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THE NATURAL PARTIAL ORDER ON LINEAR SEMIGROUPS WITH NULLITY AND CO-RANK BOUNDED BELOW

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Abstract

Higgins ['The Mitsch order on a semigroup', *Semigroup Forum* **49** (1994), 261–266] showed that the natural partial orders on a semigroup and its regular subsemigroups coincide. This is why we are interested in the study of the natural partial order on nonregular semigroups. Of particular interest are the nonregular semigroups of linear transformations with lower bounds on the nullity or the co-rank. In this paper, we determine when they exist, characterise the natural partial order on these nonregular semigroups and consider questions of compatibility, minimality and maximality. In addition, we provide many examples associated with our results.

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1. Introduction

In 1952, Wagner [14] introduced the natural partial order on inverse semigroups, and then in 1980 this relation was independently extended by Hartwig [3] and Nambooripad [10] to the class of regular semigroups. Later, in 1986, Mitsch [9] generalised the notion of the *natural partial order* $\leq$ to any semigroup S in the following fashion: for any elements a and b in S,

$$a \leq b \quad \text{if and only if } a = xb = by \quad \text{and} \quad a = ay \quad \text{for some } x, y \in S^1,$$

where S^1 is the semigroup S with an identity 1 adjoined if S has no identity, otherwise S^1 is S. Additionally, we define $a < b$ to mean $a \leq b$ and $a \neq b$.

The concept of the natural partial order on semigroups has been studied over decades. Many research articles considered various semigroups endowed with the natural partial order; for example, see [1, 5, 7, 13]. Moreover, the compatibility, minimality and maximality were also investigated. In 1994, Higgins proved the following result.

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Proposition 1.1 [4]. *Let S be a semigroup containing T as its subsemigroup and let x, y be elements in T. Then $x \leq y$ on T implies $x \leq y$ on S. In addition, the converse is true if T is regular.*

Therefore, the natural partial order on a regular semigroup can be derived from the natural partial order on any semigroup containing it. However, this is not the case for nonregular semigroups (see [1]). For this reason, we direct our attention to certain nonregular semigroups or, more precisely, nonregular semigroups contained in the semigroup of all linear transformations, one of the most well-known and important semigroups.

Throughout this paper, we let V be a vector space and $L(V)$ be the set of all linear transformations on V. Then $L(V)$ is a regular semigroup under composition. The kernel and image of α in $L(V)$ are respectively denoted by $\ker\alpha$ and $\operatorname{im}\alpha$. The dimension of V is represented by $\dim V$. For any subset A of V, the subspace spanned by A is denoted by $\langle A\rangle$. As usual, 0 and 1 are respectively the zero map and the identity map on V.

For a cardinal number κ with $\kappa \leq \dim V$, let

$$K(V,\kappa) = \{\alpha \in L(V) \mid \dim(\ker\alpha) \geq \kappa\},$$
$$CI(V,\kappa) = \{\alpha \in L(V) \mid \dim(V/\operatorname{im}\alpha) \geq \kappa\}.$$

Observe that 0 belongs to $K(V,\kappa) \cap CI(V,\kappa)$. Further, if $\dim V$ is finite, then $K(V,\kappa)$ and $CI(V,\kappa)$ are equal to each other. Otherwise, as proved by Chaopraknoi and Kemprasit [2], $K(V,\kappa)$ and $CI(V,\iota)$ are distinct whenever κ, ι are cardinal numbers such that $\kappa \neq 0$ and $\kappa, \iota \leq \dim V$. In particular, when V is an infinite-dimensional vector space, the semigroups $K(V,\aleph_0)$ and $CI(V,\aleph_0)$ are not regular (see [6, 8] for details). Furthermore, we can prove that both $K(V,\kappa)$ and $CI(V,\kappa)$ are regular if and only if $\dim V$ is finite or $\kappa = 0$. Therefore, the natural partial orders on $K(V,\kappa)$ and $CI(V,\kappa)$ when $\dim V$ is infinite and $0 < \kappa \leq \dim V$ are of interest.

