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WHEN DO FINITE BLASCHKE PRODUCTS COMMUTE?

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We study the following questions. Which finite Blaschke products are eigenvectors of the composition operators $T_u\colon f \mapsto f\circ u$, what are the possible eigenvalues, and which pairs (B, C) of finite Blaschke products commute (that is, satisfy $B \circ C = C \circ B$).

1. INTRODUCTION

We shall start by introducing some notation that will be used throughout the paper. The open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathbb{D}$ and its boundary by $\mathbb{T}$. For $p \in \mathbb{D}$, we define the automorphism S_p of $\mathbb{D}$ by

$$S_p(z) = \frac{p - z}{1 - \overline{p}z}.$$

Note that S_p is self-inverse. As usual, $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

We say that two functions $f\colon \Omega_1 \to \Omega_1$ and $g\colon \Omega_2 \to \Omega_2$ are conjugate to each other if there exists some conformal map $\Phi\colon \Omega_1 \to \Omega_2$ such that $f = \Phi^{-1} \circ g \circ \Phi$. We also say that f is Φ-conjugate to g.

It is well known (see [**13**, Chapter 0]) that a nontrivial conformal automorphism of $\mathbb{D}$ is elliptic (that is, has one fixed point in $\mathbb{D}$) or hyperbolic (that is, has two distinct fixed points on $\mathbb{T}$) or parabolic (that is, has a single fixed point on $\mathbb{T}$). Conformal automorphisms of the unit disk are also called Blaschke factors, or Blaschke products of degree 1. We denote this class by $\mathrm{Aut}(\mathbb{D})$. Finite Blaschke products are finite products of Blaschke factors; by the Schwarz–Pick lemma they have at most one fixed point in $\mathbb{D}$.

For $\alpha \in \mathbb{T}$ and $n \in \mathbb{N}^*$, define the map $R_{n,\alpha}$ by $R_{n,\alpha}(z) = \alpha z^n$. Note that $R_{1,\alpha}$ is the rotation $z \longmapsto \alpha z$, which is also denoted by R_α.

The rotation R_α is called *rational* if $\alpha^s = 1$ for some $s \in \mathbb{N}^*$. The smallest such positive integer is called the *index* of R. If u is a conformal automorphism of $\mathbb{D}$ conjugate to a rational rotation of index s, then we say that u is an elliptic Möbius transformation of index s.

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If φ is a map of $\mathbb{D}$ or $\overline{\mathbb{D}}$ into itself and n is a positive integer, the n-th iterate of φ, denoted by $\varphi^{[n]}$, is the n-fold composition $\varphi \circ \cdots \circ \varphi$ of φ.

In Section 2 we give a complete description of finite Blaschke products which are eigenvalues of composition operators of the form $T_u\colon f \longmapsto f \circ u$, where u is a conformal automorphism of $\mathbb{D}$.

In Section 3, given a finite Blaschke product B with a fixed point $p \in \mathbb{D}$, we study which finite Blaschke products C satisfy $B \circ C = C \circ B$ on $\mathbb{D}$. The problem of commuting rational functions was completely solved by Julia [**10**] and Ritt [**11**]. What we present here is another, more modern, approach to the case of finite Blaschke products. It uses results from ergodic theory in the case of Blaschke products of higher degree and is entirely elementary for the case of small degrees. At the same time, it gives a somewhat finer analysis of the solutions.

Finally, our analysis of certain commuting Blaschke products leads us to give counterexamples to three conjectures made by Cowen [**4**].

2. Finite Blaschke products which are eigenvectors of certain composition operators

Lemma 2.1. *Suppose that $a, b \in \mathbb{D}$, that $\beta \in \mathbb{T}$ and that $\lambda = S_{\overline{b}a}(\beta)$. If $a\overline{\beta} \neq b$, then $S_a \circ R_\beta \circ S_b = R_\lambda \circ S_{S_b(a\overline{\beta})}$. If $a\overline{\beta} = b$, then $S_a \circ R_\beta \circ S_b = R_\beta$.*

Proof: Since the composition of two Blaschke factors is again a Blaschke factor, we know that $S_a \circ R_\beta \circ S_b = R_\lambda \circ S_c$. To determine c, the zero of this function, we solve the equation $\beta S_b(c) = a$, which is equivalent to $S_b(\overline{\beta}a) = c$. If $a\overline{\beta} \neq b$, we see that

$$\lambda = \frac{S_a\big(\beta S_b(0)\big)}{S_b(a\overline{\beta})} = \frac{S_a(\beta b)}{S_b(a\overline{\beta})} = S_{\overline{b}a}(\beta).$$

If $a\overline{\beta} = b$, then $c = 0$. Taking derivatives on both sides and evaluating at the origin shows that $\lambda = -\beta$. ∎

