge k')\epsilon' p}}{(2^{-(k\wedge k')} + \rho(y_{\tau'}^{k',\nu'}, x))^{(d+\epsilon')p}}\,d\mu(x)\right]^{q/p}\right\}^{1/q} \\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left[\sum_{k'=-\infty}^{\infty} 2^{(k-k')sp-|k-k'|\epsilon' p-(k\wedge k')d(1-p)+k'd(1-p)}\right.\right.
\end{aligned}$$

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$$\times \left(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sp} \mu(Q_{\tau'}^{k',\nu'}) \left| E_{k'}(f)(y_{\tau'}^{k',\nu'}) \right|^p \right) \Bigg]^{q/p} \Bigg\}^{1/q}$$

$$\leq C \left\{ \sum_{k'=-\infty}^{\infty} \left[\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sp} \mu(Q_{\tau'}^{k',\nu'}) \left| E_{k'}(f)(y_{\tau'}^{k',\nu'}) \right|^p \right]^{q/p} \right\}^{1/q},$$

where we choose $\epsilon' \in (0, \epsilon)$ such that $\epsilon' > \max\{s, -s + d(1/p - 1)\}$, and in the last inequality, if $q/p \geq 1$, we use the Hölder inequality and the following estimate

$$\left(\sum_{k'=-\infty}^{\infty} + \sum_{k=-\infty}^{\infty} \right) 2^{(k-k')sp - |k-k'|\epsilon' p - (k \wedge k')d(1-p) + k'd(1-p)} \leq C.$$

If $q/p < 1$, we use (2.6) and

$$\sum_{k=-\infty}^{\infty} 2^{[(k-k')sp - |k-k'|\epsilon' p - (k \wedge k')d(1-p) + k'd(1-p)]q/p} \leq C.$$

Since $y_{\tau'}^{k',\nu'}$ is arbitrary, we can further obtain a desired estimate in this case.

If $p > 1$, (2.3)–(2.5) and the Hölder inequality yield that

$$\left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \left[\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \sup_{z \in Q_\tau^{k,\nu}} |D_k(f)(z)|^p \right]^{q/p} \right\}^{1/q}$$

$$\leq C \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) \left(\sum_{k'=-\infty}^{\infty} 2^{(k-k')sp - |k-k'|\epsilon' p} \right. \right. \right.$$

$$\times \left[\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sp} \mu(Q_{\tau'}^{k',\nu'}) \left| E_{k'}(f)(y_{\tau'}^{k',\nu'}) \right|^p \frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')} + \rho(y_{\tau'}^{k',\nu'}, y_\tau^{k,\nu}))^{d+\epsilon'}} \right]$$

$$\times \left. \left. \left. \left[\int_X \frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')} + \rho(y, y_\tau^{k,\nu}))^{d+\epsilon'}} \, d\mu(y) \right]^{p/p'} \right) \right]^{q/p} \right\}^{1/q}$$

$$\leq C \left\{ \sum_{k=-\infty}^{\infty} \left[\sum_{k'=-\infty}^{\infty} 2^{(k-k')sp - |k-k'|\epsilon' p} \right. \right.$$

$$\times \left(\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sp} \mu(Q_{\tau'}^{k',\nu'}) \left| E_{k'}(f)(y_{\tau'}^{k',\nu'}) \right|^p \right)$$

$$\times \left. \left. \int_X \frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')} + \rho(y_{\tau'}^{k',\nu'}, x))^{d+\epsilon'}} \, d\mu(x) \right]^{q/p} \right\}^{1/q}$$

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$$\leq C\left\{\sum_{k'=-\infty}^{\infty}\left[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sp}\mu(Q_{\tau'}^{k',\nu'})\left|E_{k'}(f)(y_{\tau'}^{k',\nu'})\right|^{p}\right]^{q/p}\right\}^{1/q},$$

where we choose $\epsilon' > |s|$ and we use the fact that for all $x \in X$,

$$\int_X \frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(y,x))^{d+\epsilon'}}\,d\mu(y) \leq C,$$

$$\left(\sum_{k'=-\infty}^{\infty}+\sum_{k=-\infty}^{\infty}\right)2^{(k-k')sp-|k-k'|\epsilon' p} \leq C,$$

and

$$\sum_{k=-\infty}^{\infty}2^{[(k-k')sp-|k-k'|\epsilon' p]q/p} \leq C.$$

Since $y_{\tau'}^{k',\nu'}$ is arbitrary, a desired estimate follows, and this finishes the proof of (i).

To verify (ii), by Lemma 2.6, Lemma 2.7, (2.4)–(2.6) or the Hölder inequality, we have

$$\begin{aligned}
&\left\{\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|^q\chi_{Q_\tau^{k,\nu}}(x)\right\}^{1/q}\\
&\quad\leq C\left\{\sum_{k=-\infty}^{\infty}2^{ksq}\left[\sum_{k'=-\infty}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{-|k-k'|\epsilon'}\mu(Q_{\tau'}^{k',\nu'})\left|E_{k'}(f)(y_{\tau'}^{k',\nu'})\right|\right.\right.\\
&\qquad\left.\left.\times\frac{2^{-(k\wedge k')\epsilon'}}{(2^{-(k\wedge k')}+\rho(x,y_{\tau'}^{k',\nu'}))^{d+\epsilon'}}\right]^q\right\}^{1/q}\\
&\quad\leq C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{k'=-\infty}^{\infty}2^{(k-k')s-|k-k'|\epsilon'+[(k\wedge k')-k']d(1-1/r)}\right.\right.\\
&\qquad\left.\left.\times\left[M\left(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sr}\left|E_{k'}(f)(y_{\tau'}^{k',\nu'})\right|^r\chi_{Q_{\tau'}^{k',\nu'}}\right)(x)\right]^{1/r}\right)^q\right\}^{1/q}\\
&\quad\leq C\left\{\sum_{k'=-\infty}^{\infty}\left[M\left(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sr}\left|E_{k'}(f)(y_{\tau'}^{k',\nu'})\right|^r\chi_{Q_{\tau'}^{k',\nu'}}\right)(x)\right]^{q/r}\right\}^{1/q},
\end{aligned}$$

where we choose $\epsilon' \in (0,\epsilon)$ and $r \in (0,1]$ such that $\epsilon' > \max\{d(1/r-1), s, -s+d(1/r-1)\}$ and $r < \min\{p,q\}$, and we use the fact that

$$\left(\sum_{k'=-\infty}^{\infty}+\sum_{k=-\infty}^{\infty}\right)2^{(k-k')s-|k-k'|\epsilon'+[(k\wedge k')-k']d(1-1/r)} \leq C,$$

and $\sum_{k=-\infty}^{\infty} 2^{(k-k')s-|k-k'|\epsilon'+[(k\wedge k')-k']d(1-1/r)q} \leq C$. Thus, the vector-valued inequality of Fefferman and Stein in [7] and the arbitrariness of $y_{\tau'}^{k',\nu'} \in Q_{\tau'}^{k',\nu'}$ further imply the conclusion (ii) of the theorem, which completes the proof of Theorem 2.5. $\square$

REMARK 2.8. From the proof of Theorem 2.5, it is easy to see that the key role played by $\{D_k\}_{k\in\mathbb{Z}}$ is the estimate of (2.4). However, to establish this estimate, we need only to use the regularity (iii) as in Definition 1.4 of D_k for $k \in \mathbb{Z}$; see [14, 16]. This means that if we replace the operators D_k by some other operators $\bar{D}_k$ for $k \in \mathbb{Z}$ whose kernels have the same properties as the kernels of D_k except for the regularity (ii) of Definition 1.4, then Theorem 2.5 still holds. This observation is useful in some applications.

Let us now verify that under some restrictions on β and γ, the definition of the norms $\|\cdot\|_{b\dot{B}^s_{pq}(X)}$ and $\|\cdot\|_{b\dot{F}^s_{pq}(X)}$ is independent of the choice of the distribution space, $(\mathring{\mathcal{G}}(\beta,\gamma))'$.

THEOREM 2.9. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0,\theta]$ *and* $|s| < \epsilon$.

(i) *Let* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \leq \infty$ *and* $0 < q \leq \infty$. *If* $f \in (\mathring{\mathcal{G}}_b(\beta_1,\gamma_1))'$ *with* $\max\{0, -s+d(1/p-1)_+\} < \beta_1 < \epsilon$, $\max\{0, s-d/p\} < \gamma_1 < \epsilon$, *and* $\|f\|_{b\dot{B}^s_{pq}(X)} < \infty$, *then* $f \in (\mathring{\mathcal{G}}_b(\beta_2,\gamma_2))'$ *with* $\max\{0, -s+d(1/p-1)_+\} < \beta_2 < \epsilon$ *and* $\max\{0, s-d/p\} < \gamma_2 < \epsilon$.

(ii) *Let*

$$\max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+s+\epsilon}\right\} < p < \infty \quad \textit{and} \quad \max\left\{\frac{d}{d+\epsilon}, \frac{d}{d+s+\epsilon}\right\} < q \leq \infty.$$

If $f \in (\mathring{\mathcal{G}}_b(\beta_1,\gamma_1))'$ *with* $\max\{0, -s+d(1/p-1)_+\} < \beta_1 < \epsilon$, $\max\{0, s-d/p\} < \gamma_1 < \epsilon$, *and* $\|f\|_{b\dot{F}^s_{pq}(X)} < \infty$, *then* $f \in (\mathring{\mathcal{G}}_b(\beta_2,\gamma_2))'$ *with* $\max\{0, -s+d(1/p-1)_+\} < \beta_2 < \epsilon$ *and* $\max\{0, s-d/p\} < \gamma_2 < \epsilon$.

PROOF. Let $\psi \in \mathcal{G}_b(\epsilon,\epsilon)$. With all the notation as in Lemma 2.6, we first claim for $k = 0, 1, 2, \ldots$, $\tau \in I_k$ and $\nu = 1, 2, \ldots, N(k,\tau)$ that

$$|\langle b(\cdot)\tilde{E}_k(y,\cdot), \psi\rangle| \leq C\, 2^{-k\beta_2}\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)} \frac{1}{(1+\rho(y,x_0))^{d+\gamma_2}}, \tag{2.7}$$

and for $k = -1, -2, \ldots$, $\tau \in I_k$ and $\nu = 1, 2, \ldots, N(k,\tau)$ that

$$|\langle b(\cdot)\tilde{E}_k(y,\cdot), \psi\rangle| \leq C\, 2^{k\gamma_2'}\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)} \frac{2^{-k\gamma_2}}{(2^{-k}+\rho(y,x_0))^{d+\gamma_2}}, \tag{2.8}$$

where γ_2' can be any positive number in $(0,\gamma_2)$.

