ON THE NUMBER OF CROSSINGS IN A COMPLETE GRAPH

by FRANK HARARY †‡ and ANTHONY HILL
(Received 12th August 1963)

1. Introduction

The purpose of this article is to describe two problems which involve drawing graphs in the plane. We will discuss both complete graphs and complete bicoloured graphs. The *complete graph* K_n with n points or vertices has a line or edge joining every pair of distinct points, as shown in fig. 1 for $n = 2, 3, 4, 5, 6$.

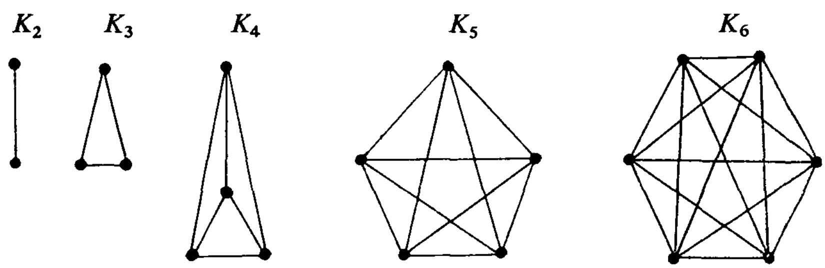

FIG. 1

In each of these complete graphs every edge is a straight line segment. In K_2, K_3 and K_4, no two distinct edges intersect. As anyone can plainly see, the number of intersections or crossings in K_5 as drawn in 5 and in K_6 is 15. It is stipulated that such an intersection involves only two edges and not more.

For any graph G, we say that the *crossing number* $c(G)$ is the minimum number of crossings with which it is possible to draw G in the plane. We note that the edges of G need not be straight line segments, and also that the result is the same whether G is drawn in the plane or on the surface of a sphere. Another invariant of G is the *rectilinear crossing number*, $\bar{c}(G)$, which is the minimum number of crossings when G is drawn in the plane in such a way that every edge is a straight line segment. We will find by an example that this is not the same number obtained by drawing G on a sphere with the edges as arcs of great circles. In drawing G in the plane, we may locate its vertices wherever it is most convenient.

† The preparation of this article was supported in part by a grant from the National Science Foundation: NSF GP-207.

‡ Present address: University of Michigan, Ann Arbor.

https://doi.org/10.1017/S0013091500025645 Published online by Cambridge University Press

Consider the case that $G = K_n$. The determination of a formula for $c(K_n)$ has been studied extensively. But $\bar{c}(K_n)$ is, to our knowledge, a new question. Neither of these questions is answered in the present note.

In fig. 2, K_5 and K_6 are drawn in the plane with straight edges. One can verify that $\bar{c}(K_5) = 1$, and $\bar{c}(K_6) = 3$ as shown.

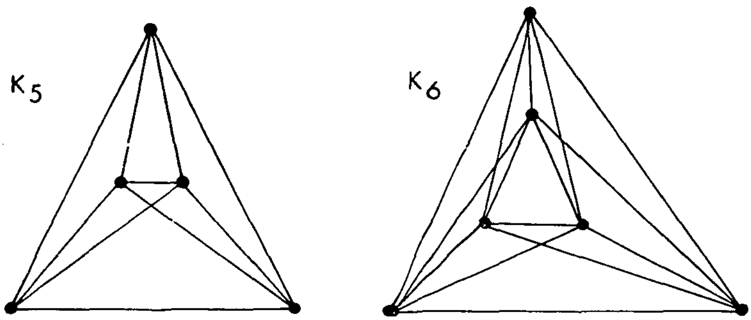

FIG. 2

A *plane graph* is one which is already drawn in the plane in such a way that no two of its edges intersect. A *planar graph* is one which can be drawn as a plane graph. In terms of the notation introduced above, a graph G is planar if and only if $c(G) = 0$. The earliest result concerning the drawing of graphs in the plane is due to Fáry (1), who showed that any planar graph (without loops or multiple edges) can be drawn in the plane in such a way that every edge is straight. Thus Fáry's result may be rephrased: if $c(G) = 0$, then $\bar{c}(G) = 0$.

