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Realizing symmetries of a shift

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Abstract. All subshifts of finite type are known to appear as basic parts of the non-wandering sets of Smale diffeomorphisms in dimensions three or more. This paper concerns the symmetries of subshifts of finite type; that is, the homeomorphisms of the shift space which commute with the shift. The group of symmetries is known to be very large for aperiodic shifts. For certain (structually stable) Smale diffeomorphisms of the sphere of dimension five or more, we show each symmetry can be extended to a homeomorphism of the sphere commuting with the diffeomorphism on the whole sphere.

0. *Introduction*

Let A be a finite, irreducible, zero-one matrix and let $\sigma_A : X_A \to X_A$ be the corresponding subshift of finite type [**F**]. Recall from [**F**] that a Smale diffeomorphism is one which satisfies the transversality condition on its hyperbolic, zero-dimensional chain recurrent set. A well-known theorem of Williams–Smale [**Wi**] says that there is a Smale diffeomorphism $F_A : S^3 \to S^3$ so that σ_A is topologically conjugate to the restriction of F_A to the basic set of index one occurring as part of the spectral decomposition. Let Aut (σ_A) denote the group of *symmetries*, or *automorphisms*, of σ_A; that is, the group of homeomorphisms of X_A which commute with σ_A. Here is the corresponding global realization results for these symmetries.

THEOREM. *Assume* $5 \le q$ *and let* $3 \le e \le q-2$. *Then there is a Smale diffeomorphism* $F_A : S^q \to S^q$ *with a basic set* Ω_e *of index* e (*along with other basic sets of index* 0, $e-1$, q) *together with a topological conjugacy between* σ_A *and* $F_A | \Omega_e$ *so that given any symmetry g in* Aut (σ_A), *there is a homeomorphism* $G : S^q \to S^q$ *satisfying*
(A) G *commutes with* F_A *on all of* S^q, *and*
(B) $G | \Omega_e = g$ *under the identification between* Aut $(F_A | \Omega_e)$ *and* Aut (σ_A).

The motivation and the idea for the proof of this geometric result came by analogy from algebraic K-theory and pseudo-isotopy theory. The proof uses Williams' notion of strong shift equivalence [**Wi, F**], the contractible simplicial complex P_A of topological Markov partitions for σ_A [**W**], and structural stability for Smale diffeomorphisms [**R, Ro**]. We would like to thank C. Pugh for useful discussions about the stability theorem. The group Aut (σ_A) is often rather large. For example, Aut (σ_2) for the Bernoulli 2-shift σ_2 has been known [**H**] for some time to contain every finite group and to have elements of infinite order not a power of σ_2. Recently,

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Boyle–Lind have shown it contains the free non-abelian group on infinitely many generators. Therefore, the group of *homeomorphisms* of S^q commuting with a certain F_2 is large when $5 \leq q$. This is to be contrasted with results of Palis–Yoccoz [**P**] which show that most diffeomorphisms F with a hyperbolic non-wandering set on which strong transversality holds have a centralizer consisting only of the powers F^k in the group of *diffeomorphisms.* Incidentally, at the present time not much is known about the structure and other algebraic or homological properties of Aut (σ_2). For some information see [**BK, BLR, W**]. An open and long standing conjecture is that Aut (σ_2) is generated by σ_2 and elements of finite order.

Here is a rough idea of the proof. Let P be an $m \times m$ zero-one matrix and let Q be an $n \times n$ zero-one matrix. Suppose there is an $m \times n$ zero-one matrix R and an $n \times m$ zero-one matrix S so that $P = RS$ and $Q = SR$. As in [**Wi**] this determines a specific conjugacy $c_R : (X_P, \sigma_P) \to (X_Q, \sigma_Q)$ sending $x = \{x_i\}$ in X_P to $c_R(x) = \{c_R(x)_i\}$ in X_Q where $c_R(x)_i$ is the unique k such that $R(x_i, k)S(k, x_{i+1}) = P(x_i, x_{i+1}) = 1$. Similarly for c_S. In fact, $c_S c_R = \sigma_P$ and $c_R c_S = \sigma_Q$ so that $c_R \sigma_P = \sigma_Q c_R$ and $c_S \sigma_Q = \sigma_P c_S$. We call c_R and c_S *elementary symbolic conjugacies.* On the topological side, let $S^q(m)$ be the standard q-sphere equipped with a fixed handle decomposition with one handle of index zero, m handles of index e, m cancelling handles of index $e-1$, and one handle of index q. Similarly for $S^q(n)$. One then constructs a Smale diffeomorphism $C_R : S^q(m) \to S^q(n)$ which is fitted both on the handles of index e and the handles of index $e-1$ according to the geometric intersection matrix R. Again, similarly for C_S. This is done in such a way that the composition $D_P = C_S C_R : S^q(m) \to S^q(m)$ is also a Smale diffeomorphism fitted on the e-handles and $(e-1)$-handles according to the matrix $P = RS$ and $D_Q = C_R C_S : S^q(n) \to S^q(n)$ is fitted according to $Q = SR$. Observe that $C_R D_P = D_Q C_R$ and $D_P C_S = C_S D_Q$, and therefore C_R and C_S are smooth conjugacies between D_P and D_Q. We call these *elementary smooth conjugacies.* Now consider a Smale diffeomorphism $F_P : S^q(m) \to S^q(m)$ which is fitted on the e-handles and $(e-1)$-handles by the matrix P. In general, of course, $F_P \neq D_P$. However, under the assumption that $3 \leq e \leq q-2$ we are able to carefully construct F_P, C_R, and C_S in such a way that there is a one-parameter family of Smale diffeomorphisms $F_P(t)$, each of which is fitted on the e-handles and $(e-1)$-handles by the matrix P, so that $F_P(0) = F_P$, $F_P(1)$ is equal to D_P on a neighborhood of the e-skeleton, and both $F_P(1)$ and D_P have the point at infinity as a source. Methods of stability theory [**PS, R, Ro**] can then be used to produce a topological (*not* smooth) conjugacy between F_P and D_P. We call this a *stability conjugacy.* Similarly, there is a stability conjugacy between D_Q and F_Q, so that we then get a topological conjugacy between F_P and F_Q. The main theorem is proved by first showing that any symmetry g in Aut (σ_A) can be obtained as the composition of a chain of elementary symbolic conjugacies and powers of shifts, and then by showing this can be mirrored compatibly with a corresponding chain of elementary smooth conjugacies, stability conjugacies, and powers of certain intermediate F_P for different matrices P. The chain starts with the original F_A which is fixed and eventually comes back to it. The composition of the various conjugacies and powers of F_P in the chain give the required homeomorphism G.

The main theorem may well be valid on S^4 also, but our argument seems to require $4<q$. There is probably a counterexample on S^3.

In § 1 we discuss elementary symbolic conjugacies, and in § 2 we discuss R-model diffeomorphisms of which the elementary smooth conjugacies C_R are a special case. R-models are obtained by the 'cobra construction' illustrated in (2.6). In § 3 we prove the main theorem.

1. *Symbolic conjugacies*

The following result was essentially proved in [**Wi**].

PROPOSITION 1.1. *Any conjugacy $f:(X_A,\sigma_A)\to(X_B,\sigma_B)$ is the composition of elementary symbolic conjugacies, their inverses, and shift powers.*

Let A be an $m\times m$ zero-one matrix. For the subshift of finite type $\sigma_A: X_A\to X_A$, we let P_A be the contractible space of Markov partitions for σ_A on X_A. See [**W**]. We will use the following reformulation of the usual definition [**F**, p. 100] of a topological Markov partition for σ_A. If $x=\{x_k\}$ and $y=\{y_k\}$ are in X_A and $x_0=y_0$, let $[x,y]$ in X_A be defined by $[x,y]_k=x_k$ for $k\le 0$ and $[x,y]_k=y_k$ for $k\ge 0$. A *rectangle* is a compact open subset Z of X_A such that $x,y\in Z$ imply $x_0=y_0$ and $[x,y]\in Z$. A *Markov partition* for $\sigma=\sigma_A$ on X_A is a finite covering $U=\{U_1,\dots,U_m\}$ of X_A by rectangles U_i such that

$$\text{if}\quad \alpha,\beta\in U_i \text{ and } \sigma(\beta)\in U_j,\qquad \text{then } [\sigma([\alpha,\beta]),\sigma(\beta)]\in U_j \tag{1.2}$$

$$\text{if}\quad \alpha\in U_i,\ \sigma(\alpha)\in U_j, \text{ and } \beta\in U_j,\qquad \text{then } [\alpha,\sigma^{-1}([\sigma(\alpha),\beta])]\in U_i \tag{1.3}$$

These conditions correspond respectively to the two properties $\sigma W^s(\alpha,U_i)\subset W^s(\sigma(x),U_j)$ and $\sigma W^u(\alpha,U_i)\supset W^u(\sigma(\alpha),U_j)$ which are usually given to define a Markov partition. See [**F**, p. 100].

The standard Markov partition for σ_A is $U^A=\{U_s^A\}$ where U_s^A consists of all those $x=\{x_k\}$ with $x_0=s\in\{1,2,\dots,m\}$. If $U=\{U_i\}$ and $V=\{V_j\}$ are in P_A, let $U\cap V=\{U_i\cap V_j\}$ where $U_i\cap V_j\ne\varnothing$. If $m,n\ge 0$ and $U\in P_A$, let $U(-m,n)=\sigma_A^m U\cap\cdots\cap\sigma_A^{-n}U$. See [**F, W**].

If $x,y\in X_A$ satisfy $x_0=y_0$, then

$$\begin{aligned}&[\sigma([x,y]),\sigma(y)]=\sigma([x,y])\\&[\sigma^{-1}(x),\sigma^{-1}([x,y])]=\sigma^{-1}([x,y]).\end{aligned}\tag{1.4}$$

An induction argument proves that for $k\ge 0$ we have

$$\begin{aligned}&[\sigma([\sigma^k([x,y]),\sigma^k(y)]),\sigma^{k+1}(y)]=\sigma^{k+1}([x,y])\\&[\sigma^{-(k+1)}(x),\sigma^{-1}([\sigma^{-k}(x),\sigma^{-k}([x,y])])]=\sigma^{-(k+1)}([x,y]).\end{aligned}\tag{1.5}$$

Now consider a Markov partition $V=\{V_1,\dots,V_n\}\in P_A$ and as in [**F**] let $B=M(V)$ be defined by $B(k,l)=1$ iff $V_k\cap\sigma^{-1}V_l\ne\varnothing$. A well known fact [**F, W, Wi**] is that there is a conjugacy of shifts

$$i=i_V:(X_A,\sigma_A)\to(X_B,\sigma_B)$$

defined by

$$i(x)_k=j\qquad \text{iff } \sigma_A^k(x)\in V_j. \tag{1.6}$$

LEMMA 1.7. *If* $W=\{W_1,\dots,W_q\}$ *is in* P_A *and refines* V, *then* $i(W)=\{i(W_1),\dots,i(W_q)\}$ *lies in* P_B.