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In fact, for (2.7), by the vanishing moment of $b(\cdot)\tilde{E}_k(y,\cdot)$, we have

$$\begin{aligned}
&\left|\langle b(\cdot)\tilde{E}_k(y,\cdot),\psi\rangle\right| \\
&\quad = \left|\int_X \tilde{E}_k(y,x)b(x)[\psi(x)-\psi(y)]\,d\mu(x)\right| \\
&\quad \le C\|\psi\|_{\mathscr{G}_b(\beta_2,\gamma_2)}\left\{\int_{\rho(y,x)\le(1+\rho(y,x_0))/2A}|\tilde{E}_k(y,x)|\frac{\rho(x,y)^{\beta_2}}{(1+\rho(y,x_0))^{d+\gamma_2+\beta}}\,d\mu(x)\right. \\
&\qquad \left.+\int_{\rho(y,x)>(1+\rho(y,x_0))/2A}|\tilde{E}_k(y,x)|\left[\frac{1}{(1+\rho(x,x_0))^{d+\gamma_2}}+\frac{1}{(1+\rho(y,x_0))^{d+\gamma_2}}\right]d\mu(x)\right\} \\
&\quad \le C\,2^{-k\beta_2}\|\psi\|_{\mathscr{G}_b(\beta_2,\gamma_2)}\frac{1}{(1+\rho(y,x_0))^{d+\gamma_2}},
\end{aligned}$$

where we choose $\epsilon'>0$ in Lemma 2.6 such that $\epsilon'>\max\{\gamma_2,\beta_2\}$. This verifies (2.7).

Similarly, the vanishing moment of $b\psi$ yields that

$$\begin{aligned}
&\left|\langle b(\cdot)\tilde{E}_k(y,\cdot),\psi\rangle\right| \\
&\quad = \left|\int_X [\tilde{E}_k(y,x)-\tilde{E}_k(y,x_0)]b(x)\psi(x)\,d\mu(x)\right| \\
&\quad \le C\|\psi\|_{\mathscr{G}_b(\beta_2,\gamma_2)}\left\{\int_{\rho(x,x_0)\le(2^{-k}+\rho(y,x_0))/2A}\frac{2^{-k\epsilon'}\rho(x,x_0)^{\gamma_2'}}{(2^{-k}+\rho(y,x_0))^{d+\epsilon'+\gamma_2'}}\frac{d\mu(x)}{(1+\rho(x,x_0))^{d+\gamma_2}}\right. \\
&\qquad +\int_{\rho(x,x_0)>(2^{-k}+\rho(y,x_0))/2A}\left[\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(y,x))^{d+\epsilon'}}+\frac{2^{-k\epsilon'}}{(2^{-k}+\rho(y,x_0))^{d+\epsilon'}}\right] \\
&\qquad \left.\times\frac{1}{(1+\rho(x,x_0))^{d+\gamma_2}}\,d\mu(x)\right\} \\
&\quad \le C\,2^{k\gamma_2'}\|\psi\|_{\mathscr{G}_b(\beta_2,\gamma_2)}\frac{2^{-k\gamma_2}}{(2^{-k}+\rho(y,x_0))^{d+\gamma_2}},
\end{aligned}$$

which is just (2.8).

Observe that if $k\ge 0$, $\tau\in I_k$, $\nu=1,\dots,N(k,\tau)$, and $y\in Q_\tau^{k,\nu}$, then

$$1+\rho(y,x_0)\sim 1+\rho(y_\tau^{k,\nu},x_0), \tag{2.9}$$

and if $k=-1,-2,\dots$, $\tau\in I_k$, $\nu=1,\dots,N(k,\tau)$, and $y\in Q_\tau^{k,\nu}$, then

$$2^{-k}+\rho(y,x_0)\sim 2^{-k}+\rho(y_\tau^{k,\nu},x_0). \tag{2.10}$$

The estimates (2.7) and (2.8) and the observations (2.9) and (2.10) tell us that

$$|\langle f,\psi\rangle| = \left|\sum_{k=-\infty}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}E_k(f)(y_\tau^{k,\nu})\int_{Q_\tau^{k,\nu}}\langle b(\cdot)\tilde{E}_k(y,\cdot),\psi\rangle b(y)\,d\mu(y)\right|$$

$$
\begin{aligned}
&\leq C\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)}\left\{\sum_{k=0}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{-k\beta_2}\mu(Q_\tau^{k,\nu})|E_k(f)(y_\tau^{k,\nu})|\right.\\
&\quad\times\frac{1}{(1+\rho(y_\tau^{k,\nu},x_0))^{d+\gamma_2}}\\
&\quad\left.+\sum_{k=-\infty}^{-1}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{k\gamma_2'}\mu(Q_\tau^{k,\nu})|E_k(f)(y_\tau^{k,\nu})|\frac{2^{-k\gamma_2}}{(2^{-k}+\rho(y_\tau^{k,\nu},x_0))^{d+\gamma_2}}\right\}.
\end{aligned}
$$

If $p \leq 1$, Lemma 2.3, Theorem 2.5, (2.6) and the Hölder inequality yield that

$$
\begin{aligned}
(2.11)\quad |\langle f,\psi\rangle| &\leq C\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)}\left\{\sum_{k=0}^{\infty}2^{-k(\beta_2+s)-kd(1-1/p)}\right.\\
&\quad\times\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksp}\mu(Q_\tau^{k,\nu})|E_k(f)(y_\tau^{k,\nu})|^p\right)^{1/p}\\
&\quad\left.+\sum_{k=-\infty}^{-1}2^{k(\gamma_2'-s+d/p)}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksp}\mu(Q_\tau^{k,\nu})|E_k(f)(y_\tau^{k,\nu})|^p\right)^{1/p}\right\}\\
&\leq C\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)}\left\{\sum_{k=-\infty}^{\infty}2^{ksq}\|E_k(f)\|_{L^p(X)}^q\right\}^{1/q}\\
&\leq C\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)}\|f\|_{b\dot{B}^s_{pq}(X)},
\end{aligned}
$$

where we choose $\gamma_2' \in (0,\gamma_2)$ such that $\gamma_2' > s - d/p$ and use the fact that

$$\beta_2 > -s + d(1/p - 1).$$

If $p > 1$, similar estimates, except that (2.4) is replaced by the Hölder inequality, lead us to

(2.12)

$$
\begin{aligned}
|\langle f,\psi\rangle| &\leq C\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)}\left\{\sum_{k=0}^{\infty}2^{-k\beta_2}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|E_k(f)(y_\tau^{k,\nu})|^p\right]^{1/p}\right.\\
&\quad\times\left[\int_X\frac{1}{(1+\rho(y,x_0))^{d+\gamma_2}}\,d\mu(y)\right]^{1/p'}\\
&\quad+\sum_{k=-\infty}^{-1}2^{k(\gamma_2'+d/p)}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})|E_k(f)(y_\tau^{k,\nu})|^p\right]^{1/p}\\
&\quad\left.\times\left[\int_X\frac{2^{-k\gamma_2}}{(2^{-k}+\rho(y,x_0))^{d+\gamma_2}}\,d\mu(y)\right]^{1/p'}\right\}\leq C\|\psi\|_{\mathcal{G}_b(\beta_2,\gamma_2)}\|f\|_{b\dot{B}^s_{pq}(X)},
\end{aligned}
$$

where we use the fact that $\beta_2 > -s$ in this case and choose $\gamma_2' > \max\{0, s - d/p\}$.

Suppose $h \in \mathring{\mathcal{G}}(\beta_2, \gamma_2)$. We choose $h_n \in \mathcal{G}(\epsilon, \epsilon)$ for any $n \in \mathbb{N}$ such that

$$\|h_n - h\|_{\mathcal{G}(\beta_2,\gamma_2)} \xrightarrow{n\to\infty} 0.$$

The estimates of (2.11) and (2.12) show that for all $n,\ m \in \mathbb{N}$,

$$|\langle f, h_n - h_m\rangle| \le C\|f\|_{b\dot{B}^s_{pq}(X)}\|h_n - h_m\|_{\mathcal{G}(\beta_2,\gamma_2)},$$

which shows $\lim_{n\to\infty}\langle f, h_n\rangle$ exists and the limit is independent of the choice of h_n. Therefore, we define

$$\langle f, h\rangle = \lim_{n\to\infty}\langle f, h_n\rangle.$$

By (2.11) and (2.12), for all $h \in \mathring{\mathcal{G}}(\beta_2, \gamma_2)$,

$$|\langle f, h\rangle| \le C\|f\|_{b\dot{B}^s_{pq}(X)}\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}.$$

Thus, $f \in \left(\mathring{\mathcal{G}}(\beta_2, \gamma_2)\right)'$. This finishes the proof of (i).

The conclusion (ii) can be deduced from (i) and the fact that

$$\|f\|_{b\dot{B}^s_{p,\max(p,q)}(X)} \le C\|f\|_{b\dot{F}^s_{pq}(X)};$$

see [37, Proposition 2.3.2/2]. This finishes the proof of Theorem 2.9. □

Now we introduce the Besov, $b\dot{B}^s_{pq}(X)$, and the Triebel-Lizorkin, $b\dot{F}^s_{pq}(X)$, spaces.

DEFINITION 2.10. Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$ and $|s| < \epsilon$. Let $\{S_k\}^b_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ as in Definition 1.4 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$.

(i) If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ and $0 < q \le \infty$, we define the space $b\dot{B}^s_{pq}(X)$ to be the set of all $f \in \left(\mathring{\mathcal{G}}_b(\beta, \gamma)\right)'$ with

$$\begin{aligned} &\max\{s, 0, -s + d(1/p - 1)_+\} < \beta < \epsilon \quad \text{and} \\ \max\{s - d/p, d(1/p-1)_+, &-s + d(1/p - 1)\} < \gamma < \epsilon \end{aligned} \tag{2.13}$$

such that

$$\|f\|_{b\dot{B}^s_{pq}(X)} = \left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k(f)\|^q_{L^p(X)}\right\}^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

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(ii) If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, we define the space $b\dot{F}^s_{pq}(X)$ to be the set of all $f \in \big(\mathring{\mathcal{G}}_b(\beta,\gamma)\big)'$ with β, γ as in (2.13) such that

$$\|f\|_{b\dot{F}^s_{pq}(X)} = \left\| \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(f)|^q \right\}^{1/q} \right\|_{L^p(X)} < \infty$$

with the usual modification made when $q = \infty$.

Theorem 2.2 and Theorem 2.9 tell us that the definitions of the Besov space $b\dot{B}^s_{pq}(X)$ and the Triebel-Lizorkin space $b\dot{F}^s_{pq}(X)$ are independent of the choice of the approximation to the identity and the distributional space, $\big(\mathring{\mathcal{G}}_b(\beta,\gamma)\big)'$, with β, γ as in (2.13).