2. Preliminaries

In this paper, every linear transformation acts on the right-hand side of vectors. For $\alpha \in L(V)$ defined by $x_i\alpha = u$ and $y_j\alpha = v_j$ for all $i \in I$, $j \in J$, we write

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} & y_j \\ u & v_j \end{pmatrix}_{j\in J},$$

where $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$ is a basis of V and I, J are index sets. Other linear transformations will also be represented in this way.

Proposition 2.1 [12]. *Let $\alpha \in L(V)$ and let B_1 be a basis of* $\ker\alpha$ *and B a basis of V containing B_1. Then:*

(i) *for each $v_1, v_2 \in B \setminus B_1$, $v_1 = v_2$ if and only if $v_1\alpha = v_2\alpha$;*
(ii) *$(B \setminus B_1)\alpha$ is a basis of* $\operatorname{im}\alpha$.

Now we give a characterisation for $K(V,\kappa)$ and $CI(V,\kappa)$ to be regular.

Theorem 2.2. *Let $S(V,\kappa)$ be either $K(V,\kappa)$ or $CI(V,\kappa)$. Then $S(V,\kappa)$ is regular if and only if* $\dim V$ *is finite or* $\kappa = 0$.

Proof. Suppose that $\dim V$ is infinite and $\kappa > 0$. Let B be a basis of V. There is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Let $\phi : B_2 \to B$ be a bijection. Define $\alpha, \beta \in L(V)$, as in [6, Theorem 6.3.13], by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2} \quad \text{and} \quad v\beta = v\phi^{-1} \quad \text{for all } v \in B.$$

Since $\dim(\ker\alpha) = |B_1| = |B| \geq \kappa$ and $\dim(V/\operatorname{im}\beta) = |B \setminus B_2| = |B_1| \geq \kappa$, we have $\alpha \in K(V,\kappa)$ and $\beta \in CI(V,\kappa)$. Let $\gamma \in L(V)$ be such that $\alpha = \alpha\gamma\alpha$. Then $\gamma\alpha = 1$, since α is onto. This implies that γ is one-to-one and hence $\gamma \notin K(V,\kappa)$, whence $K(V,\kappa)$ is not regular. Next let $\lambda \in L(V)$ be such that $\beta = \beta\lambda\beta$. Since β is one-to-one, $\beta\lambda = 1$. Then λ is onto, so $\lambda \notin CI(V,\kappa)$. Therefore, $CI(V,\kappa)$ is not regular.

For the converse, it is clear that $K(V,0) = L(V) = CI(V,0)$, which is regular. Assume that $\dim V$ is finite. Then $K(V,\kappa) = CI(V,\kappa)$. Let $\alpha \in S(V,\kappa)$ and let B_1 be a basis of $\ker\alpha$. Extend it to a basis B of V. Then, by Proposition 2.1(ii), $(B \setminus B_1)\alpha$ is a basis of $\operatorname{im}\alpha$. Let $C_1 = (B \setminus B_1)\alpha$ and let C be a basis of V containing C_1. Define $\gamma \in L(V)$ by

$$\gamma = \begin{pmatrix} C \setminus C_1 & v\alpha \\ 0 & v \end{pmatrix}_{v \in B \setminus B_1}.$$

Thus, $\dim(\ker\gamma) = \dim V - \dim(\operatorname{im}\gamma) = |B| - |B \setminus B_1| = |B_1| \geq \kappa$, so $\gamma \in S(V,\kappa)$. Clearly, $\alpha = \alpha\gamma\alpha$. Hence, $S(V,\kappa)$ is regular. □

For any $\alpha, \beta \in L(V)$, let

$$E(\alpha,\beta) = \{v \in V \mid v\alpha = v\beta\},$$

a subspace of V contained in $V\alpha\beta^{-1}$. It is called the *equaliser* of α and β. The following results about the equaliser are very useful for our paper.