Proof. Let $\sigma=\sigma_A$ and let $x,y\in V_j$. We first verify the general formula

$$i([x,y])=[i(x),i(y)]. \tag{1.8}$$

The proof is by induction.

Step 1. For $k\geq 0$, $i([x,y])_k=i(y)_k$.

This is true for $k=0$ because V_j is a rectangle so that $[x,y]\in V_j$ and hence $i([x,y])_0=j=i(y)_0$. Assume the result is true for k. This means $\sigma^k([x,y])\in V_s$ and $\sigma^k(y)\in V_s$. Assume $\sigma^{k+1}(y)\in V_t$. We must show $\sigma^{k+1}([x,y])\in V_t$. Apply (1.2) with $\alpha=\sigma^k([x,y])$, $i=s$, $j=t$ and $\beta=\sigma^k(y)$ to conclude that

$$[\sigma([\sigma^k([x,y]),\sigma^k(y)]),\sigma^{k+1}(y)]\in V_t.$$

From (1.5) we obtain $\sigma^{k+1}([x,y])\in V_t$.

Step 2. For $k\geq 0$, $i([x,y])_{-k}=i(x)_{-k}$.

This is true for $k=0$ because V_j is a rectangle. Assume it is true for k. This means $\sigma^{-k}([x,y])\in V_s$ and $\sigma^{-k}(x)\in V_s$. Assume $\sigma^{-(k+1)}(x)\in V_t$. We must show $\sigma^{-(k+1)}([x,y])\in V_t$. Apply (1.3) with $\alpha=\sigma^{-(k+1)}(x)$, $i=t$, $j=s$, and $\beta=\sigma^{-k}([x,y])$ to obtain

$$[\sigma^{-(k+1)}(x),\sigma^{-1}([\sigma^{-k}(x),\sigma^{-k}([x,y])]\in V_t.$$

From (1.5) we have $\sigma^{-(k+1)}([x,y])\in V_t$.

To continue the proof of (1.7), it follows directly from (1.8) that each $i(W_r)$ is a rectangle because each W_r lies in some V_j. We must verify (1.2) and (1.3) for $i(W)$. Let $x,y\in i(W_r)$ and $\sigma_B(y)\in i(W_s)$. Write $x=i(x')$ and $y=i(y')$ for $x',y'\in W_r$. Observe that $\sigma_A(y')\in W_s$. Then from (1.8) we have

$$\begin{aligned}[\sigma_B([x,y]),\sigma_B(y)]&=[\sigma_B([i(x'),i(y')]),\sigma_B(i(y'))]\\&=[\sigma_B(i([x',y'])),\sigma_B(i(y'))]\\&=[i(\sigma_A([x',y'])),i\sigma_A(y')]\\&=i[\sigma_A([x;y']),\sigma_A(y')]\in i(W_s)\end{aligned}$$

because W is a Markov partition. The formula (1.3) is verified similarly. □

Let $V,W\in P_A$ satisfy $U^A<V<W$. Let $B=M(V)$ and $C=M(W)$. Let $i_V:X_A\to X_B$ be as in (1.6). Then $W'=i_V(W)\in P_B$ according to (1.7). Let $C'=M(i_V(W))$ and observe C and C' are conjugate via the permutation matrix P taking W_s to $i_V(W_s)$, i.e. $C'=PCP^{-1}$. This gives an elementary symbolic conjugacy $c_R:X_C\to X_{C'}$ where $R=P^{-1}$ and $S=PC$. Let $i_W:X_A\to X_C$ and $i_{W'}:X_B\to X_{C'}$ be as in (1.6).

LEMMA 1.9. $c_Ri_W=i_{W'}i_V$.

Proof. Let $x \in X_A$. Then

$$\begin{aligned} &c_R^{-1} i_{W'} i_V(x)_n = s \\ &\text{iff } \sigma_B^n(i_V(x)) \in i_V(W_s) \\ &\text{iff } i_V^{-1}\sigma_B^n(i_V(x)) \in W_s \\ &\text{iff } \sigma_A^n(i_V^{-1} i_V(x)) \in W_s \\ &\text{iff } \sigma_A^n(x) \in W_s \\ &\text{iff } i_W(x)_n = s. \end{aligned}$$

Proof of 1.1. We first observe that it suffices to prove (1.1) for an arbitrary f of the form $f = i_V$. For consider $f:(X_A, \sigma_A) \to (X_B, \sigma_B)$. Choose a refinement $W = U^B(-n, n)$ of U^B to be small enough so that $V = f^{-1}U^B(-n, n)$ is in P_A. Let $M = M(V)$ and $N = M(W)$. Then it is easy to check there is a commutative diagram

$$\begin{tikzcd} X_A \arrow[r, "f"'] \arrow[d, "i_V"'] & X_B \arrow[d, "i_W"] \\ X_M \arrow[r, "\rho"'] & X_N \end{tikzcd}$$

where ρ is induced by the bijection $F: V_i \to f(V_i)$ between rectangles in V and rectangles in W. As above we have $N = FMF^{-1}$, and $\rho = c_R$ for $R = F^{-1}$ and $S = FM$.

Recall from [W] that for two Markov partitions $U, V \in P_A$ with $U < V$ the length $l(U, V)$ is defined to be the minimum value $m+n$ such that $U < V < U(-m, n)$. We will prove (1.1) inductively verifying the following statement for each integer $k \geq 0$:

(E_k) For any zero-one matrix A if $l(U^A, V) \leq k$, then (1.1) holds.

The statement (E_0) holds because A and $B = M(V)$ are conjugate by a permutation. See above. Next we verify (E_1). Assume $U^A < V < U^A(0, 1)$. The argument in the other case is similar. Let R and S be as in (3.1) of [W] such that $A = RS$ and $B = SR$. We will show $i_V = c_R$. Since both these homeomorphisms intertwine σ_A and σ_B, it suffices to show

$$i_V(x)_0 = c_R(x)_0$$

for each $x = \{x_n\} \in X_A$. Let $i = x_0$ and $j = x_1$. Write each V_k as a disjoint union

$$V_k = \bigcup_b U_a \cap \sigma_A^{-1} U_b.$$

Then $i_V(x)_0$ is the unique V_k containing $U_i \cap \sigma_A^{-1} U_j$. On the other hand, $c_R(x)_0$ is the unqiue V_k such that $R(i, k)S(k, j) = 1$. The condition $R(i, k) = 1$ means $V_k \subset U_i$ and the condition $S(k, j) = 1$ means $V_k \cap \sigma_A^{-1}(U_j) \neq \varnothing$. Hence V_k contains $U_i \cap \sigma_A^{-1} U_j$. Thus $i_V(x)_0 = c_R(x)_0$.

Now assume (E_k) is true and suppose $l(U^A, V) \le k+1 = m+n$ where $n \ge 1$. The case $m \ge 1$ is similar. Consider the diagram of refinements

$$\begin{array}{ccc} U^A & \xrightarrow[(-m,n)]{} & V \\ \downarrow & & \downarrow{\scriptstyle (0,1)} \\ U^A(-m, n) & \xleftarrow[(-m,n-1)]{} & V \cap \sigma_A^{-1}(U) \end{array}$$

where $U \to_{(-p,q)} V$ means $U < V < U(-p, q)$. By induction we know (1.1) holds for $V \cap \sigma_A^{-1}(U) \to U^A(-m, n)$ and $V \to V \cap \sigma_A^{-1}(U)$. One verifies directly by induction (1.1) holds for $U^A \to U^A(-m, n)$. Hence by (1.9), (1.1) holds for $U^A \to V$. □

2. *Topological conjugacies*

In this section we discuss the topological counterparts C_R and C_S of the elementary symbolic conjugacies. The construction of C_R and C_S uses certain *model* fitted diffeomorphisms of S^q. See (2.7), (2.13), and (2.15).

Let $m \ge 1$ and fix an integer e satisfying $2 \le e \le q-1$. Let $S^q(m)$ denote the q-sphere $S^q \cong R^q \cup \{\infty\}$ equipped with a fixed handlebody decomposition consisting of a single handle $H_0 \cong D^0 \times D^q$ of index zero, m handles $H_{e-1}(i) \cong D^{e-1} \times D^{q-e+1}$ of index $e-1$, m handles $H_e(i) \cong D^e \times D^{q-e}$ of index e which cancel the corresponding handles of index $e-1$, and finally a single handle $H_q \cong D^q \times D^0$ of index q. More precisely, construct $S^q(m)$ as follows.

A *k-block* $B \subset R^k$ is a product $B = I_1 \times I_2 \times \cdots \times I_k$ of intervals $I_j \subset R$ where $j = 1, \ldots, k$. A k-block *with a layer of fat* is a k-block B together with another k-block B' containing B. The *fat* of B is the closure of $B' - B$. If $B \subset R^k$ and $C \subset R^l$ are blocks with fat, then $B \times C \subset R^{k+l}$ is a block with fat also. The fat of $B \times C$ is

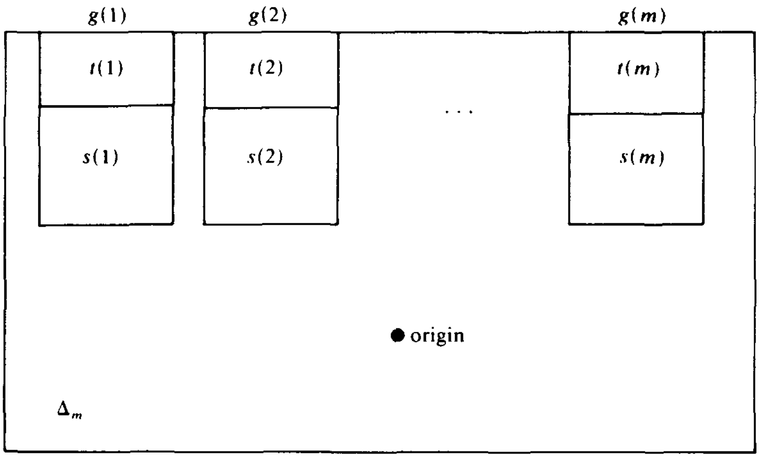
$$((\text{fat of } B) \times C) \cup (B \times (\text{fat of } C)).$$

We will write R^q in the form

$$R^q = R^{e-2} \times R^2 \times R^{q-e}.$$

Any block $B \subset R^{e-2} \times R^2 \times R^{q-e} \subset R^q$ is of the form $E \times I \times F$ where E is an $(e-1)$ block in $R^{e-2} \times R \times 0 \times 0$, F is a $q-e$ block in $0 \times 0 \times R^{q \times e}$ and $I = [a, b]$ is a 'vertical' interval in $0 \times 0 \times R \times 0$. The slices $E \times y \times F$ for $y \in I$ will be called the *horizontal levels* with $E \times a \times F$ the *bottom level* and $E \times b \times F$ the *top level*. If I is the union $I = I_1 \cap I_2$ of two nonnegative intervals meeting in the right end point of I_1 and the left end point of I_2, then B is the union of two sub-blocks $E \times I_1 \times F$ and $E \times I_2 \times F$ called the *top* and *bottom* of B respectively. Such a two tiered block will be called a *head*.