For $a \in \mathbb{D}$ and $\sigma \in \mathbb{T}$, define

$$\Phi_s(a, z) = \prod_{j=1}^{s} S_{a\sigma^{j-1}}(z).$$

Then, by Lemma 2.1, we see that

$$\Phi_s(a, \sigma z) = \sigma^s \prod_{j=0}^{s-1} S_{a\sigma^{j-1}}(z). \tag{1}$$

It is well known (and easy to prove) that two elements of $\mathrm{Aut}(\mathbb{D})$ commute if and only if they have the same fixed points (see [**2**, page 115]). In particular, there exist two commuting hyperbolic automorphisms. Take, for example,

$$F_t(z) = \frac{t+z}{1+tz},$$

where $-1 < t < 1$. This contrasts the situation dealt with in the next proposition, where, in particular, it is shown that any element of $\mathrm{Aut}(\mathbb{D})$ commuting with a Blaschke product of degree bigger than 2 is necessarily elliptic.

Proposition 2.2. *Suppose that $U \in \mathrm{Aut}(\mathbb{D})$, and that U commutes with a Blaschke product B of degree $n \geqslant 2$. Then U is an elliptic automorphism conjugate to a rational rotation R_σ. Let p be the fixed point of U, and let $s \in \mathbb{N}^*$ be the smallest positive integer such that $\sigma^s = 1$. Then s divides $n-1$. Moreover, $B = S_p \circ \Phi \circ S_p$, where*

$$\Phi(z) = e^{i\theta} z^{sk+1} \prod_{\nu=1}^{m} \Phi_s(a_\nu, z),$$

with $\theta \in [0, 2\pi)$, $a_\nu \in \mathbb{D} \setminus \{0\}$, $sk + sm = n - 1$, and $k, m \in \mathbb{N}$. (By convention, empty products of the form $\prod_{\nu=1}^{0}$ are equal to 1.) The a_ν are not necessarily different.

Conversely, every Blaschke product of the form B above commutes with U whenever U has fixed point p and is S_p-conjugate to a rational rotation with index s dividing $n-1$.

Proof: Assume that U is an elliptic automorphism with fixed point p and index s which divides $n-1$. Then $U = S_p \circ R_\sigma \circ S_p$ where $\sigma^s = 1$. Using (1), it is now straightforward to check that $B \circ U = U \circ B$. Note that $\sigma^{-1} = \overline{\sigma}$ and $\sigma^{s-1} = \overline{\sigma}$.

Conversely, suppose $B \circ U = U \circ B$. We first show that U has a fixed point in $\mathbb{D}$. If not, then U is either a hyperbolic or a parabolic Möbius transformation. In the first case, U is conjugate to a retraction $H\colon z \mapsto rz$ (where $0 < r < 1$) on the upper half plane, and in the second case, U is conjugate to a translation $T\colon z \mapsto z + r$ (where $r > 0$) on the upper half plane. In the first case, we suppose that $H = \Psi^{-1} \circ U \circ \Psi$ and $C = \Psi^{-1} \circ B \circ \Psi$, where Ψ is a conformal map from the upper half plane onto the unit disk. Then C is of the form

$$C(z) = e^{i\theta} \prod_{j=1}^{n} \frac{a_j z + b_j}{c_j z + d_j},$$

where the coefficients are real numbers and

$$\det \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} > 0$$

when $j = 1, \dots, n$. Further, $C \circ H = H \circ C$, that is, $C(rz) = rC(z)$. Then $C(0) = 0$. Suppose that C has a real pole a. Since $C(rz) = rC(z)$, all the $r^n a$ (where $n \in \mathbb{N}^*$) are poles of C, an obvious contradiction. Thus C is a polynomial. Similarly, we see that C has no zeros other than 0. Thus $C(z) = \eta z^n$. Since $C(rz) = rC(z)$, it follows that $r^{n-1} = 1$, which contradicts $0 < r < 1$. Thus U is not conjugate to a retraction.

Of course, the rational function C cannot be periodic with period r. Hence U cannot be conjugate to a translation. Thus U is elliptic, that is, it has a fixed point in $\mathbb{D}$.

