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A remark on primes in arithmetic progressions

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By generalizing a technique of Landau, the authors prove that the excess of the number of primes of the form $10x \pm 3$ over the number of primes of the form $10x \pm 1$ is infinite.

1. Introduction

In 1853, Chebyshev conjectured the still unproven assertion that there exist more primes of the form $4y + 3$ than of the form $4y + 1$ in the sense that

$$\lim_{x\to+\infty} \sum_{p>2} (-1)^{\frac{p-1}{2}} e^{-\frac{p}{x}} = -\infty .$$

Knapowski and Turán's [3] investigations into this conjecture point out that there are really *four* classes of problems, there labeled (a) - (d), involved. They also state that their investigations deal with classes (c) and (d) only.

This paper concerns itself with a problem from class (a). We begin with a weaker form of Chebyshev's conjecture. If

$$\text{(1)} \qquad \Pi(x, k, l) = \sum_{\substack{p\le x\\ p\equiv l(k)}} 1$$

$$\text{(2)} \qquad P(x, k, l_1, l_2) = \Pi(x, k, l_1) - \Pi(x, k, l_2) ,$$

then

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$$P(x,\ 4,\ 3,\ 1) \to \infty \quad \text{as} \quad x \to \infty .$$

Landau [4] proves Chebyshev's theorem. He also points out that Phragmén [6] had given the only previous correct, but very complicated, proof.

In this paper, by generalizing Landau's technique, the authors prove that

(3) $$P(x) \equiv P(x,\ 10,\ \pm 3,\ \pm 1) \to \infty \quad \text{as} \quad x \to \infty .$$

More precisely, it is proven that there exists an infinite sequence $\{x_n\}$ such that

$$x_1 < x_2 < \ldots \to \infty$$

and that as $n \to \infty$

$$\frac{\Pi(x_n,\ 10,\ \pm 3) - \Pi(x_n,\ 10,\ \pm 1)}{\left(\dfrac{\sqrt{x_n}}{\log x_n}\right)} \to 1 .$$

2. Main results

The following theorem on Dirichlet series is well known.

THEOREM 1 (Landau). *Suppose the series* $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *converges for* $s > \alpha$ *and that* $a_n \geq 0$ *for* $n \geq n_0$. *Choose* $\beta < \alpha$ *and suppose its sum function,* $f(s)$ *, is regular for* $\alpha \geq s > \beta$ *when continued along the real axis. Then the series converges for* $s > \beta$.

In the next two theorems the desired result (3) is achieved. In order to establish them the following is needed.

LEMMA 1. *For all complex* $s = \sigma + it$ *it follows that*

(4) $$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s} - \frac{s}{n^{s+1}}\right| \leq \frac{|s|\,|s+1|}{n^{\sigma+2}} .$$

Proof. For $s = 0$ inequality (4) is immediate. For $s \neq 0$ it follows from

$$\frac{1}{n^s} - \frac{1}{(n+1)^s} - \frac{s}{n^{s+1}} = s\int_n^{n+1}\left(\frac{1}{u^{s+1}} - \frac{1}{n^{s+1}}\right)du$$

$$= s(s+1)\int_n^{n+1} du \int_u^n \frac{dv}{v^{s+2}} .$$

LEMMA 2. *For* $y \geq 0$ *and* $\frac{1}{2} \leq s \leq 1$ *it follows that*

(5) $$(10y+1)^{-s} - (10y+3)^{-s} - (10y+7)^{-s} + (10y+9)^{-s} > 0 .$$

Proof. Recall the "power" inequalities, [2], that for α, β real, $0 < \beta < 1 < \alpha$ and $0 < s \neq 1$ we have

$$\alpha^s > 1 + s(\alpha-1) \quad \text{and} \quad \beta^s > 1 - s(1-\beta) .$$

By appropriate substitution and addition it follows that

$$\frac{1}{1+s(10y+2)} + \frac{1}{1+s(10y+6)} > \frac{1}{(10y+3)^s} + \frac{1}{(10y+7)^s}$$

and

$$\frac{1}{(10y+1)^s} + \frac{1}{(10y+9)^s} > 2 - s\left(\frac{10y}{10y+1}\right) - s\left(\frac{10y+8}{10y+9}\right) .$$

Clearly inequality (5) follows once we can establish the stronger

(6) $$2 > s\left(\frac{10y}{10y+1}\right) + s\left(\frac{10y+8}{10y+9}\right) + \frac{1}{1+s(10y+2)} + \frac{1}{1+s(10y+6)} .$$

We note that a simple computation shows that (6) holds for $s = 1$ and $y > 0$. Also, it is easy to establish that the right hand side of inequality (6) is a continuous, strictly increasing function of s in the interval $[\frac{1}{2}, 1]$. Hence, since (6) holds for $s = 1$ it certainly holds for $\frac{1}{2} \leq s \leq 1$ and this proves the lemma.