Write $R^q = R^{e-2} \times R^2 \times R^{q-e}$ and fix an *m-plate* $\Delta = \Delta_m \subset R^2 = 0 \times R^2 \times 0$. Namely, choose a 2-block containing the origin equipped with a set of m heads $g(1), \ldots, g(m)$ arranged along the top boundary as in the diagram:



(2.1)

FIGURE 1

Each head $g(i)$ is regarded as the union of a top block $t(i)$ and a bottom block $s(i)$. Moreover, each top will be viewed as the product of a 'horizontal' interval in $0\times R\times 0\times 0$ and a 'vertical' interval in $0\times(0\times R)\times 0$.

Let $D^k_\varepsilon=[-\varepsilon,\varepsilon]^k$ and when $\varepsilon=1$ just write D^k. Define the *thick m-plate* P_m to be

$$P_m=D^{e-2}\times\Delta_m$$

Fix a number $0<\beta\leq\frac{1}{2}$ and let

$$h(i)=D^{e-2}_\beta\times g(i),$$

$$h_0=\text{Closure of}\left[P_m-\left(\bigcup_{1\leq i\leq m}h(i)\right)\right],$$

$$H(i)=h(i)\times D^{q-e}.$$

Define the handles of $S^q(m)$ to be

$$H_0=\text{Closure of}\left[P_m\times D^{q-e}-\left(\bigcup_{1\leq i\leq m}H(i)\right)\right],$$

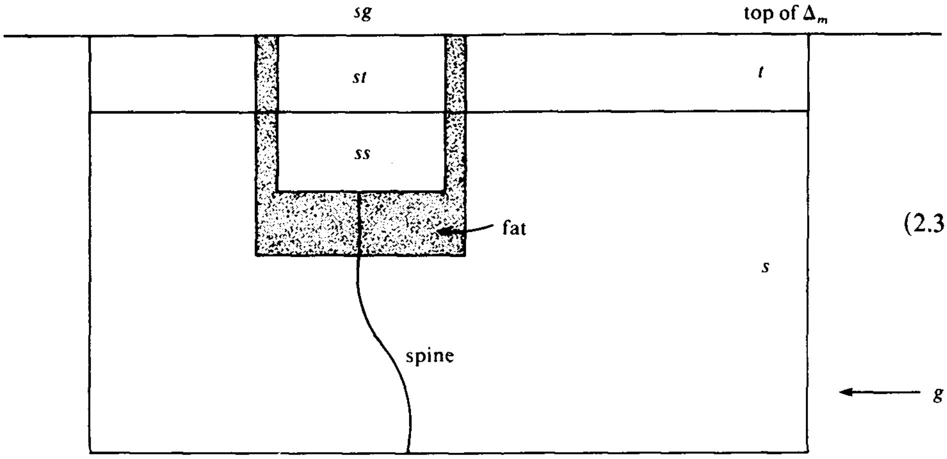
$$H_{e-1}(i)=[D^{e-2}_\beta\times(\text{horizontal of } t(i))]\times[(\text{vertical of } t(i))]\times D^{q-e}],$$

$$H_e(i)=[D^{e-2}_\beta\times s(i)]\times D^{q-e},$$

$$H_q=\text{Closure of}\,[(R^q-P_m\times D^{q-e})\cup(\infty)]. \tag{2.2}$$

Observe that $H(i)=H_{e-1}(i)\cup H_e(i)$ and $H_0=h_0\times D^{q-e}$.

The $h=h(i)$ are heads and we will use certain subheads sh of the h constructed as follows: In $0\times R^2\times 0$ choose a sub-block with fat sg of $g=g(i)$ as in the diagram:



(2.3)

FIGURE 2

and choose a small block $B \subset R^{e-2}$ with a thin layer of fat so that both B and its fat are contained in Int D_β^{e-2}. Then set

$$sh = B \times sg \tag{2.4}$$

The first part of this section *very carefully* constructs certain Smale diffeomorphisms $F: S^q(m) \to S^q(n)$ which are fitted both on the $(e-1)$-handles and e-handles according to an *arbitrary* nonnegative integer $m \times n$ matrix R. The main properties required of this construction are stated precisely in (2.7) and (2.16). We proceed in three steps.

Step I. Interval models

The horizontal model

Let I, J be two closed intervals in R. Consider an imbedding $f: I \to \mathbb{R}$ satisfying

(i) $J \subset \text{Int}\,(f(I))$

(ii) $f'(x) > 0$ for all $x \in I$

(iii) There is a $\lambda > 1$ and a $\mu \in \mathbb{R}$ such that if $x \in I$ and $f(x) \in J$, then $f(x) = \lambda x + \mu$.

For simplicity we will usually write $f: I \to J$ even though $f(I)$ is not contained in J. Such an f will be called an *h-model*. The following properties hold for h-models.

(a) Let $f: I \to J$ and $g: J \to K$ be h-models, and let $I' = f^{-1}(J)$. Then $gf: I' \to K$ is an h-model.

(b) Any two h-models $f, g: I \to J$ are isotopic through h-models $I \to J$.

The proof of (a) is clear. Here is the proof of (b). Let $[a, b] = f^{-1}(J)$ and $[c, d] = g^{-1}(J)$. Either $b - a \le d - c$ or vice-versa. Say $b - a \le d - c$. Let $\alpha_t: I \to I$, $0 \le t \le 1$, be an isotopy such that $\alpha_0 = 1$, $\alpha_1([a, b]) \subset [c, d]$, and α_t is a translation on $[a, b]$. In particular, $\alpha_t'(x) = 1$ for $x \in [a, b]$. Define $f_t = f \circ \alpha_t^{-1}$. This is an h-model $I \to K$ so that $f_1^{-1}(J) = [r, s] \subset [c, d]$ with $r - s = b - a$. Now let $\lambda = (d - c)/(r - s)$ and $\mu = c - \lambda r$. Let $\beta_t: I \to I$ for $0 \le t \le 1$ be an isotopy of $\beta_0 = 1$ so that $\beta_t'(x) > 0$ for all $x \in I$ and $\beta_t(x) = ((1-t) + t\lambda)x + t\mu$ for $x \in [r, s]$. Note that $\beta_t([r, s]) \subset [c, d]$ and $\beta_1([r, s]) = [c, d]$. Define $g_t = g \circ \beta_t$. These are also h-models $I \to J$ and $g_1^{-1}(J) =$

$[r, s]$. Finally, $(1-t)f_1 + tg_1$ for $0 \le t \le 1$ is an isotopy through h-models from f_1 and g_1.

The vertical model. Let I, J be two closed intervals in $\mathbb{R}$ and write each as the union of two closed subintervals meeting at the end points: $I = I_1 \cup I_2$ and $J_1 \cup J_2$. A *v-model* $f: I \to J$ is a smooth imbedding $f: I \to \mathbb{R}$ satisfying

(i) $f: I_1 \to J_1$ is an h-model

(ii) $f(I_2) \subset \operatorname{Int}(J_2)$

(*iii*) There is a $\mu < 1$ and a $\nu \in \mathbb{R}$ such that $f(x) = \mu x + \nu$ for $x \in I_2$.

The following properties hold:

(a) Let $f: I \to J$ and $g: J \to K$ be v-models. Let $I^* = f^{-1}(J)$, $I_1^* = I_1 \cap I^*$, and $I_2^* = I_2$. Then $gf: I^* \to K$ is a v-model.

(b) Any two v-models $f, g: I \to J$ are isotopic through v-models $I \to J$.

As above (a) is clear. To see (b) use isotopies α_t and β_t as above but satisfying $\alpha_t | I_2 = \beta_t | I_2 = 1$ to deform f and g through v-models until they satisfy $f^{-1}(J_1) = g^{-1}(J_1)$. Then take the linear isotopy $(1-t)f + tg$ from f to g.

Now we combine these interval models to obtain models for maps between heads $X = I \times J$ and $Y = K \times L$ where $I = I_1 \times \cdots \times I_{e-1}$ and $K = K_1 \times \cdots \times K_{e-1}$ are 'horizontal' $(e-1)$-blocks and the 'vertical' intervals J and L are disjoint unions $J = J_1 \cup J_2$ and $L = L_1 \cup L_2$ as for a v-model. We define a *model* $f: X \to Y$ to be a smooth imbedding of X into R^e of the form $f = (f_1, \ldots, f_e)$ where $f_e: J \to L$ is a v-model and $f_r: I_r \to K_r$ is an h-model for $r = 1, \ldots, e-1$. A picture of f looks like:

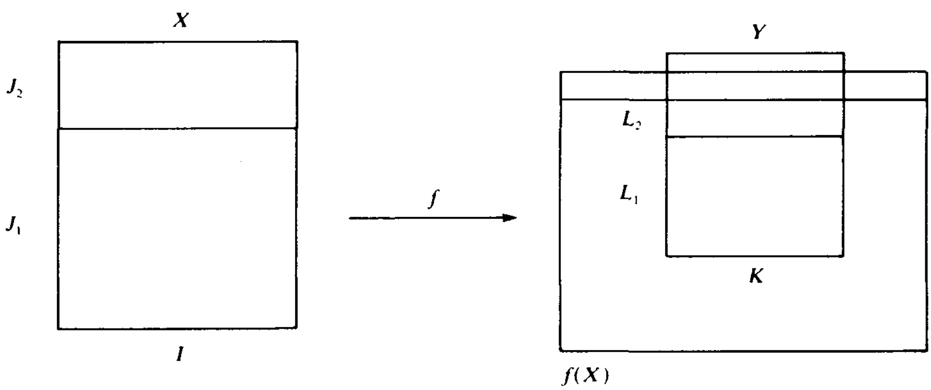


FIGURE 3

If $f: X \to Y$ is a model, we let $X^* = I^* \times J^*$ where $I^* = I_1^* \times \cdots \times I_{e-1}^*$ and the I_r^* and J^* are determined as above for h-models and v-models. X^* is a sub-head of X. The preceding material on h-models and v-models gives