Suppose that p is the fixed point of U in $\mathbb{D}$. Then

$$B(p) = B \circ U(p) = U \circ B(p).$$

The uniqueness of the fixed point implies that $B(p) = p$. Let $\widetilde{B} = S_p \circ B \circ S_p$ and $V = S_p \circ U \circ S_p$. Then $\widetilde{B}$ commutes with V and $\widetilde{B}(0) = V(0) = 0$. Moreover, by Schwarz's Lemma, $V(z) = \sigma z$ for some $\sigma \in \mathbb{T}$. So

$$\widetilde{B}(\sigma z) = \sigma \widetilde{B}(z). \tag{2}$$

Let a be a zero of $\widetilde{B}$ in $\mathbb{D} \setminus \{0\}$. Then by (2), $\sigma^k a$ is a zero of $\widetilde{B}$ when $k \in \mathbb{N}$. Since $\widetilde{B}$ has at most n zeros, we see that σ is a root of unity. Let $s \in \mathbb{N}^*$ be the smallest positive integer such that $\sigma^s = 1$. By (1) we see that $\Phi_s(a, \sigma z) = \Phi_s(a, z)$. This gives invariance under the orbit of a with respect to the rotation. The same can be done with all the remaining zeros, different from the origin. If a is a zero of $\widetilde{B}$ of order l, then every zero of the associated orbit $\{a, \sigma a, \sigma^2 a, \dots\}$ has multiplicity l too. We obtain m orbits, say, all having the same length s. If, additionally, μ denotes the order of the origin as a zero of $\widetilde{B}$, then we see that $sm + \mu = n$. Moreover, $\sigma z^\mu = (\sigma z)^\mu$ if and only if $\sigma^{\mu - 1} = 1$. Thus $\mu - 1$ is a multiple of s, say $\mu - 1 = ks$. We then obtain that $ks + ms = n - 1$ and so s divides $n - 1$. We conclude that $\widetilde{B}$ has the form

$$\Phi(z) = e^{i\theta} z^{sk+1} \prod_{\nu=1}^{m} \Phi_s(a_\nu, z),$$

as required. $\square$

Remark 2.3. We note that

$$\Phi(z) = e^{i\theta} z^{sk+1} \prod_{\nu=1}^{m} \prod_{j=1}^{s} S_{a_\nu \sigma^{j-1}}(z)$$

is independent of σ, provided $\sigma^s = 1$. In fact,

$$\Phi(z) = e^{i\theta} z^{sk+1} \prod_{\nu=1}^{m} \frac{{a_\nu}^s - z^s}{1 - \overline{a_\nu}^s z^s}.$$

If $n - 1$ is a prime number, then Φ has a very simple form. Indeed, either $\Phi(z) = e^{i\theta} z^n$, or

$$\Phi(z) = e^{i\theta} z \frac{a^{n-1} - z^{n-1}}{1 - \overline{a^{n-1}} z^{n-1}}.$$

We also obtain the following result on eigenvectors and eigenvalues, generalising special results in [**3**]. The attempt there to characterise the finite Blaschke products for which $B \circ U = B$ using the zero set of B does not seem to be manageable. Instead, if we consider fixed points, then a complete description can be given (see also [**5**]).

PROPOSITION 2.4. *Suppose that $U \in \operatorname{Aut}(\mathbb{D}) \setminus \{\mathrm{Id}\}$, and that B is a finite Blaschke product of degree $n \geqslant 2$ satisfying $B \circ U = \lambda B$ for some $\lambda \in \mathbb{C}$. Then U is an elliptic automorphism of $\mathbb{D}$, with fixed point p, say, $U = S_p \circ R_\sigma \circ S_p$, and $\sigma \in \mathbb{T} \setminus \{1\}$.*

1. *If $B(p) \neq 0$, then the index s of U divides n, $\lambda = 1$ and $B = S_{B(p)} \circ \Psi \circ S_p$, where*
$$\Psi(z) = e^{i\theta} z^{sk} \prod_{\nu=1}^{m} \Phi_s(a_\nu, z),$$
with $\theta \in [0, 2\pi)$, $a_\nu \in \mathbb{D} \setminus \{0\}$, $sk + sm = n$, $k \in \mathbb{N}^*$, and $m \in \mathbb{N}$. The a_ν are not necessarily different.
2. *If $1 \leqslant \operatorname{ord}(B, p) < n$, then U has index s, $\lambda = \sigma^q$ for some $q \in \mathbb{N}^*$, and $B = \widetilde{\Psi} \circ S_p$, where*
$$\widetilde{\Psi}(z) = e^{i\theta} z^{q} \prod_{\nu=1}^{m} \Phi_s(a_\nu, z),$$
with $\theta \in [0, 2\pi)$, $a_\nu \in \mathbb{D} \setminus \{0\}$, $q + sm = n$, $m \in \mathbb{N}$.
3. *If $\operatorname{ord}(B, p) = n$, then $\lambda = \sigma^n$ and $B = e^{i\theta} (S_p)^n$.*

PROOF: Assume that $B \circ U = \lambda B$. Suppose that U is not an elliptic transformation. Then the iterates $U^{[n]}$ converge locally uniformly to the Denjoy–Wolff fixed point $\alpha \in \mathbb{T}$. Hence, $B \circ U^{[n]} \to B(\alpha)$. Since $\alpha \in \mathbb{T}$, we have $|B(\alpha)| = 1$. But $|B \circ U^{[n]}(0)| = |\lambda^n||B(0)| \leqslant |B(0)| < 1$, hence its limit is strictly less than 1, which is a contradiction. So let $p \in \mathbb{D}$ be the fixed point of U. Put $V = S_p \circ U \circ S_p$. Then $V(z) = \sigma z$ for some $\sigma \in \mathbb{T}$.