2. Complete Bicoloured Graphs

The complete bicoloured graph $K_{m,n}$ consists of m points of one colour and n points of another colour, with every pair of points of different colour joined by a line. The exact formula for the intersection number of the complete bicoloured graph was found by Zarankiewicz (4), who showed that

$$\begin{cases} c(K_{2m,\,2n}) = (m^2-m)(n^2-n) \\ c(K_{2m,\,2n+1}) = (m^2-m)n^2 \\ c(K_{2m+1,\,2n+1}) = m^2n^2. \end{cases} \qquad \text{.......................(1)}$$

The graph $K_{3,3}$ is of special interest, since it is one of the two *courbes gauches* of Kuratowski (3), together with the complete graph K_5. He proved that a given graph G is planar if and only if it contains no subgraph homeomorphic to either of them. It is seen at once from equations (1) that $c(K_{3,3}) = 1$, as shown in fig. 3, where the colour (1 or 2) of each point is indicated by a numeral near it.

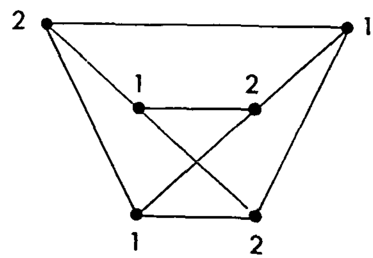

FIG. 3

3. Complete Graphs

There have been many attempts to find an explicit formula for the crossing number of a complete graph, analogous to that of equation (1) for complete bicoloured graphs. In the note by Guy (**2**), the following upper bound for $c(K_n)$ is given; it has been independently discovered several times, both before and after it appeared in print for the first time in (**2**).

$$c(K_n) \leqq \begin{cases} \frac{1}{64}(n-1)^2(n-3)^2, & n \text{ odd}, \\ \frac{1}{64}n(n-4)(n-2)^2, & n \text{ even}. \end{cases} \qquad \text{.........................(2)}$$

However, no-one has shown as yet that K_n cannot be drawn in the plane with fewer intersections than the number indicated by this formula. In fact it has not even been shown that the intersection number of K_n has the order of $n^4/64$ when n becomes large.

Conjecture. The exact value of the crossing number of K_n is given by the upper bounds stated in (2).

After many exhaustive attempts at finding a formula for $\bar{c}(K_n)$, which would even serve as an upper bound, all the information that was found is included in the following:

Table of conjectured values:

n	2	3	4	5	6	7	8	9	10
$c(K_n)$	0	0	0	1	3	9	18	36	60
$\bar{c}(K_n)$	0	0	0	1	3	9	19	36	63

The next two figures show two drawings of K_8, each with 18 crossings. In fig. 4, we have the interesting presentation of K_8 with 18 crossings in the plane, but drawn in such a way that it can be transferred onto the surface of a sphere in such a way that every edge becomes a segment of a great circle.

On the other hand, the presentation of K_8 in fig. 5 is in accordance with the method due to A. Hill, as mentioned in Guy (**2**), for drawing K_n with the number of crossings as in (2).

https://doi.org/10.1017/S0013091500025645 Published online by Cambridge University Press