LEMMA 2.5. (A) *Let $f: X \to Y$ be a model. Then f is expanding hyperbolic on the bottoms and is also hyperbolic on the tops, expanding on the horizontal levels and contracting vertically.*

(B) *Let $f: X \to Y$ and $g: Y \to Z$ be models. Then $gf: X^* \to Z$ is a model.*

(C) *Any two models from X to Y are isotopic through such models.*

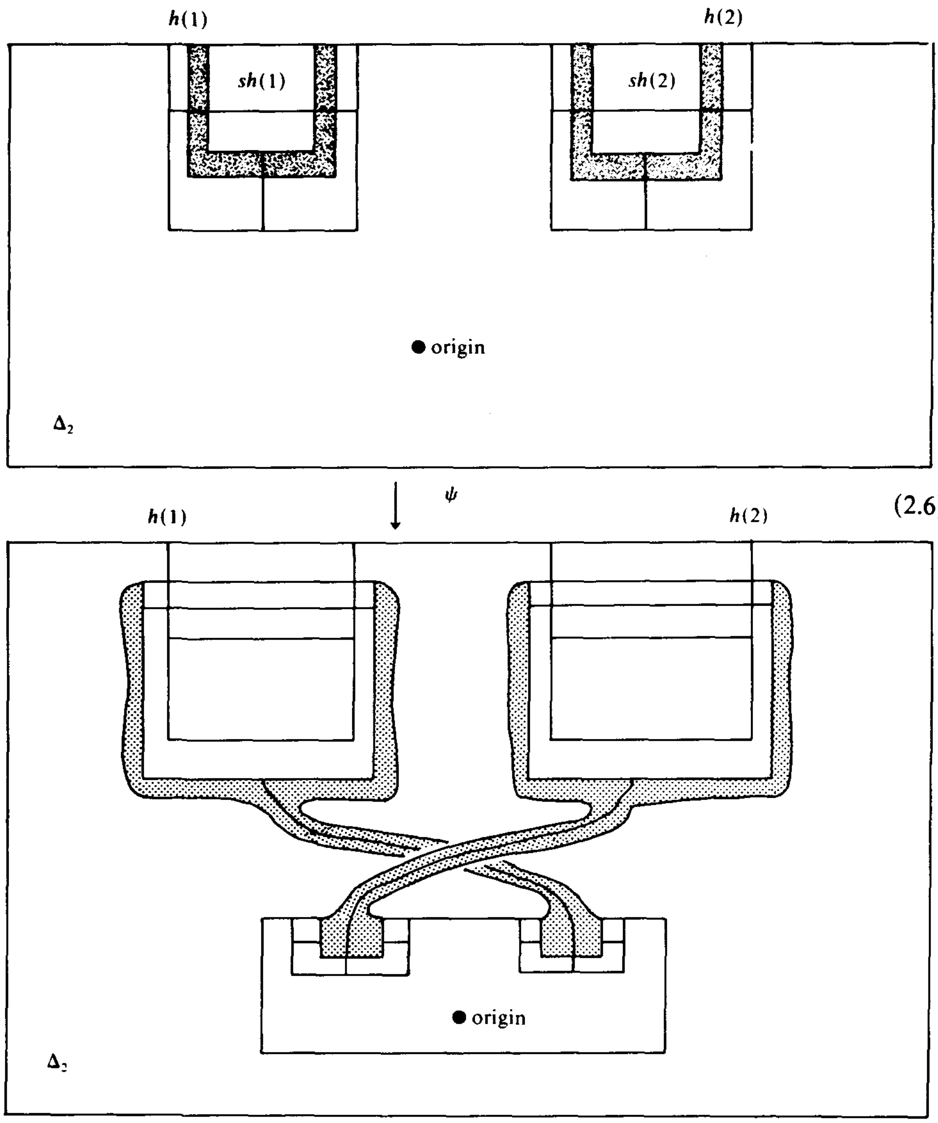
Step II. The cobra construction

Let $R = \{r_{ij}\}$ be an arbitrary $m \times n$ matrix with non-negative integer entries. We will define the notion of an R-model imbedding

$$\psi = \psi_R : P_m \times D^{q-e} \to \mathrm{Int}\, P_n \times D^{q-e}$$

which has the origin as a sink and is fitted on the $(e-1)$-handles and e-handles according to matrix R. Such models will exist when $2 \le e \le q-1$.

The diagram below illustrates the procedure for $m = n = 2$, $e = 2$, $q = 3$ and $r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We first construct an imbedding $\psi : P_2 \to P_2 \times D^1$ such that the composition of ψ and the projection $P_2 \times D^1 \to P_2$ looks like



(2.6)

FIGURE 4

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This is like a cobra with two heads. Write $\psi=(f, g)$ where $f: P_2 \to P_2$ and $g: P_2 \to D^1$. Then extend ψ to an embedding of all of $P_2\times D^1$ by letting $\psi(x, y)=(f(x), g(x)+\lambda y)$ where $0<\lambda\ll 1$.

Let $R=(r_{ij})$ be an $m\times n$ matrix with non-negative coefficients. The various parts of $S^q(n)$ will be distinguished from the corresponding parts of $S^q(m)$ by a superscript prime.

Definition 2.7. An *R-model* is a smooth imbedding

$$\psi=\psi_R: P_m\times D^{q-e}\to \operatorname{Int}(P_n\times D^{q-e})$$

satisfying the following conditions:

(a) On a neighborhood of $H_0=h_0\times D^{q-e}$, ψ has the form

$$\psi(x, y)=(\mu x, \lambda y),\qquad 0<\mu, \lambda<1.$$

(b) Everything outside a small neighborhood of $\bigcup_{j=1}^{n}\psi^{-1}(H'(j))$ is mapped into $\operatorname{Int} H_0'$ by ψ.

(c) The set $\bigcup_{j=1}^{n}\psi^{-1}(H'(j))$ is contained in a disjoint union $\bigcup_\alpha sh_\alpha\times D^{q-e}$ where sh_α runs through a finite set of sub-heads with fat of the $h(i)$. Moreover, on each $sh_\alpha\times D^{q-e}$, ψ has the form

$$\psi(x, y)=(f_\alpha(x), \lambda_\alpha y+\mu_\alpha)$$

where $0<\lambda_\alpha\le\frac{1}{2}$, $\mu_\alpha=\operatorname{Int} D^{q-e}$, and $f_\alpha: sh_\alpha\to h'(j)$ is a model imbedding taking the fat of sh_α into h_0.

(d) For a fixed pair of indices (i, j) we have

$$r_{ij}=\left\{\begin{matrix}\text{number of }\alpha\text{ with } sh_\alpha\times D^{q-e}\subset H(i)\\ \text{and } f_\alpha(sh_\alpha\times D^{q-e})\cap H'(j)\neq\varnothing\end{matrix}\right\}.$$

[Observe from (b) and (c) that for a given α there is at most one j for which $f_\alpha(sh_\alpha\times D^{q-e})\cap H'(j)\neq\varnothing$.

OBSERVATION 2.8. *If* $\psi_R: P_m\times D^{q-e}\to \operatorname{Int}(P_n\times D^{q-e})$ *and* $\psi_S: P_n\times D^{q-e}\to \operatorname{Int}(P_r\times D^{q-e})$ *are R and S models respectively. Then the composition* $\psi_S\psi_R: P_m\times D^{q-e}\to \operatorname{Int}(P_r\times D^{q-e})$ *is an RS-model.*

Now we discuss the framing invariant for an R-model *under the assumption that* $q\ge 4$. Suppose $sh_\alpha\times D^{q-e}\subset H(i)$. A *spine* for $sh_\alpha\times D^{q-e}$ is an arc c_α ascending from the bottom of the block $H(i)$ up to the bottom of $sh_\alpha\times D^{q-e}$ so that the projection onto the vertical coordinate (in $0\times 0\times R\times 0$) has no critical points. See (2.3) for an example. Thus each horizontal slice $R^{e-2}\times R\times y\times R^{q-e}$ is transverse to c_α. Note that the horizontal slice is equipped with a canonical $(q-1)$-frame. Suppose ψ satisfies the conditions:

(a) $\psi: c_\alpha\to \operatorname{Int}(P_n\times D^{q-e})\to 0\times 0\times R\times 0$ has no critical points.

(b) For each $z\in c_\alpha$ there is a neighborhood U_z of z in the horizontal slice such that ψ takes U_z diffeomorphically onto a neighborhood of $\psi(z)$ in the horizontal slice through $\psi(z)$. (2.9)

Then at each $z\in c_\alpha$ the derivative of ψ restricted to the horizontal slice at z takes the canonical framing to another framing of the horizontal slice at $\psi(z)$. This new framing varies smoothly with z. Moreover, near the top or bottom of c_α the frame lies in the contractible subspace of frames obtained from the canonical one by multiplying each frame vector by a positive number. This data therefore determines the *framing invariant*

$$\mathrm{fr}\,(\psi, c_\alpha) \in \pi_1(GL^+(q-1,\mathbb{R})) \cong \mathbb{Z}/2 \tag{2.10}$$

for ψ on the spine c_α.

In general an R-model ψ will not satisfy (2.9). However, when $q \geq 4$, ψ will be isotopic (through R-models) to an R-model satisfying (2.8). Recall the standard consequence of transversality that there are no knots or braids in R^q for $q \geq 4$. See [**B**, **Hi**]. Use this fact to deform φ until (a) of (2.9) holds. Then as in the uniqueness part of the Tubular Neighborhood Theorem [**Hi**] further deform ψ to obtain (b). This can all be done keeping ψ fixed on a neighborhood of $H_0 \cup [\bigcup_\alpha sh_\alpha \times D^{q-e}]$. Since there are also no braids in R^q when $q \geq 4$, if the above deformation is done in a different way to obtain another R-model satisfying (2.9), then this new map is isotopic through R-models to the first one through R-models satisfying (2.9). Hence the framings $\mathrm{fr}\,(\psi, c_\alpha)$ are well defined for an R-model which do not necessarily satisfy (2.9).

Definition 2.11. We say ψ has *trivial framing* (on the spines c_α) if $\mathrm{fr}\,(\psi, c_\alpha) = 0$ for every c_α.

LEMMA 2.12. *If ψ has trivial framing on one choice of spines, it has trivial framing for any other choice of spines. Moreover, the framing invariant only depends on the isotopy class of ψ in the space of R-models.*

Proof. Any two choices of spines are isotopic through a one-parameter family of spines when $q \geq 4$. This gives rise to a homotopy between the two sets of framing invariants.

Definition 2.13. A *special R-model* is an R-model with a trivial framing.

PROPOSITION 2.14. (A) *If* $2 \leq e \leq q-1$, *then there are R-models satisfying* (2.9) *with trivial framing. In particular, special R-models exist for* $q \geq 4$.