REMARK 2.11. To guarantee the definition of the spaces $b\dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X)$ is independent of the choice of the distribution space $(\mathring{\mathcal{G}}_b(\beta,\gamma))'$, we need only to restrict that $\max\{0, -s + d(1/p - 1)_+\} < \beta < \epsilon$ and $\max\{0, s - d/p\} < \gamma < \epsilon$; see Theorem 2.9. However, if we restrict $\max(0, s) < \beta < \epsilon$ and $\max\{d(1/p - 1)_+, -s + d(1/p-1)\} < \gamma < \epsilon$, we prove in the following proposition that the space of test functions, $b\mathcal{G}_b(\beta,\gamma)$, is contained in the spaces $b\dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X)$. Thus, the spaces $b\dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X)$ are non-empty if we restrict β, γ as in (2.13).

PROPOSITION 2.12. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0,\theta]$ *and* $|s| < \epsilon$.

(i) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *then* $b\dot{B}^s_{p,\min(p,q)}(X) \subset b\dot{F}^s_{pq}(X) \subset b\dot{B}^s_{p,\max(p,q)}(X)$.

(ii) *If* $f \in b\mathcal{G}_b(\beta,\gamma)$ *with* $\max(0, s) < \beta < \epsilon$ *and* $\max\{d(1/p - 1)_+, -s + d(1/p-1)\} < \gamma < \epsilon$, *then* $f \in b\dot{B}^s_{pq}(X)$ *with* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ *and* $0 < q \le \infty$, *and* $f \in b\dot{F}^s_{pq}(X)$ *with* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$. *Moreover, if* p, $q < \infty$, *then the space* $b\mathcal{G}_b(\beta,\gamma)$ *with* β, γ *as above is dense according to the norm* $\|\cdot\|_{b\dot{B}^s_{pq}(X)}$ *in the space* $b\dot{B}^s_{pq}(X)$, *and according to the norm* $\|\cdot\|_{b\dot{F}^s_{pq}(X)}$ *in the space* $b\dot{F}^s_{pq}(X)$.

(iii) *If* $1 < p < \infty$, *then* $b\dot{F}^0_{p2}(X) = L^p(X)$ *with an equivalent norm; and if* $d/(d+\epsilon) < p \le 1$, *then* $b\dot{F}^0_{p2}(X) = H^p(X)$ *with an equivalent norm.*

PROOF. The proof of (i) is trivial; see [37, Proposition 2.3.2/2] and [40, Proposition 2.3].

Let $f \in b\mathcal{G}_b(\beta,\gamma)$ with $\max(0, s) < \beta < \epsilon$ and

$$\max\{d(1/p - 1)_+, -s + d(1/p-1)\} < \gamma < \epsilon.$$

Then $f = bg$ for some $g \in \mathcal{G}_b(\beta, \gamma)$ and $\|f\|_{b\mathcal{G}_b(\beta,\gamma)} = \|g\|_{\mathcal{G}_b(\beta,\gamma)}$. Let $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Definition 2.10. The same argument as in (2.7) and (2.8) yields that for $k = 0, 1, 2, \dots, \tau \in I_k$ and $\nu = 1, 2, \dots, N(k, \tau)$,

$$|D_k(f)(x)| \le C\, 2^{-k\beta} \|f\|_{b\mathcal{G}_b(\beta,\gamma)} \frac{1}{(1 + \rho(y, x_0))^{d+\gamma}}, \tag{2.14}$$

and for $k = -1, -2, \dots, \tau \in I_k$ and $\nu = 1, 2, \dots, N(k, \tau)$,

$$|D_k(f)(x)| \le C\, 2^{k\gamma'} \|f\|_{b\mathcal{G}_b(\beta,\gamma)} \frac{2^{-k\gamma}}{(2^{-k} + \rho(y, x_0))^{d+\gamma}}, \tag{2.15}$$

where γ' can be any positive number in $(0, \gamma)$.

From (2.14), (2.15) and Definition 2.10, it follows that

$$\begin{aligned}
\|f\|_{b\dot{B}^s_{pq}(X)} &= \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \|D_k(f)\|^q_{L^p(X)} \right\}^{1/q} \\
&\le C\|f\|_{b\mathcal{G}_b(\beta,\gamma)} \left\{ \sum_{k=0}^{\infty} 2^{k(s-\beta)q} + \sum_{k=-\infty}^{-1} 2^{k[s+\gamma'-d(1/p-1)]q} \right\}^{1/q} \\
&\le C\|f\|_{b\mathcal{G}_b(\beta,\gamma)},
\end{aligned}$$

where we choose $\gamma' \in (0, \gamma)$ such that $\gamma' > -s + d(1/p - 1)$. Thus, $f \in b\dot{B}^s_{pq}(X)$. From this and (i), it is easy to deduce that f is also in $b\dot{F}^s_{pq}(X)$.

The density of the space $b\mathcal{G}_b(\beta, \gamma)$ in the space $b\dot{B}^s_{pq}(X)$ and the space $b\dot{F}^s_{pq}(X)$ can be proved by the same argument as in [14, Proposition 3.8]. This proves (ii).

If $1 < p < \infty$, David, Journé and Semmes in [6], proved that $b\dot{F}^0_{p2}(X) = L^p(X)$ with an equivalent norm. Moreover, Han, Lee and Lin in [17], proved that if $d/(d + \epsilon) < p \le 1$, then $b\dot{F}^0_{p2}(X) = H^p(X)$ with an equivalent norm. This finishes the proof of this proposition. □

REMARK 2.13. Based on Proposition 2.12 (iii), it is interesting to make clear under what restrictions on s, p and q one will have that $b\dot{B}^s_{pq}(X) = \dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X) = \dot{F}^s_{pq}(X)$, with an equivalent norm.

Lemma 2.6, Theorem 2.5 and the same argument as the proof of [16, Theorem 3] can also give the characterization of the Littlewood-Paley S-function of the Triebel-Lizorkin space $b\dot{F}^s_{pq}(X)$ as below. We omit the details.

THEOREM 2.14. *Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$ and $|s| < \epsilon$. Let $\{S_k\}^b_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ as in Definition 1.4 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. If* $\max\{d/(d + \epsilon), d/(d + s + \epsilon)\} < p < \infty$

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and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *then* $f \in b\dot{F}^s_{pq}(X)$ *if and only if* $f \in (\mathring{\mathcal{G}}_b(\beta,\gamma))'$ *with* β, γ *as in* (2.13) *and* $S^s_q(f) \in L^p(X)$, *where*

$$S^s_q(f)(x) = \left\{ \sum_{k=-\infty}^{\infty} \int_{\rho(x,y)\le 2^{-k}} 2^{kd} \left[2^{ks}|D_k(f)(y)|\right]^q \, d\mu(y) \right\}^{1/q}$$

for all $x \in X$. *Moreover, in this case,* $\|f\|_{b\dot{F}^s_{pq}(X)} \sim \|S^s_q(f)\|_{L^p(X)}$.

We now establish the frame characterizations of the Besov space $b\dot{B}^s_{pq}(X)$ and the Triebel-Lizorkin space $b\dot{F}^s_{pq}(X)$. To this end, we first introduce some spaces of sequences, $\dot{b}^s_{pq}(X)$ and $\dot{f}^s_{pq}(X)$. Let

$$\lambda = \left\{\lambda^{k,\nu}_\tau : k \in \mathbb{Z},\ \tau \in I_k,\ \nu = 1, \ldots, N(k,\tau)\right\} \tag{2.16}$$

be a sequence of complex numbers. The space $\dot{b}^s_{pq}(X)$ with $s \in \mathbb{R}$ and $0 < p, q \le \infty$ is the set of all λ as in (2.16) such that

$$\|\lambda\|_{\dot{b}^s_{pq}(X)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \left[\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \left|\lambda^{k,\nu}_\tau\right|^p \right]^{q/p} \right\}^{1/q} < \infty.$$

The space $\dot{f}^s_{pq}(X)$ with $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$ is the set of all λ as in (2.16) such that

$$\|\lambda\|_{\dot{f}^s_{pq}(X)} = \left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \left|\lambda^{k,\nu}_\tau\right|^q \chi_{Q^{k,\nu}_\tau} \right\}^{1/q} \right\|_{L^p(X)} < \infty.$$

THEOREM 2.15. *Let* b *be a para-accretive function as in Definition* 1.2, $\epsilon \in (0,\theta]$ *and* $|s| < \epsilon$. *With all the notation as in Lemma* 2.6, *let* λ *be a sequence of numbers as in* (2.16).

(i) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$, $0 < q \le \infty$ *and* $\|\lambda\|_{\dot{b}^s_{pq}(X)} < \infty$, *then the series*

$$\sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \lambda^{k,\nu}_\tau \int_{Q^{k,\nu}_\tau} b(x)\tilde{E}_k(y,x)b(y)\, d\mu(y) \tag{2.17}$$

converges to some $f \in b\dot{B}^s_{pq}(X)$ *both in the norm of* $b\dot{B}^s_{pq}(X)$ *and in* $(\mathring{\mathcal{G}}_b(\beta,\gamma))'$ *with*

$$\max\{0, -s + d(1/p - 1)_+\} < \beta < \epsilon \quad \textit{and} \quad \max\{0, s - d/p\} < \gamma < \epsilon \tag{2.18}$$

when $p,\ q < \infty$ *and only in* $(\mathring{\mathcal{G}}_b(\beta,\gamma))'$ *with* β *and* γ *as in* (2.18) *when* $\max(p,q) = \infty$. *Moreover, in all cases,* $\|f\|_{b\dot{B}^s_{pq}(X)} \le C\|\lambda\|_{\dot{b}^s_{pq}(X)}$.

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(ii) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$, $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$ *and* $\|\lambda\|_{\dot{f}^s_{pq}(X)} < \infty$, *then the series in* (2.17) *converges to some* $f \in b\dot{F}^s_{pq}(X)$ *both in the norm of* $b\dot{F}^s_{pq}(X)$ *and in* $(\mathring{\mathscr{G}}_b(\beta,\gamma))'$ *with* β, γ *as in* (2.18) *when* $p, q < \infty$ *and only in* $(\mathring{\mathscr{G}}_b(\beta,\gamma))'$ *with* β *and* γ *as in* (2.18) *when* $\max(p,q) = \infty$. *Moreover, in all cases,* $\|f\|_{b\dot{F}^s_{pq}(X)} \le C\|\lambda\|_{\dot{f}^s_{pq}(X)}$.

The proof of this theorem is similar to the proof of the frame characterizations of the Besov space $\dot{B}^s_{pq}(X)$ and the Triebel-Lizorkin space $\dot{F}^s_{pq}(X)$ in [48]; see also [23, 43]. We omit the details here.

From Lemma 2.6, Theorem 2.14 and the Plancherel-Pôlya inequalities, and Theorem 2.5, the frame characterizations of the spaces $b\dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X)$ follow.

THEOREM 2.16. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0,\theta]$ *and* $|s| < \theta$. *With all the notation as in Lemma* 2.6, *let* $\lambda^{k,\nu}_\tau = E_k(f)(y^{k,\nu}_\tau)$ *for* $k \in \mathbb{Z}$, $\tau \in I_k$ *and* $\nu = 1, \ldots, N(k,\tau)$, *where* $y^{k,\nu}_\tau$ *is any fixed element in* $Q^{k,\nu}_\tau$.