1. Assume that $B(p) \neq 0$. Then $B(p) = B(U(p)) = \lambda B(p)$ implies that $\lambda = 1$. Let $\Psi = S_{B(p)} \circ B \circ S_p$. Then $\Psi(0) = 0$ and $\Psi \circ V = \Psi$. Choose $a \in \mathbb{D}$ so that $\Psi(a) \neq 0$. Then $\Psi(\sigma^j a) = \Psi(a)$ for every $j \in \mathbb{N}$. Since the degree of Ψ is n, $\Psi(a)$ has at most n different preimages. Thus $\sigma^s = 1$ for some $s \in \mathbb{N}^*$. Let s be the smallest such positive integer, so that s is the index of U. Hence, by our previous proof,
$$\Psi(z) = e^{i\theta} z^{q} \prod_{\nu=1}^{m} \Phi_s(a_\nu, z),$$
where $q \in \mathbb{N}^*$ is the multiplicity of the origin as a zero of Ψ. Clearly $q + ms = n$. Since $\Psi(\sigma z) = \sigma^q \Psi(z)$, it follows that $\sigma^q = 1$. Thus s divides q, that is $q = sk$ for some $k \in \mathbb{N}^*$.

2. Assume that $B(p) = 0$, but $\operatorname{ord}(B, p) < n$. Let $\widetilde{\Psi} = B \circ S_p$. Then $\widetilde{\Psi}(0) = 0$ and $\widetilde{\Psi} \circ V = \lambda \widetilde{\Psi}$. Since B, and hence also $\widetilde{\Psi}$, has at least one zero a different from 0, we see that the $\sigma^j a$ (where $j \in \mathbb{N}^*$) are zeros of $\widetilde{\Psi}$. Thus $\sigma^s = 1$ for some $s \in \mathbb{N}^*$. Then, as before, it follows that
$$\widetilde{\Psi}(z) = e^{i\theta} z^{q} \prod_{\nu=1}^{m} \Phi_s(a_\nu, z),$$
where $q \in \mathbb{N}^*$ is the multiplicity of the origin as a zero of $\widetilde{\Psi}$. Clearly $q + ms = n$. Also,

$\widetilde{\Psi}(\sigma z) = \sigma^q \widetilde{\Psi}(z)$. Hence $\lambda = \sigma^q$.

3. This is clear. ∎

REMARK 2.5. If we consider solutions of $B \circ U = B$, $B(0) = 0$, $U \neq \mathrm{Id}$, where n is a prime number, then we obtain Proposition 4.3 in [3]. In fact (ii) implies that

$$B = S_a \circ R_{n,e^{i\theta}} \circ S_p = e^{i\theta} \frac{b - S_p^n}{1 - \bar{b} S_p^n},$$

where $a = e^{i\theta} p^n$ and $b = e^{-i\theta} a = p^n$. Hence

$$B = e^{i\theta} \frac{p^n - S_p^n}{1 - \overline{p^n} S_p^n},$$

which coincides with

$$B = e^{i\theta} S_{U(p)} S_{U^{[2]}(p)} \dots S_{U^{[n-1]}(p)} S_{U^{[n]}(p)}$$

and $U = S_p \circ R_\sigma \circ S_p$ where $\sigma^n = 1$, n being the index of U. Note that $U^{[n]} = \mathrm{Id}$. Here are two examples. First, suppose that

$$B(z) = z \frac{a - z}{1 - \bar{a} z} \frac{-a - z}{1 + \bar{a} z}.$$

Then $B(-z) = -B(z)$. Second, suppose that

$$B(z) = z \frac{(1/\sqrt{2}) - z}{1 - z/\sqrt{2}} \frac{(\sqrt{2} - 1) - z}{1 - (\sqrt{2} - 1) z}$$

and

$$U(z) = \frac{(1/\sqrt{2}) - z}{1 - z/\sqrt{2}}.$$

Then $B(U(z)) = -B(z)$.

Let us now consider, for a given U, the solutions of $B \circ U = B$ of minimal degree for B.

COROLLARY 2.6. *Suppose that U is an elliptic Möbius transformation with fixed point p and index n. Then the minimal degree solutions of $B \circ U = B$ are the Blaschke products*

$$B(z) = \beta \frac{b - S_p^n}{1 - \bar{b} S_p^n},$$

where $\beta \in \mathbb{T}$ and $b \in \mathbb{D}$. Their degree is n.