(B) *The composition of a special R-model and a special S-model is a special RS-model.*

(C) *If* $3 \leq e \leq q-2$, *then any two special R-models are isotopic through special R-models.*

Proof of (A). See diagram (2.6) for the idea. For each pair of indices (i, j) corresponding to the entry r_{ij} of the matrix R, let $b(i, j) \subset h(i)$ be a subhead (without fat) as in (2.3). Do this so that all the $b(i, j)$ are disjoint. Inside each $b(i, j)$ construct a collection $\{sh_\alpha\}$ of exactly r_{ij} subheads with fat as in (2.3). Choose points p_{ij} in the bottom of h'_j. Inside the interior of D^{q-e} choose a family of disjoint disks $\{t(i, j)\}$, and the inside the interior of each $t(i, j)$ choose a collection $\{z_\alpha\}$ of exactly r_{ij} points.

First we construct an imbedding $f: P_m \to \mathrm{Int}\,(P_n \times D^{q-e})$. Choose a very small $0 < \mu \ll 1$ and let $\kappa: P_m \to H'_0$ be $\kappa(x) = \mu x$. Choose r_{ij} points $\{x_\alpha\}$ in the top of $\kappa(sh_\alpha)$ and then choose 'parallel' disjoint arcs d_α ascending up from the points x_α through $\mathrm{Int}\,(P_n \times D^{q-e})$ to the points $p_{ij} \times z_\alpha$ in the bottom of $H'(j) = h'(j) \times D^{q-e}$. Fix an index α. Then pull the subhead sh_α up along d_α and map it by a model imbedding to $h'(j) \times z_\alpha \subset R^e \times z_\alpha$. This should be done by an isotopy which keeps things fixed outside the fat of sh_α and in such a way that at each time sh_α is mapped into a level $R^e \times z$, $z \in D^{q-e}$, by an imbedding which preserves axes. This insures the framing invariant will be zero along d_α. Perform the preceding construction for all the d_α for various pairs (i, j) in such a way to get the imbedding f which takes the fat of each sh_α into H_0'. Finally, choose $0 < \lambda \ll 1$ to be small enough so that

$$\psi_R : P_n \times D^{q-e} \to \operatorname{Int}(P_n \times D^{q-e})$$

defined by

$$\psi_R(x, y) = f(x) + (0, \lambda y)$$

is an imbedding. It is clear from the construction that ψ_R satisfies the required properties.

Proof of B. It has already been remarked in (2.8) that the composition of an R-model with an S-model is an RS-model. So it remains to show the composition has trivial farming. We will first show that the composition of two models $\psi_R : S^q(m) \to S^q(n)$ and $\psi_S : S^q(n) \to S^q(t)$ satisfying (2.9) also satisfies (2.9).

Let $\{sh'_\beta\}$ be the subheads of the various $h'(j)$ in P_n used to construct ψ_S. Let $\{c_\beta\}$ be the corresponding collections of spines. Similarly for $\{sh_\alpha\}$ and $\{c_\alpha\}$ used to obtain ψ_R. Then ψ_R maps c_α into $\operatorname{Int} H_0'$ so that (a) of (2.9) holds as in the diagram below which shows a 2-dimensional cross section (i.e. projection in $0 \times R^2 0$) of the special case where $h(i)$ has only subhead sh_α and $h'(j)$ has only one subhead sh'_β.

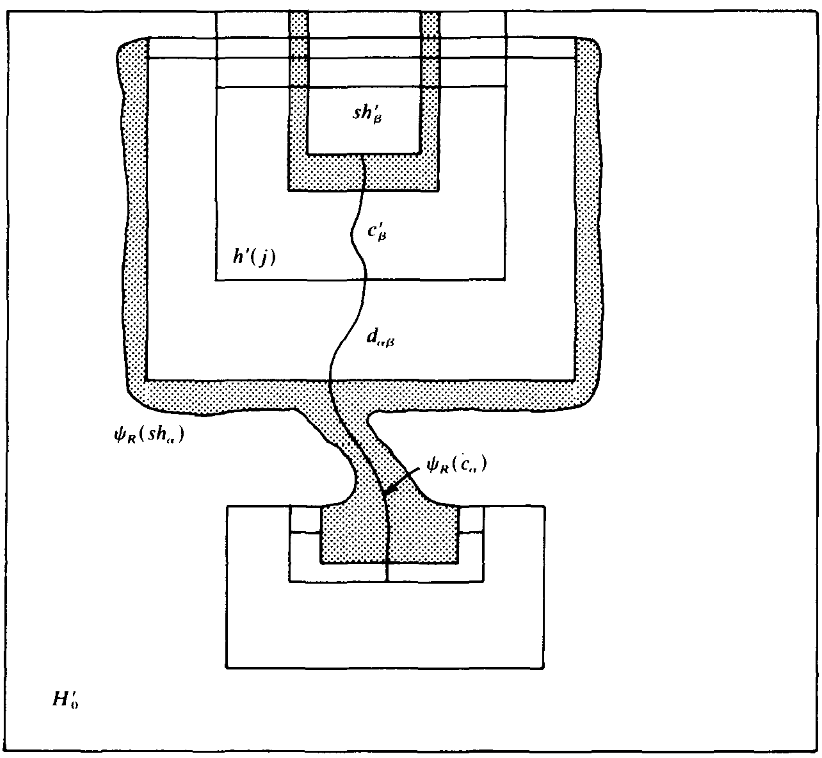

FIGURE 5

Connect the top of $\psi^R(c_\alpha)$ to the bottom of c'_β by an ascending arc $d_{\alpha\beta}$ as in the diagram. Property (c) of (2.7) then implies that the arc $c_\alpha * \psi_R^{-1}(d_{\alpha\beta}) * \psi_R^{-1}(c'_\beta)$ is a spine for the subhead $\psi_R^{-1}(sh'_\beta \times D^{q-e})$ used to get $\psi_{RS} = \psi_S\psi_R$. More generally, there may be several sh_α in an h_i and several sh_β in the $h'(j)$. In this case, fix $sh_\alpha \subset h'_\alpha$ and for each $sh'_\beta \subset h'(j)$ select a parallel copy of c_α close to c_α; then repeat the above construction. Do this in such a way that all the spines are disjoint.

We must now check that (2.9) holds for $\psi_S\psi_R$ on each spine like $c_\alpha * \psi_R^{-1}(d_{\alpha\beta}) * \psi_R^{-1}(c'_\beta)$.

Part (a). Applying $\psi_S\psi_R$ gives $\psi_S\psi_R(c_\alpha) * \psi_S(d_{\alpha\beta}) * \psi_S(c'_\beta)$. Since ψ_S satisfies (a) of (2.7) and since the ascending arc $\psi_R(c_\alpha) * d_{\alpha\beta}$ is contained in H'_0, it follows that on applying ψ_S one gets an arc ascending up to the bottom of $\psi_S(c'_\beta)$. Since ψ_S satisfies (a) of (2.9), we then continue this by the ascending arc $\psi_S(c'_\beta)$.

Part (b). This follows on $c_\alpha * \psi_R^{-1}(d_{\alpha\beta})$ because $\psi_R(c_\alpha) * d_{\alpha\beta}$ lies in H'_0 and ψ_S satisfies (a) of (2.7) there. It holds on $\psi_R^{-1}(d_{\alpha\beta})$ because ψ_R satisfies (c) of (2.7) and ψ_S satisfies (b) of (2.9) on c'_β.

Next we compute the framing fr for $\psi_S\psi_R$ on $c_\alpha * \psi_R^{-1}(d_{\alpha\beta}) * \psi_R^{-1}(c'_\beta)$. It is given by the formula

$$\begin{aligned}\mathrm{fr} &= \mathrm{fr}\,(\psi_S\psi_R, c_\alpha) + \mathrm{fr}\,(\psi_S\psi_R, \psi_R^{-1}(d_{\alpha\beta})) + \mathrm{fr}\,(\psi_S\psi_R, \psi_R^{-1}(c'_\beta))\\ &= [\mathrm{fr}\,(\psi_R, c_\alpha) + \mathrm{fr}\,(\psi_S, \psi_R(c_\alpha))] + [\mathrm{fr}\,(\psi_R, \psi_R^{-1}(d_{\alpha\beta})) + \mathrm{fr}\,(\psi_S, d_{\alpha\beta})]\\ &\quad + [(\mathrm{fr}\,(\psi_R, \psi_R^{-1}(c'_\beta)) + \mathrm{fr}\,(\psi_S, c'_\beta)].\end{aligned}$$

Then we see that $\mathrm{fr}(\psi_R, c_\alpha)$ and $\mathrm{fr}(\psi_S, c'_\beta)$ are zero by hypothesis; $\mathrm{fr}\,(\psi_S, \psi_R(c_\alpha)) = 0$ because $\psi_R(c_\alpha) \subset H'_0$ and ψ_S satisfies (a) of (2.7); $\mathrm{fr}\,(\psi_S, d_{\alpha\beta}) = 0$ because $d_{\alpha\beta} \subset H'_0$ and ψ_S satisfies (a) of (2.7); and, finally, $\mathrm{fr}\,(\psi_R, \psi_R^{-1}(d_{\alpha\beta}))$ and $\mathrm{fr}\,(\psi_R, \psi_R^{-1}(d_{\alpha\beta}))$ are zero because $\psi_R^{-1}(d_{\alpha\beta}) * \psi_R^{-1}(c'_\beta)$ lies in $sh_\alpha \times D^{q-e}$ where ψ_R satisfies (c) of (2.7).

Proof of (C). Let ψ and ϕ both be R-models.

Step 1. Both ψ and ϕ satisfy (a) of (2.7) for perhaps different sets of parameters. By smoothly shrinking these parameters to a common set of smaller ones we deform ψ and ϕ keeping everything unchanged outside a small neighborhood of H_0 until ψ and ϕ become equal on another smaller neighborhood of H_0.

Step 2. Let $\{sh_\alpha\}$ and $\{sk_\alpha\}$ respectively denote the subheads of the heads h_i used to construct ψ and ϕ as in (c) of (2.7). We want to deform ψ and ϕ through R-models to new ψ and ϕ such that

(1) $sh_\alpha = sk_\alpha$ for each α.

This will require $3 \le e$.