(i) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ *and* $0 < q \le \infty$, *then* $f \in b\dot{B}^s_{pq}(X)$ *if and only if* $f \in (\mathring{\mathscr{G}}_b(\beta,\gamma))'$ *for some* β, γ *as in* (2.13), (2.2) *holds in* $(\mathring{\mathscr{G}}_b(\beta',\gamma'))'$ *with* $\beta < \beta' < \epsilon$ *and* $\gamma < \gamma' < \epsilon$, *and* $\lambda \in \dot{b}^s_{pq}(X)$. *Moreover, in this case,* $\|f\|_{b\dot{B}^s_{pq}(X)} \sim \|\lambda\|_{\dot{b}^s_{pq}(X)}$.

(ii) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *then* $f \in b\dot{F}^s_{pq}(X)$ *if and only if* $f \in (\mathring{\mathscr{G}}_b(\beta,\gamma))'$ *for some* β, γ *as in* (2.13), (2.2) *holds in* $(\mathring{\mathscr{G}}_b(\beta',\gamma'))'$ *with* $\beta < \beta' < \epsilon$ *and* $\gamma < \gamma' < \epsilon$, *and* $\lambda \in \dot{f}^s_{pq}(X)$. *Moreover, in this case,* $\|f\|_{b\dot{F}^s_{pq}(X)} \sim \|\lambda\|_{\dot{f}^s_{pq}(X)}$.

3. Spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$

In this section, we introduce another new Besov space $b^{-1}\dot{B}^s_{pq}(X)$ and new Triebel-Lizorkin space $b^{-1}\dot{F}^s_{pq}(X)$, which are closely related to the Besov space $b\dot{B}^s_{pq}(X)$ and the Triebel-Lizorkin space $b\dot{F}^s_{pq}(X)$, respectively.

DEFINITION 3.1. Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0,\theta]$ and $|s| < \epsilon$. Let $\{S_k\}^b_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ as in Definition 1.4 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$.

(i) If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ and $0 < q \le \infty$, we define the space $b^{-1}\dot{B}^s_{pq}(X)$ to be the set of all $f \in (b\mathring{\mathscr{G}}_b(\beta,\gamma))'$ with β, γ as in (2.13) such that

$$\|f\|_{b^{-1}\dot{B}^s_{pq}(X)} = \left\{\sum_{k=-\infty}^{\infty} 2^{ksq}\|D_k(bf)\|^q_{L^p(X)}\right\}^{1/q} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

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(ii) If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, we define the space $b^{-1}\dot{F}^s_{pq}(X)$ to be the set of all $f \in (b\mathring{\mathcal{G}}_b(\beta,\gamma))'$ with β, γ as in (2.13) such that

$$\|f\|_{b^{-1}\dot{F}^s_{pq}(X)} = \left\| \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} |D_k(bf)|^q \right\}^{1/q} \right\|_{L^p(X)} < \infty$$

with the usual modification made when $q = \infty$.

To verify that Definition 3.1 is independent of the choice of the approximation to the identity, we first need to establish the following Plancherel-Pôlya inequality.

THEOREM 3.2. *Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$ and $|s| < \epsilon$. Let $\{S_k\}^b_{k\in\mathbb{Z}}$ and $\{G_k\}^b_{k\in\mathbb{Z}}$ be two approximations to the identity of order ϵ as in Definition 1.4, $D_k = S_k - S_{k-1}$ and $E_k = G_k - G_{k-1}$ for $k \in \mathbb{Z}$.*

(i) *If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ and $0 < q \le \infty$, there is a constant $C > 0$ such that for all $f \in (b\mathring{\mathcal{G}}_b(\beta,\gamma))'$ with $0 < \beta, \gamma < \epsilon$, we have*

$$\left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \left[\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \sup_{z\in Q^{k,\nu}_\tau} |D_k(bf)(z)|^p \right]^{q/p} \right\}^{1/q}$$
$$\le C \left\{ \sum_{k=-\infty}^{\infty} 2^{ksq} \left[\sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau) \inf_{z\in Q^{k,\nu}_\tau} |E_k(bf)(z)|^p \right]^{q/p} \right\}^{1/q}.$$

(ii) *If $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ and $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, there is a constant $C > 0$ such that for all $f \in (b\mathring{\mathcal{G}}_b(\beta,\gamma))'$ with $0 < \beta, \gamma < \epsilon$, we have*

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \sup_{z\in Q^{k,\nu}_\tau} |D_k(bf)(z)|^q \chi_{Q^{k,\nu}_\tau} \right\}^{1/q} \right\|_{L^p(X)}$$
$$\le C \left\| \left\{ \sum_{k=-\infty}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \inf_{z\in Q^{k,\nu}_\tau} |E_k(bf)(z)|^q \chi_{Q^{k,\nu}_\tau} \right\}^{1/q} \right\|_{L^p(X)}.$$

Theorem 3.2 can be proved in a way similar to Theorem 2.5, if we replace Lemma 2.6 by the following discrete Calderón reproducing formula in [17]. We omit the details.

LEMMA 3.3. *Let b be a para-accretive function as in Definition 1.2. Let $\epsilon \in (0, \theta]$, $\{G_k\}^b_{k\in\mathbb{Z}}$ be an approximation to the identity of order ϵ as in Definition 1.4, and*

$E_k = G_k - G_{k-1}$ for $k \in \mathbb{Z}$. Let $0 < \beta, \gamma < \epsilon$. Then there exists a family of functions $\{\tilde{E}_k(x, y)\}_{k\in\mathbb{Z}}$ such that for all fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ and all $f \in (b\mathring{\mathcal{G}}_b(\beta, \gamma))'$,

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} E_k(bf)(y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} \tilde{E}_k(y, x) b(y)\, d\mu(y), \tag{3.1}$$

where $\tilde{E}_k(x, y)$ for $k \in \mathbb{Z}$ satisfies (i) *and* (iii) *of Definition* 1.4 *with ϵ replaced by $\epsilon' \in (0, \epsilon)$, and*

$$\int_X \tilde{E}_k(x, y) b(x)\, d\mu(x) = \int_X \tilde{E}_k(x, y) b(y)\, d\mu(y) = 0.$$

Moreover the series in (3.1) *converges in the sense that for all $g \in b\mathring{\mathcal{G}}_b(\beta', \gamma')$ with $\beta < \beta' < \epsilon$ and $\gamma < \gamma' < \epsilon$,*

$$\lim_{M,\,N\to\infty} \left\langle \sum_{|k|\le M} \sum_{\substack{\tau\in I_k \\ \rho(x_0, z_\tau^k)\le N}} \sum_{\nu=1}^{N(k,\tau)} E_k(bf)(y_\tau^{k,\nu}) \int_{Q_\tau^{k,\nu}} \tilde{E}_k(y, \cdot) b(y)\, d\mu(y), g \right\rangle = \langle f, g\rangle.$$

By Lemma 3.3 and Theorem 3.2, we can also obtain a counterpart of Theorem 2.9 by the same procedure as in Theorem 2.9. Thus, Definition 3.1 is also independent of the choice of the distribution space $(b\mathring{\mathcal{G}}_b(\beta, \gamma))'$ with β, γ as in (2.13); and, therefore, Definition 3.1 is reasonable.

REMARK 3.4. To guarantee the definition of the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$ is independent of the choice of the distribution space $(b\mathring{\mathcal{G}}_b(\beta, \gamma))'$, we need only to restrict that $\max\{0, -s + d(1/p - 1)_+\} < \beta < \epsilon$ and $\max\{0, s - d/p\} < \gamma < \epsilon$; see the proof of Theorem 2.9. However, if we restrict $\max(0, s) < \beta < \epsilon$ and $\max\{d(1/p - 1)_+, -s + d(1/p - 1)\} < \gamma < \epsilon$, we can prove that the space of test functions, $\mathcal{G}_b(\beta, \gamma)$, is contained in the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$ in a way similar to that of Proposition 2.12. Thus, the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$ are non-empty if we restrict β, γ as in (2.13).

Moreover, using Lemma 3.3 and similar to proofs of Theorems 2.15 and 2.16, we can also obtain the frame characterizations of the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$. We will not give the details of these proofs.

THEOREM 3.5. *Let b be a para-accretive function as in Definition* 1.2, *$\epsilon \in (0, \theta]$ and $|s| < \epsilon$. Let all the notation be the same as in Lemma* 3.3 *and λ be a sequence of numbers as in* (2.16).

(i) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$, $0 < q \le \infty$ *and* $\|\lambda\|_{\dot{b}^s_{pq}(X)} < \infty$, *then the series*

$$\sum_{k=-\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \int_{Q_\tau^{k,\nu}} \tilde{E}_k(y, x) b(y)\, d\mu(y) \tag{3.2}$$

converges to some $f \in b^{-1}\dot{B}^s_{pq}(X)$ *both in the norm of* $b^{-1}\dot{B}^s_{pq}(X)$ *and in* $(b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *with*

$$\max\{0, -s + d(1/p - 1)_+\} < \beta < \epsilon \quad \text{and} \quad \max\{0, s - d/p\} < \gamma < \epsilon \tag{3.3}$$

when $p, q < \infty$ *and only in* $(b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *with* β *and* γ *as in* (3.3) *when* $\max(p, q) = \infty$. *Moreover, in all cases,* $\|f\|_{b^{-1}\dot{B}^s_{pq}(X)} \le C\|\lambda\|_{\dot{b}^s_{pq}(X)}$.

(ii) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$, $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *and* $\|\lambda\|_{\dot{f}^s_{pq}(X)} < \infty$, *then the series in* (3.2) *converges to some* $f \in b^{-1}\dot{F}^s_{pq}(X)$ *both in the norm of* $b^{-1}\dot{F}^s_{pq}(X)$ *and in* $(b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *with* β, γ *as in* (3.3) *when* $p, q < \infty$ *and only in* $(b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *with* β *and* γ *as in* (3.3) *when* $\max(p, q) = \infty$. *Moreover, in all cases,* $\|f\|_{b^{-1}\dot{F}^s_{pq}(X)} \le C\|\lambda\|_{\dot{f}^s_{pq}(X)}$.

THEOREM 3.6. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$ *and* $|s| < \theta$. *With all the notation as in Lemma* 3.3, *let* $\lambda_\tau^{k,\nu} = E_k(bf)(y_\tau^{k,\nu})$ *for* $k \in \mathbb{Z}$, $\tau \in I_k$ *and* $\nu = 1, \ldots, N(k, \tau)$, *where* $y_\tau^{k,\nu}$ *is any fixed element in* $Q_\tau^{k,\nu}$.

(i) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ *and* $0 < q \le \infty$, *then* $f \in b^{-1}\dot{B}^s_{pq}(X)$ *if and only if* $f \in (b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *for some* β, γ *as in* (2.13), (3.1) *holds in* $(b\mathring{\mathscr{G}}_b(\beta', \gamma'))'$ *with* $\beta < \beta' < \epsilon$ *and* $\gamma < \gamma' < \epsilon$, *and* $\lambda \in \dot{b}^s_{pq}(X)$. *Moreover, in this case,* $\|f\|_{b^{-1}\dot{B}^s_{pq}(X)} \sim \|\lambda\|_{\dot{b}^s_{pq}(X)}$.