PROOF: This is immediate from Proposition 2.4; just notice that the minimal degree of the solutions coincides with the index of U and that

$$S_a \circ R_{n,e^{i\theta}} \circ S_p = \frac{a - e^{i\theta} S_p^n}{1 - \bar{a} e^{i\theta} S_p^n} = e^{i\theta} \frac{b - S_p^n}{1 - \bar{b} S_p^n},$$

where $b = e^{-i\theta} a$. Note that b does not depend on p. ∎

3. Commuting finite Blaschke products with a fixed point in $\mathbb{D}$

The two following propositions present in an elementary way the description of Blaschke products of degree two and three which commute with Blaschke products of degree two with a fixed point in $\mathbb{D}$.

Proposition 3.1. *Suppose that B and C are Blaschke products of degree 2, and that B has a fixed point in $\mathbb{D}$. Then $B \circ C = C \circ B$ if and only if $B = C$.*

Proof: Let p be the fixed point of B in $\mathbb{D}$. Then

$$B\big(C(p)\big) = C\big(B(p)\big) = C(p).$$

Hence $C(p)$ is a fixed point of B. Due to the uniqueness of the fixed point, $C(p) = p$. Let $\widetilde{B} = S_p \circ B \circ S_p$ and $\widetilde{C} = S_p \circ C \circ S_p$. Then $\widetilde{C}$ commutes with $\widetilde{B}$ and $\widetilde{C}(0) = \widetilde{B}(0) = 0$. We show that $\widetilde{C} = \widetilde{B}$.

Case 1. Suppose that $\widetilde{B}(a) = 0$ for some $a \in \mathbb{D} \setminus \{0\}$. Since the degree of $\widetilde{B}$ is 2 and $\widetilde{B}\big(\widetilde{C}(a)\big) = \widetilde{C}\big(\widetilde{B}(a)\big) = \widetilde{C}(0) = 0$, we have either $\widetilde{C}(a) = 0$ or $\widetilde{C}(a) = a$. Since 0 is the unique fixed point of $\widetilde{C}$, the second assertion cannot hold. Since the degree of $\widetilde{C}$ is 2, we may assume that $\widetilde{B}(z) = e^{i\sigma} z S_a(z)$ and $\widetilde{C}(z) = e^{i\theta} z S_a(z)$. We then obtain

$$\widetilde{C} \circ \widetilde{B} = (R_{e^{i\theta}} S_a) \circ \widetilde{B} = e^{i\theta} \widetilde{B}(S_a \circ \widetilde{B}) = R_{e^{i(\sigma+\theta)}} S_a (S_a \circ \widetilde{B}). \tag{3}$$

On the other hand,

$$\widetilde{B} \circ \widetilde{C} = (R_{e^{i\sigma}} S_a) \circ \widetilde{C} = e^{i\sigma} \widetilde{C}(S_a \circ \widetilde{C}) = R_{e^{i(\sigma+\theta)}} S_a (S_a \circ \widetilde{C}). \tag{4}$$

Hence, by comparing (1) and (2), we see that

$$S_a \circ \widetilde{C} = S_a \circ \widetilde{B}.$$

Left-composing with S_a yields $\widetilde{C} = \widetilde{B}$.

Case 2. Suppose that $\widetilde{B}(z) = e^{i\sigma} z^2$. Assuming that $\widetilde{C}$ vanishes at a where $a \neq 0$, we see, by replacing $\widetilde{B}$ with $\widetilde{C}$ above, that $\widetilde{B}$ vanishes at a, a contradiction. Thus $\widetilde{C}(z) = e^{i\theta} z^2$. By (3) and (4), applied for $a = 0$, we obtain that $\widetilde{B} = \widetilde{C}$ in this case too. ∎

Proposition 3.2. *Suppose that B is a Blaschke product of degree 2 and C a Blaschke product of degree 3. Assume that B has a fixed point p in $\mathbb{D}$. Then $B \circ C = C \circ B$ if and only if $B = S_p \circ R_{2,\alpha} \circ S_p$ and $C = S_p \circ R_{3,\alpha^2} \circ S_p$ for some $\alpha \in \mathbb{T}$.*

Proof: Since $B(p) = p$ we see that $C(p) = C\big(B(p)\big) = B\big(C(p)\big)$. Hence $C(p) = p$ and p is a fixed point of C. Let $\widetilde{B} = S_p \circ B \circ S_p$ and $\widetilde{C} = S_p \circ C \circ S_p$. Then $\widetilde{C} \circ \widetilde{B} = \widetilde{B} \circ \widetilde{C}$ and $\widetilde{B}(0) = \widetilde{C}(0) = 0$. Write $\widetilde{B} = R_\alpha S_a$ and $\widetilde{C} = R_\beta S_b S_c$. We show that $a = b = c = 0$. By hypothesis we obtain

$$R_{\alpha\beta} S_b (S_c \circ S_a \circ \widetilde{C}) = R_{\beta\alpha} S_a (S_b \circ B)(S_c \circ \widetilde{B}). \tag{5}$$