Fix a pair of indices (i, j) and let $\{s\bar{h}_\alpha\}$ denote the subcollection of the r_{ij} subheads of h_i satisfying (d) of (2.7) for ψ. Similarly for $\{s\bar{k}_\alpha\}$ and ϕ. In fact, for this step let $s\bar{h}_\alpha$ denote both $s\bar{h}_\alpha$ together with its layer of fat. Ditto for $s\bar{k}_\alpha$. Write $s\bar{h}_\alpha = I_\alpha \times J_\alpha$ and $s\bar{k}_\alpha = K_\alpha \times L_\alpha$ as in (2.5). Now choose small subheads $I'_\alpha \times J'_\alpha$ of $s\bar{h}_\alpha$ and $K'_\alpha \times L'_\alpha$ of $s\bar{k}_\alpha$ such that the horizontal intervals of the I'_α and K'_α have the same length and the vertical intervals of J'_α and L'_α have the same length. Note that by the construction (2.3) all the top intervals of the interval intervals J'_α and L'_α have the same length automatically. Let $\theta_t : P_m \to P_m$ be an isotopy of the identity satisfying

(2) θ_t preserves each horizontal level and maps each linear interval of I'_α into the corresponding axis of $R^{e-2}\times R\times R\times 0$ by an affine imbedding with derivative at least one,

(3) θ_t is the identity outside a small neighborhood of the $s\bar{h}_\alpha$,

(4) $\theta_1(I'_\alpha\times J'_\alpha)=I_\alpha\times J'_\alpha$.

Construct a similar isotopy $\Gamma_t: P_m\to P_m$ expanding $K'_\alpha\times L'_\alpha$ out to $K_\alpha\times L'_\alpha$. Then the required deformations to new ψ and ϕ are $\psi_t=\psi(\theta_t\times id)$ and $\phi_t=\phi(\Gamma_t\times id)$. The new subheads are $s\bar{h}_\alpha=I'_\alpha\times J_\alpha$ and $s\bar{k}_\alpha=K'_\alpha\times L_\alpha$. Next let $\theta_t: P_m\to P_m$ be another isotopy satisfying.

(5) θ_t preserves each vertical line and for each α there is an imbedding $f_t: J'_\alpha\to 0\times 0\times R\times 0$ such that f_t is fixed on the top interval of J'_α, $f'_t\geq 1$ on the lower interval and is affine on it except in a very small neighborhood of the top end point, and $f_1(J'_\alpha)=J_\alpha$ so that on $I'_\alpha\times J'_\alpha$ we have $\theta_t=id\times f_t$ and $\theta_1(I'_\alpha\times J'_\alpha)=I'_\alpha\times J_\alpha$.

(6) θ_t is fixed outside of a small neighborhood of the $s\bar{h}_\alpha$.

Construct a similar isotopy Γ_t expanding $K'_\alpha\times L'_\alpha$ to $K_\alpha\times L_\alpha$. Then tne deformations to new ψ and ϕ are $\psi_t=\psi(\theta_t\times id)$ and $\phi=\phi(\Gamma_t\times id)$ and the new subheads are $s\bar{h}_\alpha=I'_\alpha\times J'_\alpha$ and $s\bar{k}_\alpha=K'_\alpha\times L'_\alpha$ for ψ and ϕ respectively.

We now bring in the hypothesis that $3\leq e$ which implies the horizontal levels of $P_m=D^{e-2}\times\Delta_m$ have dimension at least two as in the diagram below showing the top level of a head $h(i)$ together with various subheads $s\bar{h}_\alpha$ and $s\bar{k}_\alpha$.

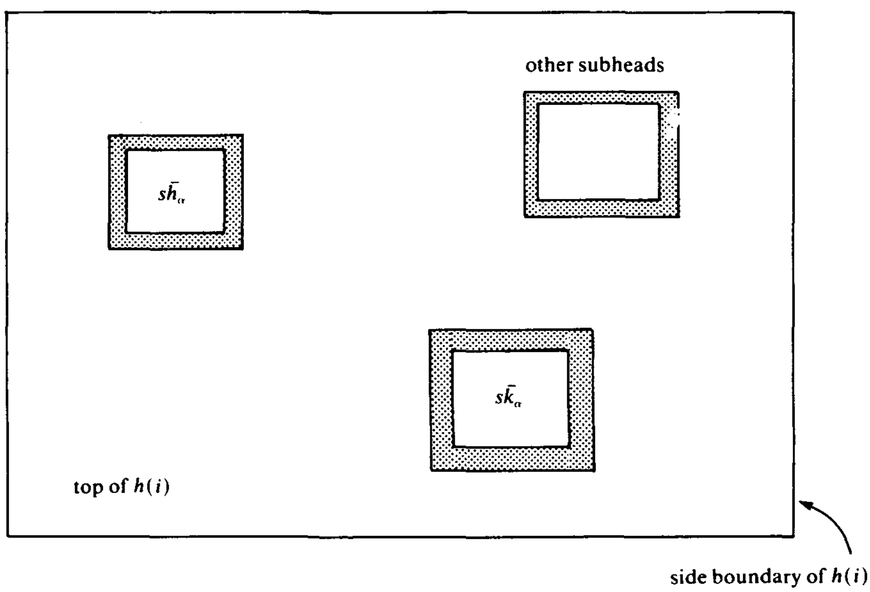

FIGURE 6

The size of $s\bar{h}_\alpha$ and $s\bar{k}_\alpha$ can be as small as desired. In particular, make them small enough so that the diameter is less than, say, half the distance between any two of them and half the distance between any one of them and the side boundary of $h(i)$.

Also make the diameter of the sh_α smaller than half the distance between any $s\bar{h}_\alpha$ and any other sh_α and half the distance between any $s\bar{k}_\alpha$ and any other sk_α. We can then construct an isotopy $\theta_t: P_m \to P_m$ satisfying

(7) θ_t preserves horizontal levels, has support contained in $h(i)$, and leaves fixed the subheads sh_α not belonging to the collection $\{s\bar{h}_\alpha\}$.

(8) θ_t restricted to $s\bar{h}_\alpha$ is a translation in the horizontal direction and $\theta_1(s\bar{h}_\alpha) = s\bar{k}_\alpha$ for each α.

Finally, we then have the isotopy $\phi_t = \phi(\theta_t \times id)$ of $\phi = \phi_0$ to a new $\phi = \phi_1$ so that (1) holds for the $s\bar{h}_\alpha$ and $s\bar{k}_\alpha$. Repeat the procedure until (1) holds for all indices α.

It is clear that the isotopies in the above procedure are deformations of ψ and ϕ through R-models.

Step 3. For this part we return to the usual notation where sh_α denotes the subhead itself and not both the subhead and its layer of fat. From Step 2 it can be assumed the subheads sh_α are the same for both ψ and ϕ. We now show ψ and ϕ can be deformed through R-models to new ψ and ϕ satisfying

(9) $\psi | sh_\alpha \times D^{q-e} = \phi | sh_\alpha \times D^{q-e}$

for each α, and it is here that we use the hypothesis $e \le q-2$. The isotopy will keep ψ and ϕ fixed on H_0 where they agree by Step 1.

For each $j = 1, \dots, n$ let $k'(j)$ be a slightly larger e-block containing $h'(j)$ in its interior. Choose the $k(j)$ to be disjoint. We claim that it is possible to deform ψ and ϕ so that

(10) $\psi(sh_\alpha \times D^{q-e})$ and $\phi(sh_\alpha \times D^{q-e})$ are contained in $\mathrm{Int}\,(k'(j) \times D^{q-e})$ for each α

and

(11) $\psi(P_m \times D^{q-e})$ and $\phi(P_m \times D^{q-e})$ only intersect each $k'(j) \times D^{q-e}$ in its interior and along the bottom horizontal level.

The image of this situation composed with the projection $\pi: P_n \times D^{q-e} \to P_n$ is illustrated by the diagram in Figure 7 below.

To obtain (10) just shrink the model imbeddings in the horizontal and vertical directions. Then for (11) push down vertically along the sides of $k'(j) \times D^{q-e}$ until the image under ψ and ϕ comes into $k'(j) \times D^{q-e}$ only through the bottom level.

As in (c) of (2.7) write

$$\psi(x, y) = (f_\alpha(x), \kappa_\alpha y + \mu_\alpha),$$

$$\phi(x, y) = (g_\alpha(x), \lambda_\alpha y + \nu_\alpha),$$

for $(x, y) \in sh_\alpha \times D^{q-e}$ where $\mu_\alpha, \nu_\alpha \in \mathrm{Int}\,(D^{q-e})$. Then use (2.5) to further deform ψ and ϕ in a small neighborhood of $sh_\alpha \times D^{q-e}$ until

(12) $f_\alpha = g_\alpha$ on sh_α.

Next use a general position argument to further move ψ and ϕ slightly so that

(13) all the μ_α are different, all the ν_α are different, and no μ_α is equal to any ν_α.

Let δ denote the minimum of the distance between any two points μ_α and/or ν_α and also the distance between any μ_α or ν_α and the boundary of D^{q-e}. Deform ψ and ϕ isotopically through R-models which keep everything fixed outside a small

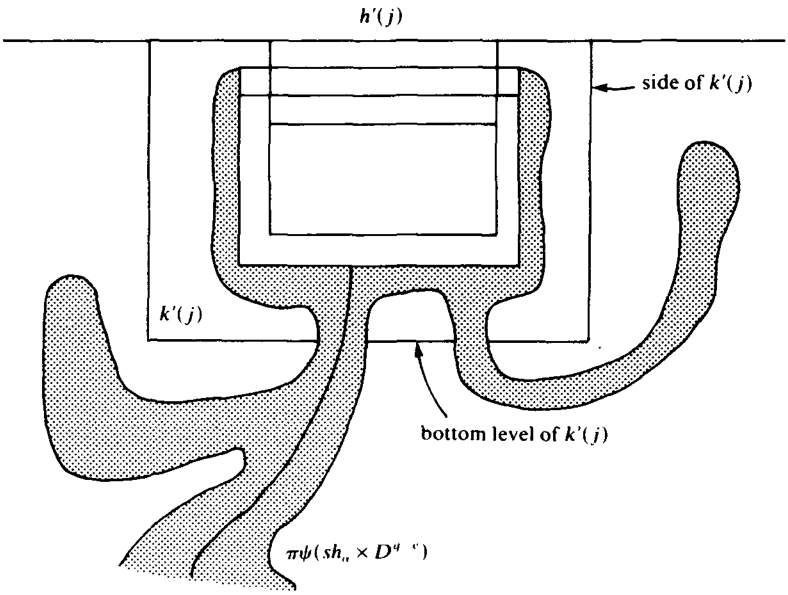

FIGURE 7

neighborhood of the $sh_\alpha \times D^{q-e}$ so as to shrink the parameters κ_α and λ_α to a common value

(14) $\kappa_\alpha = \lambda_\alpha = \delta/300$

The image of each $sh_\alpha \times D^{q-e}$ under ψ is of the form $X_\alpha \times M_\alpha \times B_\alpha$ where X_α is an $(e-1)$-block, $B_\alpha \subset \text{Int } D^{q-e}$ is a $(q-e)$-block and $M_\alpha = P_\alpha \cup Q_\alpha$ is a vertical interval. By (12) we can write the image of $sh_\alpha \times D^{q-e}$ similarly as $X_\alpha \times M_\alpha \times C_\alpha$. From (13) and (24) we know all the $(q-e)$-blocks B_α and C_α are disjoint and of the same size.