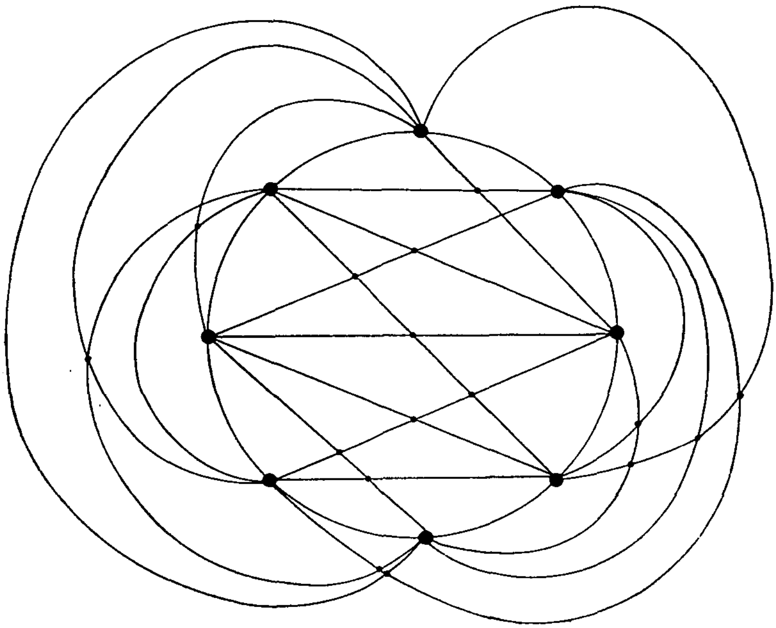

FIG. 4

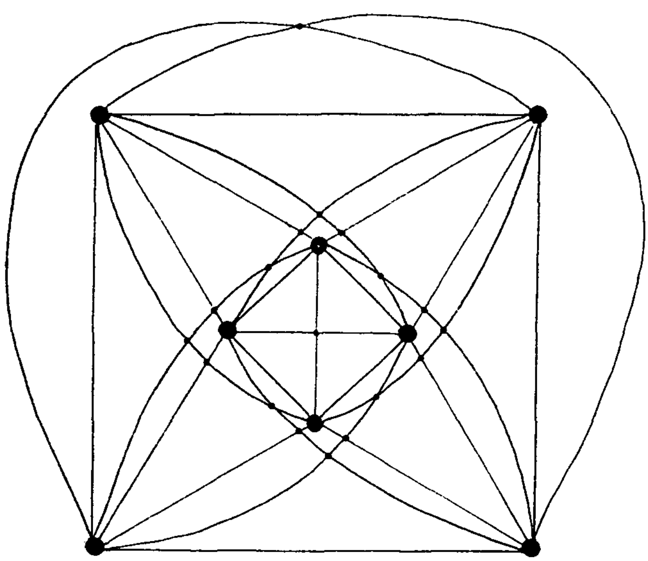

FIG. 5

https://doi.org/10.1017/S0013091500025645 Published online by Cambridge University Press

In fig. 6, K_8 is drawn rectilinearly in the plane with 19 crossings. Although no proof is known, it appears that this is the smallest possible number of rectilinear crossings. This is the example mentioned above, of a graph which can be drawn geodesically on a sphere with 18 crossings, but requiring 19

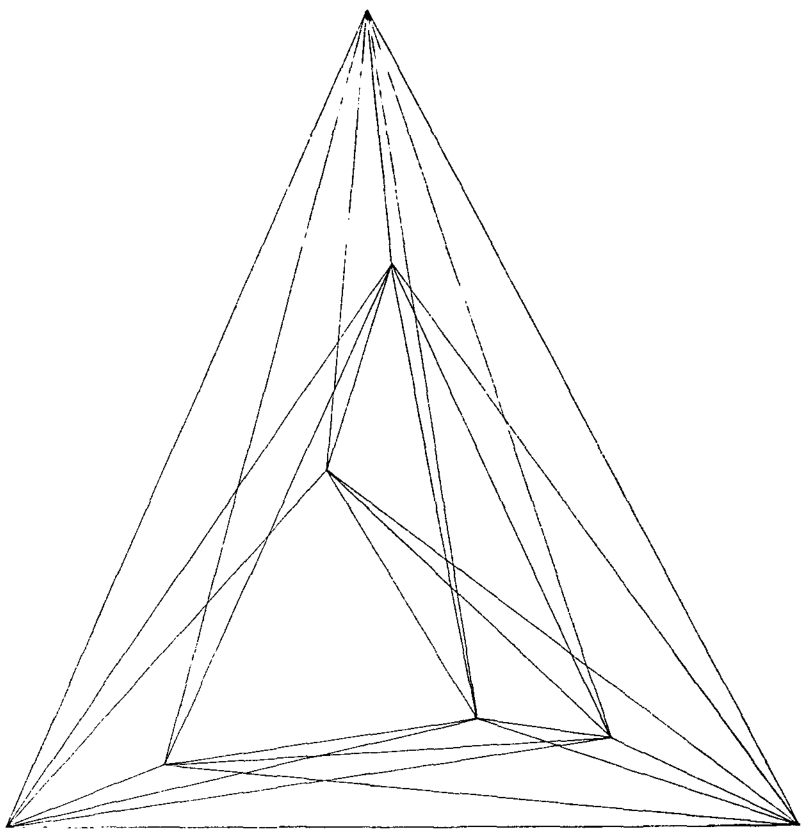

FIG. 6

crossings when drawn geodesically in the plane. One may also ask about geodesic drawings of graphs on other surfaces.