(ii) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *then* $f \in b^{-1}\dot{F}^s_{pq}(X)$ *if and only if* $f \in (b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *for some* β, γ *as in* (2.13), (3.1) *holds in* $(b\mathring{\mathscr{G}}_b(\beta', \gamma'))'$ *with* $\beta < \beta' < \epsilon$ *and* $\gamma < \gamma' < \epsilon$, *and* $\lambda \in \dot{f}^s_{pq}(X)$. *Moreover, in this case,* $\|f\|_{b^{-1}\dot{F}^s_{pq}(X)} \sim \|\lambda\|_{\dot{f}^s_{pq}(X)}$.

We also have the characterization of the Littlewood-Paley S-function of the Triebel-Lizorkin space $b^{-1}\dot{F}^s_{pq}(X)$, which can be proved in a way similar to Theorem 2.14.

THEOREM 3.7. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$ *and* $|s| < \epsilon$. *Let* $\{S_k\}^b_{k\in\mathbb{Z}}$ *be an approximation to the identity of order* ϵ *as in Definition* 1.4 *and* $D_k = S_k - S_{k-1}$ *for* $k \in \mathbb{Z}$. *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *then* $f \in b^{-1}\dot{F}^s_{pq}(X)$ *if and only if* $f \in (b\mathring{\mathscr{G}}_b(\beta, \gamma))'$ *with* β, γ *as in* (2.13) *and* $S^s_{q,b}(f) \in L^p(X)$, *where*

$$S^s_{q,b}(f)(x) = \left\{ \sum_{k=-\infty}^{\infty} \int_{\rho(x,y) \le 2^{-k}} 2^{kd} \left[2^{ks}|D_k(bf)(y)|\right]^q d\mu(y) \right\}^{1/q}$$

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for all $x \in X$. *Moreover, in this case,* $\|f\|_{b^{-1}\dot{F}^s_{pq}(X)} \sim \|S^s_{q,b}(f)\|_{L^p(X)}$.

From Definition 2.10, Definition 3.1 and (1.3), it is easy to deduce the below relations between the space $b\dot{B}^s_{pq}(X)$ and the space $b^{-1}\dot{B}^s_{pq}(X)$, and between the space $b\dot{F}^s_{pq}(X)$ and the space $b^{-1}\dot{F}^s_{pq}(X)$.

On the other hand, for $d/(d+\epsilon) < p \le 1$, let $H^p(X)$ be the Hardy spaces studied by Macías and Segovia in [28]. We define the Hardy space $H^p_b(X)$ to be the set of all $f \in (b\mathring{\mathcal{G}}_b(\beta,\gamma))'$ with $d(1/p-1)_+ < \beta, \gamma < \epsilon$ such that $bf \in H^p(X)$. Moreover, we define $\|f\|_{H^p_b(X)} = \|bf\|_{H^p(X)}$. When $X = \mathbb{R}^n$, $H^1_b(X)$ was first introduced by Meyer in [31] and $H^p_b(X)$ was first introduced by Han, Lee and Lin in [17]. Based on the results in [16, 17], we can easily obtain the relation between the Hardy spaces $H^p_b(X)$ and the Triebel-Lizorkin spaces $b^{-1}\dot{F}^s_{pq}(X)$.

PROPOSITION 3.8. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0,\theta]$ *and* $|s| < \epsilon$.

(i) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p \le \infty$ *and* $0 < q \le \infty$, *then* $f \in b^{-1}\dot{B}^s_{pq}(X)$ *if and only if* $bf \in b\dot{B}^s_{pq}(X)$. *Moreover, in this case,*

$$\|f\|_{b^{-1}\dot{B}^s_{pq}(X)} = \|bf\|_{b\dot{B}^s_{pq}(X)}.$$

(ii) *If* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+s+\epsilon)\} < q \le \infty$, *then* $f \in b^{-1}\dot{F}^s_{pq}(X)$ *if and only if* $bf \in b\dot{F}^s_{pq}(X)$. *Moreover, in this case,*

$$\|f\|_{b^{-1}\dot{B}^s_{pq}(X)} = \|bf\|_{b\dot{B}^s_{pq}(X)}.$$

(iii) *If* $1 < p < \infty$, *then* $b^{-1}\dot{F}^0_{p2}(X) = L^p(X)$ *with an equivalent norm; and if* $d/(d+\epsilon) < p \le 1$, *then* $b^{-1}\dot{F}^0_{p2}(X) = H^p_b(X)$ *with an equivalent norm.*

4. Some applications

We first consider real interpolations of the spaces $b\dot{B}^s_{pq}(X)$, $b\dot{F}^s_{pq}(X)$, $b^{-1}\dot{B}^s_{pq}(X)$, and $b^{-1}\dot{F}^s_{pq}(X)$. Let us now recall the general background of the real interpolation method; see [1] and [38, pages 62–64].

Let $\mathcal{H}$ be a linear complex Hausdorff space, and let $\mathcal{A}_0$ and $\mathcal{A}_1$ be two complex quasi-Banach spaces such that $\mathcal{A}_0 \subset \mathcal{H}$ and $\mathcal{A}_1 \subset \mathcal{H}$. Let $\mathcal{A}_0 + \mathcal{A}_1$ be the set of all elements $a \in \mathcal{H}$ which can be represented as $a = a_0 + a_1$ with $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$. If $0 < t < \infty$ and $a \in \mathcal{A}_0 + \mathcal{A}_1$, then Peetre's celebrated K-functional is given by

$$K(t,a) = K(t,a;\mathcal{A}_0,\mathcal{A}_1) = \inf\left(\|a_0\|_{\mathcal{A}_0} + t\|a_1\|_{\mathcal{A}_1}\right),$$

where the infimum is taken over all representations of a having the form $a = a_0 + a_1$ with $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$.

DEFINITION 4.1. Let $0 < \sigma < 1$. If $0 < q < \infty$, then

$$(\mathscr{A}_0, \mathscr{A}_1)_{\sigma,q} = \left\{ a : a \in \mathscr{A}_0 + \mathscr{A}_1,\ \|a\|_{(\mathscr{A}_0,\mathscr{A}_1)_{\sigma,q}} = \left\{ \int_0^\infty \left[t^{-\sigma} K(t,a)\right]^q \frac{dt}{t} \right\}^{1/q} < \infty \right\}.$$

If $q = \infty$, then

$$(\mathscr{A}_0, \mathscr{A}_1)_{\sigma,\infty} = \left\{ a : a \in \mathscr{A}_0 + \mathscr{A}_1,\ \|a\|_{(\mathscr{A}_0,\mathscr{A}_1)_{\sigma,\infty}} = \sup_{0<t<\infty} t^{-\sigma} K(t,a) < \infty \right\}.$$

Using Lemma 2.6, Theorems 2.15–2.16, Lemma 3.3, Theorems 3.5–3.6, and the method of retraction and coretraction as in the proofs of [37, Theorem 2.4.1 and Theorem 2.4.2], we obtain theorems on the real interpolations of the spaces $b\dot{B}^s_{pq}(X)$, $b\dot{F}^s_{pq}(X)$, $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$. We omit the details; see also the proofs of [43, Theorem 3.2 and Theorem 3.3].

THEOREM 4.2. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$ *and* $\sigma \in (0, 1)$.

(i) *Let* $-\epsilon < s_0, s_1 < \epsilon$, $s_0 \neq s_1$, $1 \le p \le \infty$, *and* $0 < q_0, q_1, q \le \infty$. *Then*

$$\left(b\dot{B}^{s_0}_{p,q_0}(X), b\dot{B}^{s_1}_{p,q_1}(X)\right)_{\sigma,q} = b\dot{B}^s_{pq}(X)$$

and

$$\left(b^{-1}\dot{B}^{s_0}_{p,q_0}(X), b^{-1}\dot{B}^{s_1}_{p,q_1}(X)\right)_{\sigma,q} = b^{-1}\dot{B}^s_{pq}(X),$$

where $s = (1-\sigma)s_0 + \sigma s_1$.

(ii) *Let* $-\epsilon < s < \epsilon$, $1 \le p \le \infty$, $0 < q_0, q_1 \le \infty$, *and* $q_0 \neq q_1$. *Then*

$$\left(b\dot{B}^{s}_{p,q_0}(X), b\dot{B}^{s}_{p,q_1}(X)\right)_{\sigma,q} = b\dot{B}^s_{pq}(X)$$

and

$$\left(b^{-1}\dot{B}^{s}_{p,q_0}(X), b^{-1}\dot{B}^{s}_{p,q_1}(X)\right)_{\sigma,q} = b^{-1}\dot{B}^s_{pq}(X),$$

where $1/q = (1-\sigma)/q_0 + \sigma/q_1$.

(iii) *Let* $-\epsilon < s_0, s_1 < \epsilon$ *and* $1 \le p_0, p_1 \le \infty$. *Then*

$$\left(b\dot{B}^{s_0}_{p_0,p_0}(X), b\dot{B}^{s_1}_{p_1,p_1}(X)\right)_{\sigma,p} = b\dot{B}^s_{p,p}(X)$$

and

$$\left(b^{-1}\dot{B}^{s_0}_{p_0,p_0}(X), b^{-1}\dot{B}^{s_1}_{p_1,p_1}(X)\right)_{\sigma,p} = b^{-1}\dot{B}^s_{p,p}(X),$$

where $1/p = (1-\sigma)/p_0 + \sigma/p_1$.

THEOREM 4.3. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$ *and* $\sigma \in (0, 1)$. *Let* $-\epsilon < s_0, s_1 < \epsilon$, $\max(d/(d+\epsilon), d/(d+\epsilon+s_0)) < p_0 < \infty$, $\max(d/(d+\epsilon), d/(d+\epsilon+s_1)) < p_1 < \infty$, $1 \le q_0, q_1 \le \infty$, $s = (1-\sigma)s_0 + \sigma s_1$, $1/p = (1-\sigma)/p_0 + \sigma/p_1$, *and* $1/q = (1-\sigma)/q_0 + \sigma/q_1$.