Proposition 2.3. *Let $\alpha, \beta \in L(V)$ be such that* $\ker\beta \subseteq \ker\alpha$ *and let A_1, A_2, A_3 be disjoint linearly independent sets such that A_1, $A_1 \cup A_2$, $A_1 \cup A_2 \cup A_3$ are bases of* $\ker\beta$, $\ker\alpha$, *V, respectively. If $v\alpha = v\beta$ for all $v \in A_3$, then $V\alpha\beta^{-1} = E(\alpha,\beta)$.*

Proof. Let $v \in V\alpha\beta^{-1}$. Then $v\beta = v'\alpha$ for some $v' \in V$. We write

$$v = \sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k z_k \quad \text{and} \quad v' = \sum_i a'_i x_i + \sum_j b'_j y_j + \sum_k c'_k z_k$$

for some $x_i \in A_1$, $y_j \in A_2$, $z_k \in A_3$ and some scalars $a_i, a'_i, b_j, b'_j, c_k, c'_k$, where $i \in I$, $j \in J$, $k \in K$ and I, J, K are finite index sets. Then

$$v\beta = \sum_j b_j y_j\beta + \sum_k c_k z_k\beta \quad \text{and} \quad v'\alpha = \sum_k c'_k z_k\alpha = \sum_k c'_k z_k\beta.$$

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Notice that $(A_2 \cup A_3)\beta$ is linearly independent by Proposition 2.1(ii). Since $v\beta = v'\alpha$, $b_j = 0$ for all $j \in J$. Hence, $v\alpha = \sum_k c_k z_k \alpha = \sum_k c_k z_k \beta = v\beta$, so $v \in E(\alpha, \beta)$. Therefore, $V\alpha\beta^{-1} = E(\alpha, \beta)$. □

Observe that for each $\alpha, \beta \in L(V)$, $V\alpha\beta^{-1} = E(\alpha, \beta)$ implies $\ker\beta \subseteq \ker\alpha$. The next lemma is extracted from [13, proof of Theorem 2.5].

Lemma 2.4. *Let $\alpha, \beta \in L(V)$ be such that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$. Then*

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & y_j\beta & u_k \end{pmatrix}_{j\in J, k\in K},$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$ and V, respectively.

In addition, one can see the following result.

Lemma 2.5. *Let $\alpha, \beta \in L(V)$ be such that $V\alpha\beta^{-1} = E(\alpha, \beta)$.*

(i) *If $\operatorname{im}\alpha = \operatorname{im}\beta$, then $\alpha = \beta$.*
(ii) *If $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $\ker\alpha = \ker\beta$, then $\alpha = \beta$.*

For the remainder of this paper, unless stated otherwise, we assume that V is an infinite-dimensional vector space and that κ is a nonzero cardinal number not greater than $\dim V$. Note that both $K(V, \kappa)$ and $CI(V, \kappa)$ do not contain the identity. Thus, $K(V, \kappa)$ is not equal to $K(V, \kappa)^1$, and similarly for $CI(V, \kappa)$.

3. The natural partial order

In this section, we characterise the natural partial order on $K(V, \kappa)$ and $CI(V, \kappa)$. We first state a significant property of $(L(V), \leq)$.

Theorem 3.1 [13]. *Let $\alpha, \beta \in L(V)$. Then $\alpha \leq \beta$ on $L(V)$ if and only if $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha, \beta)$.*

The following example shows that $(K(V, \kappa), \leq)$ cannot be obtained from $(L(V), \leq)$.

Example 3.2. Let $\kappa > 1$ and let B be a basis of V. Then there is a partition $\{B_1, B_2\}$ of B such that $|B| = |B_1| = |B_2|$. Let $u \in B_2$. Thus, there exists a bijection $\phi : B_2 \setminus \{u\} \to B \setminus \{u\}$. Define $\alpha, \beta \in K(V, \kappa)$ by

$$\alpha = \begin{pmatrix} B_1 \cup \{u\} & v \\ 0 & v\phi \end{pmatrix}_{v\in B_2\setminus\{u\}} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & v & u \\ 0 & v\phi & u \end{pmatrix}_{v\in B_2\setminus\{u\}}.$$

Obviously, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$. Substituting $A_1 = B_1$, $A_2 = \{u\}$ and $A_3 = B_2 \setminus \{u\}$ in Proposition 2.3, we have $V\alpha\beta^{-1} = E(\alpha, \beta)$. Therefore, by Theorem 3.1, $\alpha \leq \beta$ on $L(V)$. Let $\mu \in L(V)$ be such that $\alpha = \beta\mu$. Observe that $0 = u\alpha = u\beta\mu = u\mu$. Let $v \in B_2 \setminus \{u\}$. It follows that $v\phi = v\alpha = v\beta\mu = v\phi\mu$. Since $(B_2 \setminus \{u\})\phi = B \setminus \{u\}$,

$$\mu = \begin{pmatrix} u & v \\ 0 & v \end{pmatrix}_{v\in B\setminus\{u\}}.$$

Hence, $\dim(\ker\mu) = 1 < \kappa$, so $\mu \notin K(V, \kappa)^1$. Therefore, $\alpha \nleq \beta$ on $K(V, \kappa)$.