Assume that $a \neq 0$ and $a \neq b$. Then $\widetilde{C}(a) \neq a$ (note that 0 is the fixed point of C). Hence $S_a(\widetilde{C}(a)) \neq 0$. Since the right side of (5) vanishes at a, either $S_b(a) = 0$ (which is not possible, since $a \neq b$), or $S_c(a) = 0$, hence $c = a$. It follows from (5) that

$$S_b(S_a \circ \widetilde{C}) = (S_b \circ B)(S_a \circ \widetilde{B}). \tag{6}$$

Evaluate (6) at a. Then $S_b(a)S_a(\widetilde{C}(a)) = S_b(0)S_a(0)$. But $\widetilde{C}(a) = 0$. Hence $S_b(a)a = ba$. Division by a yields $S_b(a) = b$, hence $a = 0$, a contradiction. So assuming $a \neq 0$, we see that $a = b = c$. But in that case, too, we get the following contradiction, by evaluating (6) at a:

$$0 = \left[S_a(\widetilde{B}(a))\right]^2 \Longrightarrow \widetilde{B}(a) = a \Longrightarrow a = 0.$$

We conclude that $a = 0$. Note that $S_0 = -\,\mathrm{Id}$. In this case, by replacing α by $-\alpha$, we may write $\widetilde{B}$ as $\widetilde{B}(z) = \alpha z^2$. Then

$$R_{2,\alpha\beta^2} S_b^2 S_c^2 = R_{2,\beta\alpha}(S_b \circ R_{2,\alpha})(S_c \circ R_{2,\alpha}). \tag{7}$$

It follows from (7) that

$$\beta S_b^2 S_c^2 = (S_b \circ R_{2,\alpha})(S_c \circ R_{2,\alpha}). \tag{8}$$

Evaluating at 0 shows that $\beta b^2 c^2 = bc$; hence $bc = 0$. Say $b = 0$. By (8),

$$R_{2,\beta} S_c^2 = R_{2,-\alpha}(S_c \circ R_{2,\alpha}). \tag{9}$$

Therefore we get $\beta S_c^2 = -\alpha(S_c \circ R_{2,\alpha})$. Evaluating at 0 gives $\beta c^2 = -\alpha c$. Hence $c = 0$, too.

So $a = b = c = 0$. This implies that $R_{2,\alpha} \circ R_{3,\beta} = R_{6,\alpha\beta^2}$ and $R_{3,\beta} \circ R_{2,\alpha} = R_{6,\beta\alpha^3}$. Hence $\beta = \alpha^2$. ∎

The next theorem generalises the previous results in this section. Its proof is modelled along the lines of the main theorem of Arteaga [**1**]. We use two results from ergodic theory.

For a positive integer q, denote by T_q the map of $\mathbb{T}$ given by $T_q(z) = z^q$. Recall that an endomorphism (that is, a continuous selfmap) f of $\mathbb{T}$ is said to be expanding if there exist $K > 0$ and $\lambda > 1$ such that

$$\left|f^{[n]}(z)\right| > K\lambda^n \text{ for all } z \in \mathbb{T}, n \in \mathbb{N}^*.$$

In this special case we should depart from our usual style in Arteaga [**1**, Lemma 2.1]. Let f and g be two commuting expanding $C^r(\mathbb{T})$-endomorphisms of the unit circle $\mathbb{T}$ (where $r \geqslant 2$). Assume that f and g have degrees n and m respectively and that they have a common fixed point on $\mathbb{T}$. Then there exists a homeomorphism φ of $\mathbb{T}$ that conjugates f to T_n and g to T_m.

The Main Theorem of Johnson and Rudolph [**9**]. Let f and g be two commuting orientation preserving endomorphisms in $C^2(\mathbb{T})$ with a common fixed point. Assume they are both expanding and that f is p-to-1 and g is q-to-1, where p and q are not contained in a common singly generated multiplicative semigroup of $\mathbb{N}$. Then there exists a diffeomorphism $\varphi \in C^2(\mathbb{T})$ such that φ conjugates f to T_n and g to T_m.