(i) *If* $s_0 \neq s_1$, *then*

$$\left(b\dot{F}^{s_0}_{p_0,q_0}(X), b\dot{F}^{s_1}_{p_1,q_1}(X)\right)_{\sigma,p} = b\dot{B}^{s}_{p,p}(X)\left(= b\dot{F}^{s}_{p,p}(X)\right)$$

and

$$\left(b^{-1}\dot{F}^{s_0}_{p_0,q_0}(X), b^{-1}\dot{F}^{s_1}_{p_1,q_1}(X)\right)_{\sigma,p} = b^{-1}\dot{B}^{s}_{p,p}(X)\left(= b^{-1}\dot{F}^{s}_{p,p}(X)\right).$$

(ii) *If* $s_0 = s_1 = s$, $p_0 = q_0$, $p_1 = q_1$, *and* $q_0 \neq q_1$, *then*

$$\left(b\dot{F}^{s}_{p_0,p_0}(X), b\dot{F}^{s}_{p_1,p_1}(X)\right)_{\sigma,p} = b\dot{B}^{s}_{p,p}(X)$$

and

$$\left(b^{-1}\dot{F}^{s}_{p_0,p_0}(X), b^{-1}\dot{F}^{s}_{p_1,p_1}(X)\right)_{\sigma,p} = b^{-1}\dot{B}^{s}_{p,p}(X).$$

(iii) *If* $s_0 = s_1 = s$, $q_0 = q_1 = q$, *and* $p_0 \neq p_1$, *then*

$$\left(b\dot{F}^{s}_{p_0,q}(X), b\dot{F}^{s}_{p_1,q}(X)\right)_{\sigma,p} = b\dot{F}^{s}_{pq}(X)$$

and

$$\left(b^{-1}\dot{F}^{s}_{p_0,q}(X), b^{-1}\dot{F}^{s}_{p_1,q}(X)\right)_{\sigma,p} = b^{-1}\dot{F}^{s}_{pq}(X).$$

Moreover, by Lemma 2.6, Lemma 3.3, and some similar computations to the proof of [47, Theorem 2.1], we can further establish the following general interpolation theorem; see also [37, Theorem 2.4.2].

THEOREM 4.4. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$, $\sigma \in (0, 1)$, $s_0, s_1 \in (-\epsilon, \epsilon)$, $s_0 \neq s_1$, *and* $s = (1-\sigma)s_0 + \sigma s_1$.

(i) *If* $\max(d/(d+\epsilon), d/(d+s_0+\epsilon), d/(d+s_1+\epsilon)) < p \leq \infty$ *and* $0 < q_0, q_1, q \leq \infty$, *then*

$$\left(b\dot{B}^{s_0}_{p,q_0}(X), b\dot{B}^{s_1}_{p,q_1}(X)\right)_{\sigma,q} = b\dot{B}^{s}_{pq}(X)$$

and

$$\left(b^{-1}\dot{B}^{s_0}_{p,q_0}(X), b^{-1}\dot{B}^{s_1}_{p,q_1}(X)\right)_{\sigma,q} = b^{-1}\dot{B}^{s}_{pq}(X).$$

(ii) *If* $\max(d/(d+\epsilon), d/(d+s_0+\epsilon), d/(d+s_1+\epsilon)) < p < \infty$, $\max(d/(d+\epsilon), d/(d+s_i+\epsilon)) < q_i \leq \infty$ *for* $i = 0, 1$, *and* $0 < q \leq \infty$, *then*

$$\left(b\dot{F}^{s_0}_{p,q_0}(X), b\dot{F}^{s_1}_{p,q_1}(X)\right)_{\sigma,q} = b\dot{B}^{s}_{pq}(X)$$

and

$$\left(b^{-1}\dot{F}^{s_0}_{p,q_0}(X), b^{-1}\dot{F}^{s_1}_{p,q_1}(X)\right)_{\sigma,q} = b^{-1}\dot{B}^{s}_{pq}(X).$$

By Lemma 2.6 and Lemma 3.3 with the estimate (2.4), and the same argument as in [18] (see also [15, 23, 44]), we can also obtain the following embedding theorem. In the sequel, for two quasi-Banach spaces $\mathscr{A}_1$ and $\mathscr{A}_2$, $\mathscr{A}_1 \subset \mathscr{A}_2$ means a linear and continuous embedding.

THEOREM 4.5. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$ *and* $-\epsilon < s_2 < s_1 < \epsilon$.

(i) *If* $0 < q \le \infty$, $\max(d/(d+\epsilon), d/(d+\epsilon+s_i)) < p_i \le \infty$ *for* $i = 1, 2$, *and* $s_1 - d/p_1 = s_2 - d/p_2$, *then* $b\dot{B}^{s_1}_{p_1,q}(X) \subset b\dot{B}^{s_2}_{p_2,q}(X)$ *and* $b^{-1}\dot{B}^{s_1}_{p_1,q}(X) \subset b^{-1}\dot{B}^{s_2}_{p_2,q}(X)$;

(ii) *If* $\max(d/(d+\epsilon), d/(d+\epsilon+s_i)) < p_i < \infty$ *and* $\max(d/(d+\epsilon), d/(d+\epsilon+s_i)) < q_i \le \infty$ *for* $i = 1, 2$, *and* $s_1 - d/p_1 = s_2 - d/p_2$, *then* $b\dot{F}^{s_1}_{p_1,q_1}(X) \subset b\dot{F}^{s_2}_{p_2,q_2}(X)$ *and* $b^{-1}\dot{F}^{s_1}_{p_1,q_1}(X) \subset b^{-1}\dot{F}^{s_2}_{p_2,q_2}(X)$.

REMARK 4.6. In [15, 18, 23, 44], all the results similar to Theorem 4.5 asked that $-\epsilon < s_1 - d/p_1 = s_2 - d/p_2 < \epsilon$. However, $s_2 - d/p_2 < \epsilon$ is automatically true since $s_2 < \epsilon$. A careful check of their proofs shows that one has not used the condition $-\epsilon < s_1 - d/p_1$.

We now turn to consider the Tb theorems on the spaces $b\dot{B}^s_{pq}(X)$, $b\dot{F}^s_{pq}(X)$, $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$. We first recall some notation. In what follows, for $\eta \in (0, \theta]$, we let $C^\eta_0(X)$ be the set of all functions having compact support such that

$$\|f\|_{C^\eta_0(X)} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\eta} < \infty.$$

Endow $C^\eta_0(X)$ with the natural topology and let $(C^\eta_0(X))'$ be its dual space. Moreover, if b is a para-accretive function as in Definition 1.2, in what follows, we will use M_b to denote the corresponding multiplication operator and $bC^\eta_0(X)$ to denote the image of $C^\eta_0(X)$ under M_b with the natural topology. This means that $f \in bC^\eta_0(X)$ if and only if $f = bg$ for some $g \in C^\eta_0(X)$ and we define $\|f\|_{bC^\eta_0(X)} = \|g\|_{C^\eta_0(X)}$.

Let b_1 and b_2 be two para-accretive functions as in Definition 1.2. A continuous complex-valued function $K(x, y)$ on $\Omega = \{(x, y) \in X \times X : x \neq y\}$ is called a *Calderón-Zygmund kernel of type* ϵ if there exist $\epsilon \in (0, \theta]$ and $C_6 > 0$ such that for $\rho(x, y) \neq 0$,

$$|K(x, y)| \le C_6 \rho(x, y)^{-d}, \tag{4.1}$$

and for $\rho(x, x') \le \rho(x, y)/(2A)$,

$$|K(x, y) - K(x', y)| \le C_6 \rho(x, x')^\epsilon \rho(x, y)^{-d-\epsilon}, \tag{4.2}$$

and for $\rho(y, y') \le \rho(x, y)/(2A)$,

$$|K(x, y) - K(x, y')| \le C_6 \rho(y, y')^\epsilon \rho(x, y)^{-d-\epsilon}; \tag{4.3}$$

and a continuous linear operator $T : b_1C^\eta_0(X) \to (b_2C^\eta_0(X))'$ is a *Calderón-Zygmund singular integral operator of type* ϵ if there is a Calderón-Zygmund kernel $K(x, y)$ of type ϵ such that

$$\langle Tf, g\rangle = \int_X \int_X g(x) b_2(x) K(x, y) b_1(y) f(y)\, d\mu(x)\, d\mu(y)$$

for all $f, g \in C_0^\eta(X)$ with disjoint supports. Moreover, a Calderón-Zygmund singular integral operator T is said to have the weak boundedness property, if there exist $\eta \in (0, \theta]$ and $C_7 > 0$ such that

$$|\langle Tf, g\rangle| \leq C_7 r^{d+2\eta} \|f\|_{C_0^\eta(X)} \|g\|_{C_0^\eta(X)}$$

for all $f, g \in C_0^\eta(X)$ with $\operatorname{diam}(\operatorname{supp} f) \leq r$ and $\operatorname{diam}(\operatorname{supp} g) \leq r$, and we denote this by $T \in WBP(X)$.

THEOREM 4.7. *Let b be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$, *and* $|s| < \epsilon$. *Suppose T is a Calderón-Zygmund singular integral operator of type ϵ, $T(b) = 0 = T^*(b)$, $M_b T M_b \in WBP$, and its kernel $K(x, y)$ satisfies* (4.1), (4.2), *and* (4.3).

(i) *If* $\max(d/(d+\epsilon), d/(d+s+\epsilon)) < p \leq \infty$ *and* $0 < q \leq \infty$, *then T is bounded from $b\dot{B}^s_{pq}(X)$ to $b^{-1}\dot{B}^s_{pq}(X)$ with an operator norm not larger than* $C\max(C_6, C_7)$;

(ii) *If* $\max(d/(d+\epsilon), d/(d+s+\epsilon)) < p < \infty$ *and* $\max(d/(d+\epsilon), d/(d+s+\epsilon)) < q \leq \infty$, *then T is bounded from $b\dot{F}^s_{pq}(X)$ to $b^{-1}\dot{F}^s_{pq}(X)$ with an operator norm not larger than* $C\max(C_6, C_7)$.

PROOF. We only give an outline of the proof. Let $\{D_j\}_{j\in\mathbb{Z}}$ be as in Definition 2.10. With all the notation as in Lemma 2.6, under the assumptions of the theorem, we can verify that for all $j, k \in \mathbb{Z}$ and all $x, y \in X$,

$$\left|\left[D_j M_b T M_b \tilde{E}_k(y, \cdot)\right](x)\right| \leq C 2^{-|k-j|\epsilon'} \frac{2^{-(k\wedge j)\epsilon'}}{(2^{-(k\wedge j)} + \rho(x, y))^{d+\epsilon'}}, \tag{4.4}$$

where ϵ' can be any positive number in $(0, \epsilon)$; see [14, Lemma 3.13] and [46, Lemma 2.3] for details. From the estimate (4.4), Lemma 2.6 and Lemma 2.7, and by an argument similar to the proof of Theorem 2.5, we can prove (ii).

The estimate (4.4) and some trivial computation also lead to the conclusion (i) with $p = q = \infty$. This, together with (ii) and Theorem 4.4, will then give (i); see also [46] for details. This completes the proof of Theorem 4.7. □

Finally, we consider the boundedness of Riesz potentials on the spaces $b\dot{B}^s_{pq}(X)$, $b\dot{F}^s_{pq}(X)$, $b^{-1}\dot{B}^s_{pq}(X)$, and $b^{-1}\dot{F}^s_{pq}(X)$.