Now fix an index α and an index j. Write $k'(j) = E \times I$ where E is an $(e-1)$-block in $R^{e-2} \times R \times 0 \times 0$ and $I = [a, c]$ is a vertical interval in $0 \times 0 \times R \times 0$. Let $M = M_\alpha$, $B = B_\alpha$, and $C = C_\alpha$, $\mu = \mu_\alpha$, and $\nu = \nu_\alpha$. Write $M = [b, d]$ with $a < b < d < c$. Write $I = P \cup Q$ where $P = [a, b]$ and $Q = [b, c]$. Since $e \le q-2$ and all the blocks B_α and C_α have diameter no bigger than, say, $\delta/50$, there is an isotopy $\theta: P \times D^{q-e} \to D^{q-e}$ such that

(15) θ_t has support in Int D^{q-e} and does not move any B_α or C_α except B and C,

(16) θ_t is a translation on B,

(17) $\theta_t = id$ for t near a and $\theta_t = \theta_b$ for t near b, $\theta_b(B) = C$, and $\theta_b(\mu_\alpha) = \nu_\alpha$.

Use this to define a diffeomorphism $\bar\theta$ of $E \times I \times D^{q-e}$ by the formula

$$\bar\theta(x, t, y) = \begin{cases} (x, t, \theta(t, y)), & a \le t \le b \\ (x, t, \theta(b, y)), & b \le t \le c. \end{cases} \tag{18}$$

Define an isotopy $\bar\theta_s$ from the identity on $E \times I \times D^{q-e}$ to $\bar\theta$ by the formula

$$\bar\theta_s(x, t, y) = \begin{cases} (x, t, \theta((1-s)a + st, y)), & a \le t \le b \\ (x, t, \theta((1-s)a + sb, y)), & b \le t \le c \end{cases} \tag{19}$$

for $0 \le s \le 1$. Next define an isotopy ψ_s of ψ on $P_m \times D^{q-e}$ by the formula

$$\psi_s(z) = \begin{cases} \bar\theta_s\psi(z), & \text{for } z \in (sh_\alpha \times D^{q-e}) \cap \psi^{-1}(k'(j) \times D^{q-e}) \\ \psi(z), & \text{otherwise.} \end{cases} \tag{20}$$

Then (9) is satisfied for the particular index α we have just fixed. Continue to do this procedure for other sh_α until (9) holds for all of them. At each stage things are not moved on those $sh_\alpha \times D^{q-e}$ where (9) already holds.

Step 4. At this point we have deformed ψ and ϕ until they agree on a neighborhood of $H_0 \cup (\bigcup_\alpha sh_\alpha \times D^{q-e})$. The last part in proving (C) of (2.14) is to show ψ can be deformed to be equal to ϕ on the region

$$\text{Closure of}\left\{P_m \times D^{q-e} - H_0 - \bigcup_\alpha (sh_\alpha \times D^{q-e})\right\}.$$

The deformation in this step will not move anything near $H_0 \cup (\bigcup_\alpha sh_\alpha \times D^{q-e})$. Let $\{c_\alpha\}$ be the collection of spines used to define $\mathrm{fr}(\psi, c_\alpha)$. In view of (2.12) we can also use $\{c_\alpha\}$ to define $\mathrm{fr}(\phi, c_\alpha)$. Since $5 \le q$ (and hence $4 \le q$) and since there are no knots in R^q for $4 \le q$, we can deform ψ through R-models until $\psi \mid c_\alpha = \phi \mid c_\alpha$ on each c_α. Moreover, since $\mathrm{fr}(\psi, c_\alpha) = \mathrm{fr}(\phi, c_\alpha) = 0$, we can further deform ψ until it agrees with ϕ on a neighborhood of the c_α. Let U be a neighborhood of

$$X = H_0 \cup \left(\bigcup_\alpha c_\alpha\right) \cup \left(\bigcup_\alpha sh_\alpha \times D^{q-e}\right)$$

on which ψ is equal to ϕ. Observe as in the diagram below that X has a closed neighborhood V inside U so that $P_m \times D^{q-e}$ is diffeomorphic to $V \cup (\partial V \times I)$.

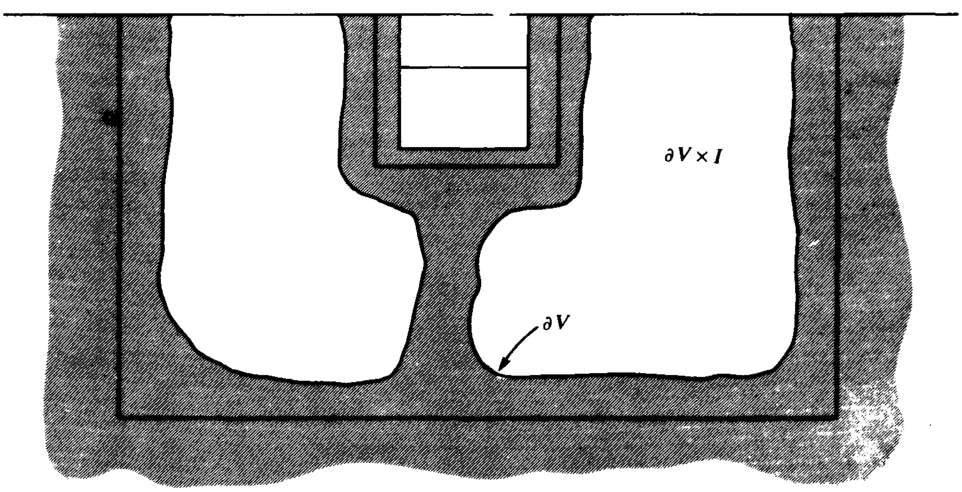

FIGURE 8

Hence we can choose an isotopy $\theta_t : P_m \times D^{q-e} \to P_m \times D^{q \times e}$ which is fixed on V and such that $\theta_1(P_m \times D^{q-e}) \subset U$. Then the final deformations of ψ and ϕ are $\psi\theta_t$ and $\phi\theta_t$.

This completes the proof of (C) of (2.14).

Step III. Extending to S^q. As usual let $R=\{r_{ij}\}$ be an arbitrary $m\times n$ non-negative integer matrix.

Definition 2.15. A diffeomorphism $\psi=\psi_R: S^q(m)\to S^q(n)$ is an *R-model* provided it satisfies

(a) $\psi(0)=0$, $\psi(\infty)=\infty$.

(b) ψ is already an R-model from $P_m\times D^{q-e}$ into $\operatorname{Int}(P_n\times D^{q-e})$.

(c) there is a neighborhood $U_\infty\subset \operatorname{Int} H_q$ of infinity and a number $\mu>1$ such that $H'_q\subset \operatorname{Int}\psi(U_\infty)$ and for $x\in U_\infty$ we have $\|T_x\psi\|\geq\mu$.

We similarly define the notion of an *R-model satisfying* (2.9) *with trivial framing* and a *special R-model.*

The global version of (2.14) is

PROPOSITION 2.16. (A) *If* $2\leq e\leq q-1$, *then there are R-models satisfying* (2.9) *with trivial framing. In particular, special R-models exist for* $q\geq 3$.

(*B*) *The composition of a special R-model and a special S-model is a special RS-model.*

(C) *Let* $3\leq e\leq q-2$ *and let* ψ *and* ϕ *be two special R-models. Then there is an isotopy* ψ_t *of* ψ *through special R-models such that* $\psi_1\,|\,P_m\times D^{q-e}=\phi\,|\,P_m\times D^{q-e}$.

Proof of A. We are identifying S^q with $R^q\cup\infty$ by stereographic projection. Let $0<\lambda\ll 1$ and let $\theta_\lambda: S^q\to S^q$ come from multiplication by λ on R^q. This takes $P_m\times D^{q-e}$ into a neighborhood of the origin in $\operatorname{Int}(P_n\times D^{q-e})$. Then as in (A) of (2.14) let $\psi_t: P_n\times D^{q-e}\to P_n\times D^{q-e}$ be an isotopy realizing the cobra construction so that $\psi=\psi_1\circ\theta_\lambda$ satisfies (2.9) with trivial framing.

Proof of (B). Conditions (a), (b), (c) of (2.15) are clearly preserved under composition, and the trivial framing property was verified in (B) of (2.14).

Proof of (C). By (C) of (2.14) there is an isotopy ψ_t between $\psi\,|\,P_m\times D^{q-e}$ and $\phi\,|\,P_m\times D^{q-e}$. Use the isotopy extension theorem to obtain an isotopy $\theta_t: S^q(n)\to S^q(n)$ having support in $\operatorname{Int}(P_n\times D^{q-e})$ so that on $P_m\times D^{q-e}$ we have $\psi_t=\theta_t\circ\psi$. The required deformation of $\psi: S^q(m)\to S^q(n)$ is therefore $\psi_t=\theta_t\circ\psi$.

3. *Proof of the Main Theorem*

Throughout this section assume A is an *irreducible, zero-one* matrix. Let $F=F_A: S^q(m)\to S^q(m)$ be a fixed choice of a special A-model as in § 2. It is a fitted Smale diffeomorphism as in [F, § 4] with a non-wandering set $\Omega=\Omega(F)$ the disjoint union

$$\Omega(F)=\Omega_0(F)\cup\Omega_{e-1}(F)\cup\Omega_e(F)\cup\Omega_q(F),$$

where each $\Omega_k(F)$ is a basic set of index k, $\Omega_0(F)=\{0\}$, $\Omega_q(F)=\{\infty\}$, and F restricted to both $\Omega_{e-1}(F)$ and $\Omega_e(F)$ is conjugate to σ_A. We shall concentrate attention on Ω_e. There is a similar statement for Ω_{e-1}. In this section, change notation slightly and let the collection of m handles of index e be denoted by $K(i)$ for $1\leq i\leq m$.