THEOREM 3.3. *Suppose that B and C are two finite Blaschke products of degrees n and m respectively, where $m \geqslant n$, and suppose that B and C satisfy the following conditions:*

1. *$B \circ C = C \circ B$ on $\mathbb{D}$ and*
2. *there exists $p \in \mathbb{D}$ such that $B(p) = p$.*

Then there exist $\Phi \in \operatorname{Aut}(\mathbb{D})$ and a positive integer m_0 such that

$$\begin{cases} B = \Phi \circ R_{n,1} \circ \Phi^{-1} \\ C^{[m_0]} = \Phi \circ R_{mm_0,1} \circ \Phi^{-1} \end{cases}$$

(and thus there exists $\alpha \in \mathbb{T}$ such that C is conjugate to $z \longmapsto \alpha z^m$) or there exist $n_1, n_2 \in \mathbb{N}^*$ such that $C^{[m_0 n_1]} = B^{[n_2]}$.

In the particular case where $n = 2$, one can take $m_0 = 1$ and $n_1 = 1$. If B and C have a common fixed point on $\mathbb{T}$, then one may take $m_0 = 1$.

PROOF: Let $\widetilde{B} = S_p \circ B \circ S_p$. Then $\widetilde{B}$ is a Blaschke product of degree n such that $\widetilde{B}(0) = 0$, and thus there exist $\gamma \in \mathbb{T}$ and $a_j \in \mathbb{D}$ (where $2 \leqslant j \leqslant n$) such that

$$\widetilde{B}(z) = \gamma z \prod_{j=2}^{n} \frac{z - a_j}{1 - \overline{a_j} z}.$$

Therefore, for $z \in \mathbb{T}$, we have

$$\left|\widetilde{B}'(z)\right| = 1 + \sum_{j=2}^{n} \frac{1 - |a_j|^2}{|z - a_j|^2} > 1,$$

and so $\widetilde{B}$ is expanding. Now set $\widetilde{C} = S_p \circ C \circ S_p$. Since $B \circ C = C \circ B$, we have $\widetilde{B} \circ \widetilde{C} = \widetilde{C} \circ \widetilde{B}$, and therefore $\widetilde{B} \circ \widetilde{C}(0) = \widetilde{C}(0)$. The uniqueness of the fixed point in $\mathbb{D}$ of $\widetilde{B}$ implies that $\widetilde{C}(0) = 0$. It follows that $\widetilde{C}$ is also expanding. Since $B \circ C = C \circ B$, we have, for every fixed point ξ of B in $\mathbb{T}$, that

$$C^{[j]}(\xi) = C^{[j]}\big(B(\xi)\big) = B\big(C^{[j]}(\xi)\big), \quad j \in \mathbb{N}^*.$$

Hence, for every positive integer j, $C^{[j]}(\xi)$ is a fixed point of B. Since B has only a finite number of fixed points on $\mathbb{T}$, there exists a fixed point ξ_0 of B such that there exists a strictly increasing sequence $(j_k)_{k \geqslant 1}$ of positive integers such that $C^{[j_k]}(\xi) = \xi_0$, $k \geqslant 1$. Therefore

$$C^{[j_{k+1}]-[j_k]}(\xi_0) = C^{[j_{k+1}]-[j_k]}\big(C^{[j_k]}(\xi)\big) = C^{[j_{k+1}]}(\xi) = \xi_0, \quad j \geqslant 1.$$

Hence there exists an iterate $C^{[m_0]}$ of C which shares a common fixed point with B on $\mathbb{T}$. Obviously, if B and C have a common fixed point on $\mathbb{T}$, one may take $m_0 = 1$. Note that in the particular case where $n = 2$, the uniqueness of the fixed point $\xi \in \mathbb{T}$ of B implies that $C(\xi) = \xi$. Therefore if $n = 2$, one can take $m_0 = 1$. It follows from Lemma 2.1 in [**1**] that there exists a homeomorphism $\Phi\colon \mathbb{T} \to \mathbb{T}$ such that

$$\widetilde{B}_{|\mathbb{T}} = \Phi \circ T_n \circ \Phi^{-1} \quad \text{and} \quad \widetilde{C}^{[m_0]}_{|\mathbb{T}} = \Phi \circ T_{m^{m_0}} \circ \Phi^{-1}.$$

If Φ is a diffeomorphism, it follows from a result of [**12**] that Φ is a Möbius transformation, that is, an automorphism of the unit disk. Interested readers are referred to [**6**, **7**, **8**] for further generalisations. If Φ is not a diffeomorphism, the main theorem of [**9**] implies that m^{m_0} and n are contained in a singly generated semigroup of the integers. Thus, m^{m_0} and n are powers of a number $l \in \mathbb{N}$ with $l > 1$. Then there exist $n_1, n_2 \in \mathbb{N}^*$ satisfying

$$n = l^{n_1}, \quad m^{m_0} = l^{n_2}.$$

Therefore $m^{m_0 n_1} = n^{n_2}$ and thus $C^{[m_0 n_1]}{}_{|\mathbb{T}} = B^{[n_2]}{}_{|\mathbb{T}}$. Clearly $C^{[m_0 n_1]} = B^{[n_2]}$.