We define the non-principal character, $\chi(n)$, (mod 10) as

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{10} \\ -1 & \text{if } n \equiv \pm 3 \pmod{10} \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2. *Let* $s = \sigma + it$ *and put*

(7) $$F(s) = \sum_{n=2}^{\infty} \frac{P(n)-(\sqrt{n}/\log n)}{n^{s+1}}$$

so that $F(s)$ *is analytic for* $\sigma > 1$. *Then by direct continuation* $F(s)$ *is regular for* $1 \geq s > \frac{1}{2}$. *The point* $s = \frac{1}{2}$ *is a singular point of* $F(x)$ *and yet* $\lim_{s \to \frac{1}{2}^+} F(x)$ *exists.*

Proof. Using the notation of Landau [5] we let $R_1(s), R_2(s), \ldots$ denote functions of s which are regular for $\sigma > 1$ and which can be analytically continued along the real segment $\frac{1}{2} \leq s \leq 1$. Recalling the character, $\chi(n)$, we let

$$L(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = 1 - \frac{1}{3^s} - \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \ldots +$$

$$+ \frac{1}{(10y+1)^s} - \frac{1}{(10y+3)^s} - \frac{1}{(10y+7)^s} + \frac{1}{(10y+9)^s} + \ldots .$$

It follows, by Lemma 2, that $L(s) > 0$ for $\frac{1}{2} \leq s \leq 1$. Hence, in this interval, we can define

$$\log L(s) = \sum_{p,m} \frac{\chi(p^m)}{mp^s} = R_1(s) .$$

For $\sigma > 1$ we have

$$\sum_{p,m} \frac{\chi(p^m)}{mp^s} = \sum_p \frac{\chi(p)}{p^s} + \frac{1}{2} \sum_p \frac{\chi(p^2)}{p^{2s}} + R_2(s) .$$

Thus,

$$\sum_p \frac{\chi(p)}{p^s} = - \frac{1}{2} \sum_p \frac{\chi(p^2)}{p^{2s}} + R_3(s) = - \frac{1}{2} \sum_p \frac{1}{p^{2s}} + R_4(s) .$$

On the other hand letting $\zeta(s)$ be the Riemann zeta function, it follows that

$$\log\left((s - \tfrac{1}{2})\zeta(2s)\right) = \log\zeta(2s) + \log(s - \tfrac{1}{2}) = R_5(s)$$

and

$$\log\zeta(2s) = \sum_p \frac{1}{p^{2s}} + R_6(s) .$$

This implies that

$$\sum_p \frac{1}{p^{2s}} = - \log(s - \tfrac{1}{2}) + R_7(s)$$

and

$$\sum_p \frac{\chi(p)}{p^s} = \tfrac{1}{2}\log(s-\tfrac{1}{2}) + R_8(s) .$$

This says that the series on the left, in the last two equations, are convergent for $s > \frac{1}{2}$.

Recalling the definitions of $\Pi(x, 10, \pm 1)$ and $\Pi(x, 10, \pm 3)$ it follows that

$$\Pi(n, 10, \pm 1) - \Pi(n-1, 10, \pm 1) = \begin{cases} 1 & \text{if } n \text{ is a prime } 10y \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Pi(n, 10, \pm 3) - \Pi(n-1, 10, \pm 3) = \begin{cases} 1 & \text{if } n \text{ is a prime } 10y \pm 3 \\ 0 & \text{otherwise.} \end{cases}$$

Recalling (2) it now follows that

$$\begin{aligned} -\sum_{n=2}^{\infty} \frac{P(n)-P(n-1)}{n^s} &= \sum_{n=2}^{\infty} \frac{[\Pi(n, 10, \pm 1)-\Pi(n-1, 10, \pm 1)] - [\Pi(n, 10, \pm 3)-\Pi(n-1, 10, \pm 3)]}{n^s} \\ &= \sum_{\substack{p \equiv \pm 1 \\ (\mathrm{mod}\ 10)}} \frac{1}{p^s} - \sum_{\substack{p \equiv \pm 3 \\ (\mathrm{mod}\ 10)}} \frac{1}{p^s} = \sum_p \frac{\chi(p)}{p^s} . \end{aligned}$$

By Lemma 1, the sum

$$\sum_{n=2}^{\infty} P(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s} - \frac{s}{n^{s+1}}\right)$$

represents a regular function for $\sigma > 0$. Since $P(1) = 0$ it follows for $\sigma > 1$, that