The basic set $\Omega_e(F)$ is the intersection

$$\Omega_e(F)=\bigcap_{-\infty<k<\infty} F^{-k}\Bigl(\bigcup_{1\leq i\leq m} K(i)\Bigr)$$

and the standard formula [F] for the topological conjugacy

$$\chi_F : X_A \to \Omega_e(F)$$

between σ_A on X_A and F on $\Omega_e(F)$ is given on $x = \{x_i\} \in X_A$ by

$$\chi_F(x) = \bigcap_{-\infty < l < \infty} F^{-l}(K(x_l)). \tag{3.1}$$

We will sometimes use the notation $\chi(F)$ for χ_F. A key point here is that A is a zero-one matrix. This implies that $K(i) \cap F^{\pm 1}K(j)$ has at most one component for each pair (i, j). Hence as in [F], the D^{q-e}-coordinate of a point in $\bigcap_{0 \le l \le n} F^l(K(x_l))$ converges to a single value as $n \to \infty$ because F is contracting in the D^{q-e} factor. Similarly, the R^e-coordinate of a point in $\bigcap_{0 \le l \le n} F^{-l}(K(x_l))$ converges to a single value because F^{-1} is contracting in the R^e factor.

Let P, Q, R, S be zero-one matrices with $P = RS$ and $Q = SR$. Assume P and Q are irreducible. Let $c_R : X_P \to X_Q$ and $c_S : X_Q \to X_P$ be the elementary symbolic conjugacies between σ_P and σ_Q as defined in the introduction. Let $C_R : S^q(m) \to S^q(m)$ be an R-model and $C_S : S^q(n) \to S^q(m)$ be an S-model. Let $D_P = C_S C_R : S^q(m) \to S^q(m)$ and $D_Q = C_R C_S : S^q(n) \to S^q(n)$. Then as discussed in the introduction, C_R and C_S are what we call elementary smooth conjugacies between the P-model D_P and the Q-model D_Q.

LEMMA 3.2. *There is a commutative diagram*

$$\begin{tikzcd}
X_P \arrow[r, shift left, "c_S"', leftarrow] \arrow[r, shift right, "c_R"'] \arrow[d, "\chi(D_P)"'] & X_Q \arrow[d, "\chi(D_Q)"] \\
\Omega_e(D_P) \arrow[r, shift left, "C_S"', leftarrow] \arrow[r, shift right, "C_R"'] & \Omega_e(D_Q)
\end{tikzcd}$$

Proof. As usual, the handles in $S^q(n)$ will be distinguished from those of $S^q(m)$ by a superscript 'prime'. Let $x \in X_P$, $y = c_R(x) \in X_Q$, $u = \chi(D_Q)(y) \in \Omega_e(D_Q)$, $z = \chi(D_P)(x) \in \Omega_e(D_P)$, and $v = C_R(z) \in \Omega_e(D_Q)$. Observe that u is characterized as the unique point such that $D_Q^k(u) \in K'(y_j)$ for all $j \in \mathbb{Z}$. To prove $v = u$ we must therefore verify v satisfies this condition. However, the homeomorphisms in the diagram of (3.2) commute with the appropriate shifts, the D_p, or the D_Q, and therefore it suffices to show both u and v lie in $K'(y_0)$. The definition of $\chi(D_P)$ implies $z \in K(x_0)$ and $D_P(z) \in K(x_1)$. Since $D_P = C_S C_R$ the image of z under C_R must lie in one of the $K'(j)$ which it $K(x_1)$ under C_S. However, $C_S C_R$ is an RS-model as observed in (2.8) and so j must satisfy $R(x_0, j)S(j, x_1) = P(x_0, x_1) = 1$. Since R, S, and P are zero-one matrices, there is exactly one such j and from the definition of c_R we must have $j = y_0$.

PROPOSITION 3.3. *Let F, $G : S^q(m) \to S^q(m)$ be special A-models and assume $3 \le e \le q-2$. Then there is a topological conjugacy $\theta = \theta(F, G) : S^q(m) \to S^q(m)$ such that the* *diagram*

$$\begin{tikzcd}[row sep=small] & \Omega_e(F) \arrow[dd, "\theta"] \\ X_A \arrow[ur, "\chi_F"] \arrow[dr, "\chi_G"'] & \\ & \Omega_e(G) \end{tikzcd}$$

is commutative.

Definition 3.4. The homeomorphism θ will be called a *stability conjugacy.*

Before completing (3.3) we need several lemmas.

LEMMA 3.5. *Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that if $F, G: S^q(m) \to S^q(m)$ are A-models which are δ-close in the C^0-topology, then*

$$\chi_G \circ \chi_F^{-1}: \Omega_e(F) \to \Omega_e(G) \subset S^q(m)$$

is ε-close to the inclusion $\Omega_e(F) \hookrightarrow S^q(m)$ in the C^0-topology.

Proof. It suffices to show χ_G and χ_F are near each other whenever G is sufficiently close to F. Let

$$\delta = \max_{x \in S^q} |G(x) - F(x)|$$

be the C^0-distance between G and F. Let $\{sh_\alpha^F\}$ and $\{sh_\alpha^G\}$ denote for F and G respectively the subheads appearing in the definition (2.7) of a model. If δ is small enough, then the $\{\lambda_\alpha^G, \mu_\alpha^G\}$ will be close to the $\{\lambda_\alpha^F, \mu_\alpha^F\}$. Choose δ so small that $\mathrm{Int}\,(sh_\alpha^F) \cap \mathrm{Int}\,(sh_\alpha^G) \neq \emptyset$ for each α. Now suppose $X \subset (sh_\alpha^F \cup sh_\alpha^G) \times D^{q-e} \subset K(j)$ is contained in $(sh_\alpha^F \cup sh_\alpha^G) \times D$ where $D \subset \mathrm{Int}\, D^{q-e}$, D a $(q-e)$-disc of diameter ρ. Let $\lambda = \max_\alpha \{\lambda_\alpha^F, \lambda_\alpha^G\}$. Since A is a zero-one matrix each intersection $K(i) \cap F^{\pm 1}(K(j))$ has at most one component. Similarly for G. It then follows from (c) of (2.7) that for each i there is a $(q-e)$-disc $B \subset \mathrm{Int}\, D^{q-e}$ of diameter $\delta + \lambda\rho$ such that each of $F(X) \cap K(i)$ and $G(X) \cap K(i)$ is contained in $h(i) \times B$. Let $\beta =$ diameter D^{q-e}. An induction arguement then shows that the intersections

$$K(x_0) \cap F(K(x_1)) \cap \cdots \cap F^n(K(x_n))$$

and

$$K(x_0) \cap G(K(x_1)) \cap \cdots \cap G^n(K(x_n)),$$

are contained in the product of an e-block and a $(q-e)$-disc D of diameter at most

$$\delta\left(\sum_{i=0}^{n-1} \lambda^i\right) + \lambda^n \beta.$$

Letting n go to infinity, we see that for any point $x \in X_A$ the distance between the R^{q-e} coordinates of $\chi_F(x)$ and $\chi_G(x)$ is at most $\delta/(1-\lambda)$. There is a similar argument for the R^e-coordinates of points in the intersections

$$K(x_0) \cap F^{-1}(K(x_1)) \cap \cdots \cap F^{-n}(K(x_n))$$

and

$$K(x_0) \cap G^{-1}(K(x_1)) \cap \cdots \cap G^{-n}(K(x_n)).$$

LEMMA 3.6. *Let $F, G: S^q(m) \to S^q(m)$ be A-models. If F and G are sufficiently close* *in the C^2-topology, then there is a topological conjugacy $\theta=\theta(F,G):S^q(m)\to S^q(m)$ from F to G which is close to the identity and such that the diagram*

$$\begin{tikzcd} & \Omega_e(F) \arrow[d, "\theta"] \\ X_A \arrow[ur, "\chi_F"] \arrow[dr, "\chi_G"'] & \\ & \Omega_e(G) \end{tikzcd}$$

is commutative.

Proof. If F and G are sufficiently close then the stability theorem [**R**, **Ro**] says there is a topological conjugacy θ from F to G which is very close to the identity. Hence from (3.5) we see that $\theta^{-1}\circ\chi_G\circ\chi_F^{-1}:\Omega_e(F)\to\Omega_e(F)$ can be made arbitrarily close to the identity. Since σ_A is expansive it follows that $\theta^{-1}\circ\chi_G\circ\chi_F^{-1}$ is equal to the identity if this approximation is good enough.

Let $F, G: S^q(m)\to S^q(m)$ be A-models such that $F|P_m\times D^{q-e}=G|P_m\times D^{q-e}$ as in the conclusion (C) of (2.16).

LEMMA 3.7. *There is a topological conjugacy $\theta=\theta(F,G):S^q(m)\to S^q(m)$ from F to G which is the identity on $\Omega(F)=\Omega(G)$.*

Proof. The argument is the same as [**PS**, (4.2)]. Define

$$\theta(x)=\lim_{n\to\infty}G^{-n}F^n(x)$$

for $x\neq\infty$ and $\theta(\infty)=\infty$. If $x\neq\infty$, then for n sufficiently large $F^n(x)\in \operatorname{Int}(P_m\times D^{q-e})$ and $\theta(y)=G^{-n}F^n(y)$ for y in a neighborhood of x. In particular, θ is a diffeomorphism of $S^q(m)-\{\infty\}$ to itself and since $\theta(\infty)=\infty$, it must be continuous at ∞.

Proof of Proposition 3.3. Let F_t be the isotopy from $F=F_0$ to F_1 as in (2.16). Break up the interval from $t=0$ to $t=1$ into steps small enough to apply (3.6) to a finite number of successive A-models F_t and then take the composition of the various θ to get a topological conjugacy as required between F and F_1. Then apply (3.7) to F_1 and G and take the composition again to get the final θ as required between F and G.

Proof of the Main Theorem. This follows immediately from (1.1), (3.2), and (3.3) which show precisely how to mirror a composition of elementary symbolic conjugacies and shift powers with a composition of elementary smooth conjugacies, stability conjugacies, and powers of the intermediate D_P.

The key point is how to realize the composition of two elementary symbolic conjugacies: suppose we have three irreducible zero-one matrices A, B, C. Assume there are zero-one matrices, R, S, P, Q satisfying

$$A=RS,\quad SR=B=PQ,\quad C=QP.$$

Construct the special models C_R, C_S, C_P, C_Q, and let $D_A=C_SC_R$, $D'_B=C_RC_S$,

$D''_B = C_Q C_P$, $D_C = C_P C_Q$. Then there is the commutative diagram

$$\begin{tikzcd} X_A \arrow[r, "c_R"] \arrow[d, "\chi(D_A)"'] & X_B \arrow[r, "c_P"] \arrow[d, "\chi(D'_B)"'] \arrow[dr, "\chi(D''_B)"] & X_C \arrow[dr, "\chi(D_C)"] & \\ \Omega_e(D_A) \arrow[r, "C_R"'] & \Omega_e(D'_B) \arrow[r, "\theta"'] & \Omega_e(D''_B) \arrow[r, "C_P"'] & \Omega_e(D_C) \end{tikzcd}$$

Thus even though we will usually have $D'_B \neq D''_B$, a stability conjugacy θ can be used to bridge the gap.

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