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Theorem 3.3. *Let $\alpha,\beta \in K(V,\kappa)$. Then $\alpha \le \beta$ on $K(V,\kappa)$ if and only if:*

(i) $\alpha = \beta$; *or*

(ii) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, $V\alpha\beta^{-1} = E(\alpha,\beta)$ *and* $\alpha \in CI(V,\kappa)$.

Proof. Assume that $\alpha < \beta$ on $K(V,\kappa)$. Then $\alpha < \beta$ on $L(V)$ and therefore $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by Theorem 3.1. Next we show that $\alpha \in CI(V,\kappa)$. Since $\alpha < \beta$ on $K(V,\kappa)$, $\alpha = \alpha\mu$ for some $\mu \in K(V,\kappa)$. Let B_1 be a basis of $\operatorname{im}\alpha$ and B_2 a basis of $\ker\mu$. If there exists $v \in B_1 \cap B_2$, then $v = u\alpha = u\alpha\mu = v\mu = 0$ for some $u \in V$, which is a contradiction. Hence, $B_1 \cap B_2 = \emptyset$. We claim that $B_1 \cup B_2$ is linearly independent. Suppose that

$$\sum_i a_i v_i + \sum_j b_j w_j = 0$$

for some $v_i \in B_1$, $w_j \in B_2$ and suitable scalars a_i, b_j, where $i \in I$, $j \in J$ and I, J are finite index sets. Notice that for each $i \in I$, $v_i = u_i\alpha$ for some $u_i \in V$. Thus,

$$0 = \sum_i a_i v_i + \sum_j b_j w_j = \sum_i a_i u_i \alpha + \sum_j b_j w_j,$$

so

$$0 = \Big(\sum_i a_i u_i \alpha + \sum_j b_j w_j\Big)\mu = \sum_i a_i u_i \alpha + 0 = \sum_i a_i v_i.$$

Hence, $a_i = 0 = b_j$ for all $i \in I$, $j \in J$, and we have the claim. Now extend $B_1 \cup B_2$ to a basis B of V. Since $\mu \in K(V,\kappa)$, $\dim(V/\operatorname{im}\alpha) = |B \setminus B_1| \ge |B_2| \ge \kappa$. Hence, $\alpha \in CI(V,\kappa)$, as desired.

Conversely, suppose that the condition (ii) holds. By Lemma 2.4,

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & y_j\beta & u_k \end{pmatrix}_{j\in J, k\in K},$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$ and V, respectively. Extend the linearly independent set $\{y_j\beta\}_{j\in J} \cup \{u_k\}_{k\in K}$ to a basis of V by joining $\{w_l\}_{l\in L}$. Define $\lambda, \mu \in L(V)$ by

$$\lambda = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & z_k \end{pmatrix}_{k\in K} \quad \text{and} \quad \mu = \begin{pmatrix} \{y_j\beta\}_{j\in J} \cup \{w_l\}_{l\in L} & u_k \\ 0 & u_k \end{pmatrix}_{k\in K}.$$

Then $\dim(\ker\lambda) = \dim(\ker\alpha)$. Also, $\dim(\ker\mu) = \dim(V/\operatorname{im}\alpha) \ge \kappa$, as $\alpha \in CI(V,\kappa)$. Hence, $\lambda, \mu \in K(V,\kappa)$. Since $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$, we have $\alpha \le \beta$ on $K(V,\kappa)$. □

The next example gives a reason why the partially ordered set $(CI(V,\kappa), \le)$ will be determined.