In the particular case where $n = 2$, necessarily $l = 2$, and then there exists $n_2 \in \mathbb{N}^*$ such that $m = n^{n_2}$. It follows that $C = B^{[n_2]}$. ∎

The following example shows that, if $n > 2$, then the index m_0 need not be equal to 1.

EXAMPLE. Take $B(z) = z^3$ and $C(z) = -z^5$. Then $B(0) = C(0) = 0$ and $B \circ C(z) = C \circ B(z)$. Note that B and C have no common fixed point on $\mathbb{T}$, which implies that $m_0 \neq 1$. Since $C^{[2]}(z) = z^{25}$, one can take $m_0 = 2$ and we obtain that B and $C^{[2]}$ are conjugate to $z \longmapsto z^3$ and $z \longmapsto z^{25}$ respectively by the identity map.

As a corollary of Arteaga [**1**], we obtain the following result.

COROLLARY 3.4. *Suppose that B and C are finite Blaschke products of degrees n and m respectively, where $m \geqslant n \geqslant 3$. Suppose that*

1. *$B \circ C = C \circ B$ on $\mathbb{D}$*
2. *there exists $p \in \mathbb{D}$ such that $B(p) = p$*
3. *$B'(\xi) \neq B'(\xi')$ for all different fixed points ξ and ξ' on $\mathbb{T}$.*

Then m is a power of n. Moreover, if $m = n^k$ with $k \in \mathbb{N}^*$, then $C = B^{[k]}$.

PROOF: Since the degree of B is greater than 3, we deduce from Arteaga [**1**] that $C = B^{[k]}$ for some $k \in \mathbb{N}^*$. Hence $m = n^k$. ∎

We note that the condition on the derivatives of B implies that B cannot be conjugate to $z \longmapsto z^n$. In fact, the derivative of this function is equal to n at each of its fixed points $e^{2\pi i k/(n-1)}$.

Moreover, let us point out that in order to keep the assertion of the previous corollary, one cannot omit the third condition, as the following example shows.

Example. Take $B(z) = e^{i\theta}z^3$ and $C = -B$ with $\theta \in (0, 2\pi)$. Then $B(0) = C(0) = 0$ and $B \circ C(z) = C \circ B(z) = -e^{4i\theta}z^9$. Obviously we have $B \neq C$ and $B^{[2]}(z) = C^{[2]}(z) = e^{4i\alpha}z^9$. Note that $B'(\alpha) = B'(\beta)$ for the two fixed points α and β of B on $\mathbb{T}$.

Our analysis of commuting finite Blaschke products leads us to a better intuition about the role of fixed points. In particular we are now able to disprove conjectures of [**4**].

4. Counterexamples to C.C. Cowen's conjectures

Suppose that f and g are nonconstant analytic functions mapping the unit disk $\mathbb{D}$ into itself and denote by a the Denjoy–Wolff point of f. Assume also that neither f nor g is a conformal automorphism of $\mathbb{D}$. Recall that whenever $\alpha \in \mathbb{T}$, then $f'(\alpha)$ denotes the angular derivative at α. It is well-known that then $0 < f'(\alpha) < 1$.

Conjecture 1 of [**4**] says that if $\alpha \in \mathbb{T}$ and if $f'(\alpha) < 1$ then every g that commutes with f has the same fixed point set as f. This is not true. Take $f(z) = (z + 1/2)^2(1 + z/2)^{-2}$ and $g = f^{[2]}$. Then we have $\alpha = 1$, $f'(1) = 2/3 < 1$, $f \circ g = g \circ f$ but f has three distinct fixed points $(1, \omega, \overline{\omega})$ whereas g has five distinct fixed points $(1, \omega, \overline{\omega}, -7/8 + i\sqrt{15}/8, -7/8 - i\sqrt{15}/8)$, where $\omega = e^{2\pi i/3}$.

Conjecture 2 of [**4**] says that if $f'(a) \neq 0$, and if f and g are commuting functions with two fixed points in common, then the fixed point sets of f and g are the same. This is not true. Take $f(z) = z(z + 1/2)(1 + z/2)^{-1}$ and $g = f^{[2]}$. Then $a = 0$, $f'(a) \neq 0$, the fixed points of f (1 and 0) are also fixed points of g, but -1 and $-1/2$ are additional fixed points of g.

Conjecture 3 of [**4**] says that if $|a| < 1$ and $f'(a) \neq 0$ then there is an integer n such that for every g that commutes with f, the fixed point sets of f and $g^{[n]}$ are the same. This is not true, either. Take, as before, $f(z) = z(z + 1/2)(1 + z/2)^{-1}$ and $g = f \circ f$. Since the fixed point set of $g^{[n]}$ contains -1 and $-1/2$ for every positive integer n, Conjecture 3 is not correct.

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