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$$\sum \frac{\chi(p)}{p^s} = - \sum_{n=2}^{\infty} P(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right)$$

$$= - s \sum_{n=2}^{\infty} \frac{P(n)}{n^{s+1}} + R_9(s) ,$$

$$\sum_{n=2}^{\infty} \frac{P(n)}{n^{s+1}} = - \frac{1}{2s} \log(s - \tfrac{1}{2}) + R_{10}(s) .$$

Moreover, for $\sigma > \frac{1}{2}$ and C a constant,

$$\sum_{n=2}^{\infty} \frac{(\sqrt{n}/\log n)}{n^{s+1}} = \sum_{n=2}^{\infty} \frac{1}{(\log n) n^{s+\frac{1}{2}}}$$

$$= \sum_{n=2}^{\infty} \frac{1}{\log n}\left(\frac{1}{n^{s+\frac{1}{2}}} - \frac{1}{n^{2+\frac{1}{2}}}\right) + C$$

$$= \int_s^2 \left(\zeta(n + \tfrac{1}{2})-1\right)du + C = \int_s^2 \frac{du}{u+\frac{1}{2}} + R_{11}(s)$$

$$= -\log(s - \tfrac{1}{2}) + R_{12}(s) .$$

Returning to the definition of $F(s)$, we have

$$F(s) = \left(\frac{-1}{2s} + 1\right)\log(s - \tfrac{1}{2}) + R_{10}(s) - R_{12}(s)$$

$$= \frac{1}{s}(s - \tfrac{1}{2})\log(s - \tfrac{1}{2}) + R_{13}(s) .$$

The last expression clearly displays the asserted properties of $F(s)$.

Q.E.D.

THEOREM 3. *The following inequality holds for every* $\delta > 0$.

$$\left|\frac{P(x)}{(\sqrt{x}/\log x)} - 1\right| < \delta ; \tag{8}$$

that is, given any δ *and* $\xi > 0$, *there always exists an* $x = x(\delta, \xi) > \xi$ *such that*

$$-\delta \frac{\sqrt{x}}{\log x} < P(x) - \frac{\sqrt{x}}{\log x} < \delta \frac{\sqrt{x}}{\log x} . \tag{9}$$

Proof (by contradiction). Suppose false and let

$$Q(x) = \frac{P(x)}{(\sqrt{x}/\log x)} - 1 .$$

Then from a certain point on, $Q(x)$ will never belong to the interval $(-\delta, \delta)$. Considered as a function of increasing x, $Q(x)$ has the following properties:

(i) it is continuous on the open interval between two consecutive integers;

(ii) it is continuous on the right at every integer;

(iii) it either remains continuous or makes a jump equal to

$$\lim_{\varepsilon\to 0} \big(Q(x) - Q(x-\varepsilon)\big) = \pm \frac{1}{(\sqrt{x}/\log x)}$$

when x is an integer.

Thus,

$$\lim_{x\to\infty} \big(Q(x) - Q(x-\varepsilon)\big) = 0 ,$$

and we have two cases to consider, namely

$$Q(x) > \delta \quad \text{or} \quad Q(x) \le -\delta .$$

Since the treatment of the two cases is quite similar we need only consider Case 1 in this paper. Thus, we let

$$x \ge n_0 \ge 2 , \quad n_0 \text{ an integer and } Q(x) \ge \delta .$$

Then the Dirichlet series $F(s)$ of Theorem 2, which converges for $\sigma > 1$, satisfies the hypothesis of Theorem 1 for $\alpha = 1$ and $\beta = \frac{1}{2}$. Thus $F(s)$ converges for $s > \frac{1}{2}$. Also, in the interval $[\frac{1}{2}, 1]$ we have

$$F(s) = \sum_{n=2}^{n_0-1} \frac{|P(n)-(\sqrt{n}/\log n)|}{n^{s+1}} + \sum_{n=n_0}^{\infty} \frac{\delta(\sqrt{n}/\log n)}{n^{s+1}}$$
$$- \sum_{n=2}^{n_0-1} \frac{|P(n)-(\sqrt{n}/\log n)|}{n^{\frac{3}{2}}} + \delta \sum_{n=n_0}^{\infty} \frac{1}{n^{s+\frac{1}{2}}\log n} .$$

Since, $\sum_{n=2}^{\infty} 1/(n\log n)$ diverges it follows that

$$\lim_{s\to\frac{1}{2}^+} F(s) = \infty .$$

But this contradicts Theorem 2 and hence proves Theorem 3.

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