Example 3.4. Let $\kappa > 1$ and $\{B_1, B_2\}$ be a partition of a basis B of V with $|B| = |B_1| = |B_2|$. Choose $u \in B_1$ and let $\phi : B \setminus \{u\} \to B_2$ be a bijection. Define distinct $\alpha, \beta \in CI(V,\kappa)$ by

$$\alpha = \begin{pmatrix} u & v \\ 0 & v\phi \end{pmatrix}_{v\in B\setminus\{u\}} \quad \text{and} \quad \beta = \begin{pmatrix} u & v \\ u & v\phi \end{pmatrix}_{v\in B\setminus\{u\}}.$$

Then $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$. Furthermore, $V\alpha\beta^{-1} = E(\alpha,\beta)$ by choosing $A_1 = \emptyset$, $A_2 = \{u\}$ and $A_3 = B \setminus \{u\}$ in Proposition 2.3. Hence, $\alpha \leq \beta$ on $L(V)$ by Theorem 3.1. Suppose that $\alpha = \lambda\beta$ for some $\lambda \in L(V) \setminus \{1\}$. Let $v \in B \setminus \{u\}$. Thus, $v\beta = v\phi = v\alpha = v\lambda\beta$. Since β is one-to-one, $v\lambda = v$. Hence, $\dim(V/\operatorname{im}\lambda) \leq 1 < \kappa$. Therefore, $\lambda \notin CI(V,\kappa)$, so $\alpha \nleq \beta$ on $CI(V,\kappa)$.

Theorem 3.5. *Let $\alpha,\beta \in CI(V,\kappa)$. Then $\alpha \leq \beta$ on $CI(V,\kappa)$ if and only if:*

(i) *$\alpha = \beta$; or*
(ii) *$\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, $V\alpha\beta^{-1} = E(\alpha,\beta)$ and $\alpha \in K(V,\kappa)$.*

Proof. Assume that $\alpha < \beta$ on $CI(V,\kappa)$. From Theorem 3.1, it remains to show that $\alpha \in K(V,\kappa)$. Let $\lambda \in CI(V,\kappa)$ be such that $\alpha = \lambda\beta$. It follows from Lemma 2.4 that

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad\text{and}\quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & y_j\beta & u_k \end{pmatrix}_{j\in J,k\in K},$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$ and V, respectively. We claim that for each $k \in K$, $z_k + v_k \in \operatorname{im}\lambda$ for some v_k, a linear combination of the x_i. Let $k_0 \in K$. We write

$$z_{k_0}\lambda = \sum_i a_i x_i + \sum_j b_j y_j + \sum_k c_k z_k$$

for some scalars a_i, b_j, c_k, where $i \in I' \subseteq I$, $j \in J' \subseteq J$, $k \in K' \subseteq K$ and I', J', K' are finite. Then

$$u_{k_0} = z_{k_0}\alpha = z_{k_0}\lambda\beta = \sum_j b_j y_j\beta + \sum_k c_k u_k,$$

so $c_{k_0} = 1, b_j = 0$ and $c_k = 0$ for all $j \in J'$ and $k \in K' \setminus \{k_0\}$. Thus, $z_{k_0}\lambda = \sum_i a_i x_i + z_{k_0}$ and the claim is proven. It is easy to see that $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k + v_k\}_{k\in K}$ is a basis of V. Since $\lambda \in CI(V,\kappa)$ and $\{z_k + v_k\}_{k\in K} \subseteq \operatorname{im}\lambda$,

$$\dim(\ker\alpha) = |\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}| \geq \dim(V/\operatorname{im}\lambda) \geq \kappa.$$

Hence, $\alpha \in K(V,\kappa)$, as desired.

On the other hand, suppose that the condition (ii) holds. Then, by Lemma 2.4,

$$\alpha = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & u_k \end{pmatrix}_{k\in K} \quad\text{and}\quad \beta = \begin{pmatrix} \{x_i\}_{i\in I} & y_j & z_k \\ 0 & y_j\beta & u_k \end{pmatrix}_{j\in J,k\in K},$$

where $\{x_i\}_{i\in I}$, $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J}$, $\{u_k\}_{k\in K}$ and $\{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} \cup \{z_k\}_{k\in K}$ are bases of $\ker\beta$, $\ker\alpha$, $\operatorname{im}\alpha$ and V, respectively. Notice that $\{y_j\beta\}_{j\in J} \cup \{u_k\}_{k\in K}$ is linearly independent. Extend this to a basis $\{y_j\beta\}_{j\in J} \cup \{u_k\}_{k\in K} \cup \{w_l\}_{l\in L}$ of V. Define $\lambda, \mu \in L(V)$, as in Theorem 3.3, by

$$\lambda = \begin{pmatrix} \{x_i\}_{i\in I} \cup \{y_j\}_{j\in J} & z_k \\ 0 & z_k \end{pmatrix}_{k\in K}, \quad \mu = \begin{pmatrix} \{y_j\beta\}_{j\in J} \cup \{w_l\}_{l\in L} & u_k \\ 0 & u_k \end{pmatrix}_{k\in K}.$$