DEFINITION 4.8. Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$, $\{D_k\}_{k\in\mathbb{Z}}$ be the same as in Definition 2.10, and $\alpha \in \mathbb{R}$. Then the Riesz operator I_α for $f \in \mathcal{G}_b(\beta, \gamma)$ with $0 < \beta, \gamma < \epsilon$ is defined by $I_\alpha(f)(x) = \sum_{l=-\infty}^{\infty} D_l(f)(x)$ for all $x \in X$.

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Obviously, when $\alpha > 0$ and $b \equiv 1$, I_α is the discrete version of the fractional integrals introduced in [9–11]; while when $\alpha < 0$ and $b \equiv 1$, I_α is the discrete version of the fractional derivatives introduced there. When $\alpha = 0$ and $b \equiv 1$, I_α is just the identity. We also mention that in [32, 33], Nahmod considered some discrete and inhomogeneous fractional integrals and derivatives similar to those above.

THEOREM 4.9. *Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$, $|\alpha| < \epsilon$, $|s| < \epsilon$, and $|s + \alpha| < \epsilon$.*

(i) *If $\max\{d/(d+\epsilon), d/(d+\epsilon+s+\alpha\wedge 0), d/(d+\epsilon-\alpha)\} < p \le \infty$, and $0 < q \le \infty$, then I_α is bounded from $b\dot{B}^s_{pq}(X)$ into $b^{-1}\dot{B}^{s+\alpha}_{pq}(X)$.*

(ii) *If $\max\{d/(d+\epsilon), d/(d+\epsilon+s+\alpha\wedge 0), d/(d+\epsilon-\alpha)\} < p < \infty$, and $\max\{d/(d+\epsilon), d/(d+\epsilon+s+\alpha\wedge 0), d/(d+\epsilon-\alpha))\} < q \le \infty$, then I_α is bounded from $b\dot{F}^s_{pq}(X)$ into $b^{-1}\dot{F}^{s+\alpha}_{pq}(X)$.*

PROOF. We only give an outline; see [45, 46] for details. Let $\{D_k\}_{k\in\mathbb{Z}}$ be as in Definition 2.10. With all the notation as in Lemma 2.6, under the assumptions of the theorem, we can verify that for all $k, k' \in \mathbb{Z}$ and all $x, y \in X$,

$$\left|\left[D_k M_b I_\alpha M_b \tilde{E}_{k'}(y,\cdot)\right](x)\right| \le C\,2^{-(k\wedge k')\alpha}2^{-|k-k'|(\epsilon'+\alpha\wedge 0)}\frac{2^{-(k\wedge k')(\epsilon'-\alpha)}}{(2^{-(k\wedge k')}+\rho(x,y))^{d+\epsilon'-\alpha}}, \tag{4.5}$$

where ϵ' can be any positive number in $(0, \epsilon)$; see [45, Lemma 2] for details. The estimate (4.5), Lemma 2.6, and an argument similar to the proof of Theorem 2.5 yield (ii).

From the estimate (4.5) and some trivial computation, the conclusion (i) with $p = q = \infty$ can be deduced. This, together with (ii) and Theorem 4.4 will then give (i); see also [45] for details. This completes the proof of Theorem 4.9. □

REMARK 4.10. Theorem 1 and Theorem 2 in [45] also ask that $s < \theta + \alpha\wedge 0$. This is superfluous and the mistake is caused by the factor $2^{-k\alpha}$ in (2.1) there, which should be $2^{-(k\wedge k')\alpha}$ as in (4.5).

From Theorem 4.5 and Theorem 4.9, we can deduce the following interesting conclusion.

COROLLARY 4.11. *Let b be a para-accretive function as in Definition 1.2, $\epsilon \in (0, \theta]$, $0 \le \alpha < \epsilon$ and $|s| < \epsilon$.*

(i) *If $\max\{d/(d+\epsilon), d/(d+\epsilon+s), d/(d+\epsilon-\alpha)\} < p_1 \le \infty$, $0 < q \le \infty$, and $1/p_2 = 1/p_1 - \alpha/d$, then I_α is bounded from $b\dot{B}^s_{p_1,q}(X)$ into $b^{-1}\dot{B}^s_{p_2,q}(X)$.*

(ii) *If* $\max\{d/(d+\epsilon), d/(d+\epsilon+s), d/(d+\epsilon-\alpha)\} < p_1 < \infty$, $\max\{d/(d+\epsilon), d/(d+\epsilon+s), d/(d+\epsilon-\alpha)\} < q \le \infty$, *and* $1/p_2 = 1/p_1 - \alpha/d$, *then* I_α *is bounded from* $b\dot{F}^s_{p_1,q}(X)$ *into* $b^{-1}\dot{F}^s_{p_2,q}(X)$.

We remark that Corollary 4.11 (ii) is specially interesting by noting Proposition 2.12 (iii) and Proposition 3.8 (iii). It means that I_α is bounded from the Hardy space $H^{p_1}(X)$ into the Hardy space $H_b^{p_2}(X)$, where p_1 and p_2 are as in Corollary 4.11 (ii).

Using Theorem 4.7, we can also establish the converse of Theorem 4.9.

THEOREM 4.12. *Let* b *be a para-accretive function as in Definition* 1.2, $\epsilon \in (0, \theta]$, $|\alpha| < \epsilon$, $|s| < \epsilon$ *and* $|s+\alpha| < \epsilon$.

(i) *If* $\max\{d/(d+\epsilon), d/(d+\epsilon+\alpha), d/(d+\epsilon+s-|\alpha|)\} < p \le \infty$ *and* $0 < q \le \infty$, *then there are* $\alpha_0(s) \in (0, \epsilon)$ *and a constant* $C > 0$ *such that if* $|\alpha| < \alpha_0(s)$, *for all* $f \in b\dot{B}^s_{pq}(X)$, $\|f\|_{b\dot{B}^s_{pq}(X)} \le C\|I_\alpha(f)\|_{b^{-1}\dot{B}^{s+\alpha}_{pq}(X)}$.

(ii) *If* $\max\{d/(d+\epsilon), d/(d+\epsilon+\alpha), d/(d+\epsilon+s-|\alpha|)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+\epsilon+\alpha), d/(d+\epsilon+s-|\alpha|)\} < q \le \infty$, *then there are* $\alpha_0(s) \in (0, \epsilon)$ *and a constant* $C > 0$ *such that if* $|\alpha| < \alpha_0(s)$, *for all* $f \in b\dot{F}^s_{pq}(X)$, $\|f\|_{b\dot{F}^s_{pq}(X)} \le C\|I_\alpha(f)\|_{b^{-1}\dot{F}^{s+\alpha}_{pq}(X)}$.

PROOF. We only give an outline of the proof. The key of the proof is to verify that the operator $I_{-\alpha}M_bI_\alpha M_b$ is invertible in the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$, respectively. To this end, we need to show that the operator $T = I - I_{-\alpha}M_bI_\alpha M_b$ is bounded on the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$ with an operator norm less than 1 when α is small. We show this by using Theorem 4.7. In fact, by using Coifman's idea in [6], for any given $N \in \mathbb{N}$, we write

$$\begin{aligned} T = I - I_{-\alpha}M_bI_\alpha M_b &= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}(1-2^{-l\alpha})D_kM_bD_{k+l}M_b \\ &= \left\{\sum_{k=-\infty}^{\infty}\sum_{|l|\le N}(1-2^{-l\alpha})D_kM_bD_{k+l} + \sum_{k=-\infty}^{\infty}\sum_{|l|>N}(1-2^{-l\alpha})D_kM_bD_{k+l}\right\}M_b = \tilde{T}M_b. \end{aligned}$$

It is easy to see that $\tilde{T}(b) = 0 = \tilde{T}^*(b)$. For any $\epsilon' \in (0, \epsilon)$, all $k, l \in \mathbb{Z}$ and all $x, y \in X$, recall

$$|D_kM_bD_{k+l}(x, y)| \le C\,2^{-|l|\epsilon'}\frac{2^{-[k\wedge(k+l)]\epsilon}}{(2^{-[k\wedge(k+l)]} + \rho(x, y))^{d+\epsilon}}, \tag{4.6}$$

and if $\rho(y, y') \le (1/4A^2)\rho(x, y)$, for all $\sigma \in (0, 1)$,

$$\begin{aligned} &\left|D_kM_bD_{k+l}(x, y) - D_kM_bD_{k+l}(x, y')\right| \\ &\quad + \left|D_kM_bD_{k+l}(y, x) - D_kM_bD_{k+l}(y', x)\right| \end{aligned} \tag{4.7}$$

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$$\leq C\, 2^{-|l|\epsilon'\sigma} \left(\frac{\rho(y, y')}{2^{-[k\wedge(k+l)]} + \rho(x, y)} \right)^{(1-\sigma)\epsilon} \frac{2^{-[k\wedge(k+l)]\epsilon}}{(2^{-[k\wedge(k+l)]} + \rho(x, y))^{d+\epsilon}};$$

see [14] for details.

From the estimates (4.6) and (4.7), we can verify that $\tilde{T}$ is a Calderón-Zygmund singular integral operator of type $(1-\sigma)\epsilon$ with

$$C_6, C_7 \leq C_8 \sum_{|l| \leq N} \left|1 - 2^{-l\alpha}\right| 2^{-|l|\sigma\epsilon'} + C_9 2^{-\delta N},$$

where $\delta = \min\{\sigma\epsilon + \alpha, \sigma\epsilon - \alpha\}$, C_8 is independent of α and N, C_9 is independent of N, and if $|\alpha| < \alpha_1$, where $\alpha_1 > 0$ (and its value will be chosen later), then C_9 is also independent of α, but it may depend on α_1.