Finally, in fig. 7, we observe K_9 drawn with 36 rectilinear crossings, a number which agrees with (2).

Based on these data, we make the following conjectures. The rectilinear crossing number $\bar{c}(K_n)$ exceeds $c(K_n)$ for $n = 8$ and all $n \geq 10$. We are not as yet able to formulate a likely result for the limiting ratio $\bar{c}(K_n)/c(K_n)$ as n increases.

It is hoped that this problem will eventually come to the attention of someone who can settle the above conjecture which asserts that (2) is in fact an equation. Until now, the attempts at proving this to be the case have consisted mainly of a construction which verifies that the formulæ constitute

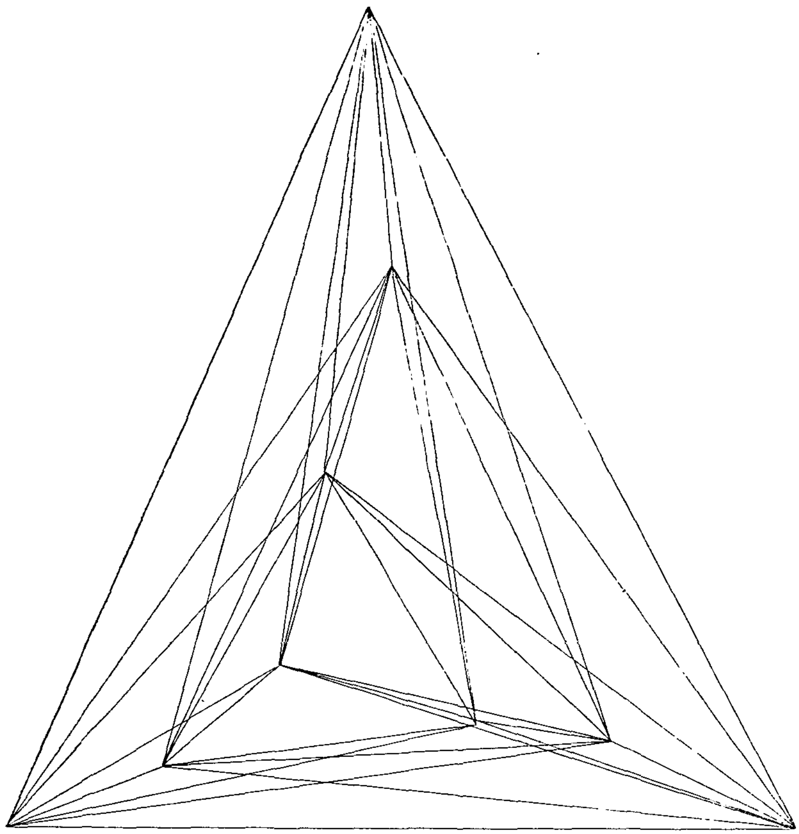

Fig. 7

an upper bound, followed by assertions rather than proofs that they are also a lower bound.

In general, it would be interesting to express for any graph G the numbers $c(G)$ and $\bar{c}(G)$ in terms of other invariants of G.

REFERENCES

(1) I. Fáry, On straight line representation of planar graphs, *Acta Univ. Szeged*, **11** (1948), 229-233.

(2) R. Guy, A combinatorial problem, *Bull. Malayan Math. Soc.*, **7** (1960), 68-72.

(3) C. Kuratowski, Sur le problème des courbes gauches en Topologie, *Fund. Math.*, **16** (1930), 271-283.

(4) K. Zarankiewicz, On a problem of P. Turán concerning graphs, *Fund. Math.*, **41** (1954), 137-145.

University College and Regent Street Polytechnic
London