Then $\dim(V/\operatorname{im}\lambda) = \dim(\ker\alpha) \geq \kappa$, since $\alpha \in K(V,\kappa)$. As $\operatorname{im}\mu \subseteq \operatorname{im}\beta$ and $\beta \in CI(V,\kappa)$, $\dim(V/\operatorname{im}\mu) \geq \dim(V/\operatorname{im}\beta) \geq \kappa$; it follows that $\lambda,\mu \in CI(V,\kappa)$. Since $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$, we have $\alpha \leq \beta$ on $CI(V,\kappa)$. □

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By Theorems 3.3 and 3.5, we have the following result.

COROLLARY 3.6. *Let $\alpha, \beta \in K(V,\kappa) \cap CI(V,\kappa)$. Then $\alpha \leq \beta$ on $K(V,\kappa) \cap CI(V,\kappa)$ if and only if* $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$ *and* $V\alpha\beta^{-1} = E(\alpha, \beta)$.

COROLLARY 3.7.

(i) *For each $\alpha, \beta \in K(V,\kappa)$, $\alpha < \beta$ on $K(V,\kappa)$ if and only if $\alpha < \beta$ on $L(V)$ and $\alpha \in CI(V,\kappa)$.*

(ii) *For each $\alpha, \beta \in CI(V,\kappa)$, $\alpha < \beta$ on $CI(V,\kappa)$ if and only if $\alpha < \beta$ on $L(V)$ and $\alpha \in K(V,\kappa)$.*

We use the condition (ii) in Theorems 3.3 and 3.5 to pursue another example in which we can show that the natural partial orders on $K(V,\kappa)$ and $CI(V,\kappa)$ are totally different.

EXAMPLE 3.8. Let $\{B_1, B_2, B_3\}$ be a partition of a basis of V such that $|B_1| = |B_2| = |B_3| = \dim V$ and B_1, B_2, B_3 are disjoint.

(i) Let $\phi : B_2 \to B_1 \cup B_2$ be a bijection. Define $\alpha, \beta \in K(V,\kappa)$ by

$$\alpha = \begin{pmatrix} B_1 \cup B_2 & v \\ 0 & v \end{pmatrix}_{v \in B_3} \quad \text{and} \quad \beta = \begin{pmatrix} B_1 & w & v \\ 0 & w\phi & v \end{pmatrix}_{w \in B_2, v \in B_3}.$$

Then $\alpha \in CI(V,\kappa)$ and $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. Choosing $A_1 = B_1$, $A_2 = B_2, A_3 = B_3$ and applying Proposition 2.3, we have $V\alpha\beta^{-1} = E(\alpha, \beta)$. Therefore, $\alpha \leq \beta$ on $K(V,\kappa)$ by Theorem 3.3. Since β is onto, $\beta \notin CI(V,\kappa)$.

(ii) Let $\varphi : B_1 \to B_2$ and $\phi : B_2 \cup B_3 \to B_3$ be bijections. Define $\alpha, \beta \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\phi \end{pmatrix}_{v \in B_2 \cup B_3} \quad \text{and} \quad \beta = \begin{pmatrix} w & v \\ w\varphi & v\phi \end{pmatrix}_{w \in B_1, v \in B_2 \cup B_3}.$$

Then $\alpha, \beta \in CI(V,\kappa)$, $\alpha \in K(V,\kappa)$ and $\operatorname{im} \alpha \subseteq \operatorname{im} \beta$. Substituting $A_1 = \emptyset$, $A_2 = B_1$ and $A_3 = B_2 \cup B_3$ in Proposition 2.3, we get $V\alpha\beta^{-1} = E(\alpha, \beta)$. Hence, $\alpha \leq \beta$ on $CI(V,\kappa)$ by Theorem 3.5. As β is one-to-one, $\beta \notin K(V,\kappa)$.