By Theorem 4.7, we know that $\tilde{T}$ is bounded from $b\dot{B}^s_{pq}(X)$ to $b^{-1}\dot{B}^s_{pq}(X)$ and from $b\dot{F}^s_{pq}(X)$ to $b^{-1}\dot{F}^s_{pq}(X)$ with an operator norm no more than $C_{10} = C\max\{C_6, C_7\}$. Note that M_b is bounded from $b^{-1}\dot{B}^s_{pq}(X)$ to $b\dot{B}^s_{pq}(X)$, and from $b^{-1}\dot{F}^s_{pq}(X)$ to $b\dot{F}^s_{pq}(X)$ with an operator norm to be 1. Thus, T is bounded on $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$ with an operator norm no more than C_{10}. Now, if we choose α_1 small enough and if $|\alpha| < \alpha_1$, then $I_{-\alpha}M_bI_\alpha M_b$ is invertible in the spaces $b^{-1}\dot{B}^s_{pq}(X)$ and $b^{-1}\dot{F}^s_{pq}(X)$. Thus, by Theorem 4.9, we have that for all $f \in b^{-1}\dot{B}^s_{pq}(X)$,

$$\begin{aligned} \|f\|_{b^{-1}\dot{B}^s_{pq}(X)} &= \left\|(I_{-\alpha}M_bI_\alpha M_b)^{-1}\, I_{-\alpha}M_bI_\alpha M_b(f)\right\|_{b^{-1}\dot{B}^s_{pq}(X)} \\ &\leq C\, \|I_{-\alpha}M_bI_\alpha M_b(f)\|_{b^{-1}\dot{B}^s_{pq}(X)} \\ &\leq C\, \|M_bI_\alpha M_b(f)\|_{b\dot{B}^{s+\alpha}_{pq}(X)} \\ &\leq C\, \|I_\alpha M_b(f)\|_{b^{-1}\dot{B}^{s+\alpha}_{pq}(X)}, \end{aligned}$$

and for all $f \in b^{-1}\dot{F}^s_{pq}(X)$,

$$\begin{aligned} \|f\|_{b^{-1}\dot{F}^s_{pq}(X)} &= \left\|(I_{-\alpha}M_bI_\alpha M_b)^{-1}\, I_{-\alpha}M_bI_\alpha M_b(f)\right\|_{b^{-1}\dot{F}^s_{pq}(X)} \\ &\leq C\, \|I_{-\alpha}M_bI_\alpha M_b(f)\|_{b^{-1}\dot{F}^s_{pq}(X)} \\ &\leq C\, \|M_bI_\alpha M_b(f)\|_{b\dot{F}^{s+\alpha}_{pq}(X)} \\ &\leq C\, \|I_\alpha M_b(f)\|_{b^{-1}\dot{F}^{s+\alpha}_{pq}(X)}. \end{aligned}$$

Proposition 3.8 then tells us the conclusion of the theorem. □

By combining Theorem 4.9 with Theorem 4.12, we obtain the following simple conclusion.

COROLLARY 4.13. *Let* $|\alpha| < \epsilon$, $|s| < \epsilon$, *and* $|s + \alpha| < \epsilon$.

(i) *If* $\max\{d/(d+\epsilon), d/(d+\epsilon-|\alpha|), d/(d+\epsilon+s-|\alpha|)\} < p \le \infty$ *and* $0 < q \le \infty$, *then there is an* $\alpha_0(s) \in (0,\epsilon)$ *such that if* $|\alpha| < \alpha_0(s)$, *for all* $f \in b\dot{B}^s_{pq}(X)$, $\|f\|_{b\dot{B}^s_{pq}(X)} \sim \|I_\alpha(f)\|_{b^{-1}\dot{B}^{s+\alpha}_{pq}(X)}$.

(ii) *If* $\max\{d/(d+\epsilon), d/(d+\epsilon-|\alpha|), d/(d+\epsilon+s-|\alpha|)\} < p < \infty$ *and* $\max\{d/(d+\epsilon), d/(d+\epsilon+\alpha), d/(d+\epsilon+s-|\alpha|)\} < q \le \infty$, *then there is an* $\alpha_0(s) \in (0,\epsilon)$ *such that if* $|\alpha| < \alpha_0(s)$, *for all* $f \in b\dot{F}^s_{pq}(X)$, $\|f\|_{b\dot{F}^s_{pq}(X)} \sim \|I_\alpha(f)\|_{b^{-1}\dot{F}^{s+\alpha}_{pq}(X)}$.

From Corollary 4.13, it is easy to see that I_α can be used as a lifting operator for the spaces $b\dot{B}^s_{pq}(X)$ and $b\dot{F}^s_{pq}(X)$ when α is small; see also [38] for $\mathbb{R}^n$ case. Moreover, by Corollary 4.13, we can also see that I_α is independent of the choice of the approximation to the identity; see also [23, 45].

References

[1] J. Bergh and J. Löfström, *Interpolation spaces* (Springer, Berlin, 1976).

[2] M. Christ, 'A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral', *Colloq. Math.* **40/41** (1990), 601–628.

[3] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. 242 (Springer, Berlin, 1971).

[4] ———, 'Extensions of Hardy spaces and their use in analysis', *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[5] G. David and J. L. Journé, 'A boundedness criterion for generalized Calderón-Zygmund operators', *Ann. of Math. (2)* **120** (1984), 371–397.

[6] G. David, J. L. Journé and S. Semmes, 'Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation', *Rev. Mat. Iberoam.* **1** (1985), 1–56.

[7] C. Fefferman and E. M. Stein, 'Some maximal inequalities', *Amer. J. Math.* **93** (1971), 107–116.

[8] M. Frazier and B. Jawerth, 'A discrete transform and decompositions of distribution spaces', *J. Funct. Anal.* **93** (1990), 34–170.

[9] A. E. Gatto, 'Product rule and chain rule estimates for fractional derivatives on spaces that satisfy the doubling condition', *J. Funct. Anal.* **188** (2002), 27–37.

[10] A. E. Gatto, C. Segovia and S. Vági, 'On fractional differentiation and integration on spaces of homogeneous type', *Rev. Mat. Iberoam.* **12** (1996), 111–145.

[11] A. E. Gatto and S. Vági, 'On Sobolev spaces of fractional order and ϵ-families of operators on spaces of homogeneous type', *Studia Math.* **133** (1999), 19–27.

[12] A. Grigor'yan, J. Hu and K. Lau, 'Heat kernels on metric-measure spaces and an application to semi-linear elliptic equations', *Trans. Amer. Math. Soc.* **355** (2003), 2065–2095.

[13] P. Hajłasz and P. Koskela, 'Sobolev met Poincaré', *Mem. Amer. Math. Soc.* **145** (2000), 1–101.

[14] Y. Han, 'Calderón-type reproducing formula and the Tb theorem', *Rev. Mat. Iberoam.* **10** (1994), 51–91.

[15] ———, 'The embedding theorem for the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type', *Proc. Amer. Math. Soc.* **123** (1995), 2181–2189.

[16] ———, 'Plancherel-Pôlya type inequality on spaces of homogeneous type and its applications', *Proc. Amer. Math. Soc.* **126** (1998), 3315–3327.

[17] Y. Han, M. Lee and C. Lin, 'Hardy spaces and the Tb theorem', *J. Geom. Anal.* **14** (2004), 291–318.

[18] Y. Han and C. Lin, 'Embedding theorem on spaces of homogeneous type', *J. Fourier Anal. Appl.* **8** (2002), 291–307.

[19] Y. Han, S. Lu and D. Yang, 'Inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type', *Approx. Theory Appl.* **15** (3) (1999), 37–65.

[20] ———, 'Inhomogeneous Triebel-Lizorkin spaces on spaces of homogeneous type', *Math. Sci. Res. Hot-Line* **3** (1999), 1–29.

[21] ———, 'Inhomogeneous discrete Calderón reproducing formulas for spaces of homogeneous type', *J. Fourier Anal. Appl.* **7** (2001), 571–600.

[22] Y. Han and E. T. Sawyer, 'Littlewood-Paley theory on spaces of homogeneous type and classical function spaces', *Mem. Amer. Math. Soc.* **110** (1994), 1–126.

[23] Y. Han and D. Yang, 'New characterizations and applications of inhomogeneous Besov and Triebel-Lizorkin spaces on homogeneous type spaces and fractals', *Dissertationes Math. (Rozprawy Mat.)* **403** (2002), 1–102.

[24] ———, 'Some new spaces of Besov and Triebel-Lizorkin type on homogeneous spaces', *Studia Math.* **156** (2003), 67–97.

[25] J. Heinonen, *Lectures on analysis on metric spaces* (Springer, Berlin, 2001).

[26] J. Kigami, *Analysis on fractals*, Cambridge Tracts in Math. 143 (Cambridge University Press, Cambridge, 2001).

[27] Y. Liu, G. Lu and R. L. Wheeden, 'Some equivalent definitions of high order Sobolev spaces on stratified groups and generalizations to metric spaces', *Math. Ann.* **323** (2002), 157–174.

[28] R. A. Macías and C. Segovia, 'A decomposition into atoms of distributions on spaces of homogeneous type', *Adv. Math.* **33** (1979), 271–309.

[29] ———, 'Lipschitz functions on spaces of homogeneous type', *Adv. in Math.* **33** (1979), 257–270.

[30] A. McIntosh and Y. Meyer, 'Algèbres d'opèrateurs dèfinis par des intègrales singulières', *C. R. Acad. Sci. Paris Sèr. I Math.* **301** (1985), 395–397.

[31] Y. Meyer and R. R. Coifman, *Wavelets. Calderón-Zygmund and multilinear operators*, Cambridge studies in advanced mathematics 48 (Cambridge Univ. Press, Cambridge, 1997).

[32] A. R. Nahmod, 'Geometry of operators and spectral analysis on spaces of homogeneous type', *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991), 721–725.

[33] ———, 'Generalized uncertainty principles on spaces of homogeneous type', *J. Funct. Anal.* **119** (1994), 171–209.

[34] S. Semmes, 'An introduction to analysis on metric spaces', *Notice Amer. Math. Soc.* **50** (2003), 438–443.

[35] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals* (Princeton Univ. Press, Princeton, N.J., 1993).

[36] R. S. Strichartz, 'Function spaces on fractals', *J. Funct. Anal.* **198** (2003), 43–83.

[37] H. Triebel, *Theory of function spaces* (Birkhäuser, Basel, 1983).

[38] ———, *Interpolation theory, function spaces, differential operators*, 2nd edition (Johann Ambrosius Barth, Leipzig, 1995).

[39] ———, *Fractals and spectra* (Birkhäuser, Basel, 1997).

[40] ———, *The structure of functions* (Birkhäuser, Basel, 2001).

[41] ———, 'Function spaces in Lipschitz domains and onLipschitz manifolds. Characteristic functions as pointwise multipliers', *Rev. Mat. Complut.* **15** (2002), 1–50.

[42] H. Triebel and D. Yang, 'Spectral theory of Riesz potentials on quasi-metric spaces', *Math. Nachr.* **238** (2002), 160–184.

[43] D. Yang, 'Frame characterizations of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and their applications', *Georgian Math. J.* **9** (2002), 567–590.

[44] ———, 'Embedding theorems of Besov and Triebel-Lizorkin spaces on spaces of homogeneous type', *Sci. China (Ser. A)* **46** (2003), 187–199.

[45] ———, 'Riesz potentials in Besov and Triebel-Lizorkin spaces over spaces of homogeneous type', *Potential Anal.* **19** (2003), 193–210.

[46] ———, '$T1$ theorems on Besov and Triebel-Lizorkin spaces on spaces of homogeneous type and their applications', *Z. Anal. Anwendungen* **22** (2003), 53–72.

[47] ———, 'Real interpolations for Besov and Triebel-Lizorkin spaces on spaces of homogeneous type', *Math. Nachr.* **273** (2004), 96–113.

[48] ———, 'Some new Triebel-Lizorkin spaces on spaces of homogeneous type and their frame characterizations', *Sci. China (Ser. A)* **48** (2005), 12–39.

Department of Mathematics
Zhongshan University
Guangzhou 510275
People's Republic of China
e-mail: stsdd@zsu.edu.cn

School of Mathematical Sciences
Beijing Normal University
Beijing 100875
People's Republic of China
e-mail: dcyang@bnu.edu.cn