4. The left and the right compatibility

For a semigroup S with a partial order ρ, an element $c \in S$ is said to be *left* (*right*) *compatible* with respect to ρ on S or, in short, on (S, ρ), if for any elements $a, b \in S$, $a\rho b$ implies $ca\rho cb$ ($ac\rho bc$). Moreover, c is said to be *compatible* on (S, ρ) if c is left and right compatible on (S, ρ). In what follows, we describe the compatible elements of $(K(V,\kappa), \leq)$ and $(CI(V,\kappa), \leq)$.

THEOREM 4.1 [13]. *Let $\gamma \in L(V)$ be nonzero. Then:*

(i) *γ is left compatible on $(L(V), \leq)$ if and only if γ is an epimorphism;*

(ii) *γ is right compatible on $(L(V), \leq)$ if and only if γ is a monomorphism.*

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The following facts are helpful.

Lemma 4.2 [11].

(i) *$K(V,\kappa)$ is a right ideal of $L(V)$.*
(ii) *$CI(V,\kappa)$ is a left ideal of $L(V)$.*

Recall that for each $\alpha,\beta \in L(V)$, if $V\alpha\beta^{-1} = E(\alpha,\beta)$, then $\ker\beta \subseteq \ker\alpha$.

Theorem 4.3. *Let $\gamma \in K(V,\kappa)$ be nonzero. Then:*

(i) *γ is left compatible on $(K(V,\kappa),\leq)$ if and only if γ is an epimorphism;*
(ii) *γ is not right compatible on $(K(V,\kappa),\leq)$.*

Proof. (i) Assume that γ is not an epimorphism. Let B_1 be a basis of $\ker\gamma$ and B a basis of V containing B_1. Then $(B\setminus B_1)\gamma$ is a basis of $\operatorname{im}\gamma$ and we let $C_1 = (B\setminus B_1)\gamma$. Extend C_1 to a basis C of V. Let $u \in C\setminus C_1$ and $w \in C_1$. Thus, $w = w_0\gamma$ for some $w_0 \in B\setminus B_1$. Define $\alpha,\beta \in K(V,\kappa)\cap CI(V,\kappa)$ by

$$\alpha = \begin{pmatrix} \{u,w\} & C\setminus\{u,w\} \\ w & 0 \end{pmatrix} \quad\text{and}\quad \beta = \begin{pmatrix} u & w & C\setminus\{u,w\} \\ w & u & 0 \end{pmatrix}.$$

It follows that $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by letting $A_1 = C\setminus\{u,w\}$, $A_2 = \{u-w\}$ and $A_3 = \{u\}$ in Proposition 2.3. Hence, by Theorem 3.3, $\alpha \leq \beta$ on $K(V,\kappa)$. By Proposition 2.1(i), we have $v\gamma \neq w$ for all $v \in B\setminus(B_1\cup\{w_0\})$, as $w_0\gamma = w$. For each $v \in B\setminus(B_1\cup\{w_0\})$, $v\gamma \in C_1\setminus\{w\} \subseteq C\setminus\{u,w\}$, so $v\gamma\alpha = 0 = v\gamma\beta$. Since $w = w\alpha = w_0\gamma\alpha$ and $u = w\beta = w_0\gamma\beta$,

$$\gamma\alpha = \begin{pmatrix} w_0 & B\setminus\{w_0\} \\ w & 0 \end{pmatrix} \quad\text{and}\quad \gamma\beta = \begin{pmatrix} w_0 & B\setminus\{w_0\} \\ u & 0 \end{pmatrix}.$$

Then $\operatorname{im}\gamma\alpha \nsubseteq \operatorname{im}\gamma\beta$ and so, by Theorem 3.3, we get $\gamma\alpha \nleq \gamma\beta$ on $K(V,\kappa)$.

Conversely, suppose that γ is an epimorphism. By Theorem 4.1(i), γ is left compatible on $(L(V),\leq)$. Let $\alpha,\beta \in K(V,\kappa)$ be such that $\alpha < \beta$ on $K(V,\kappa)$. Then $\alpha < \beta$ on $L(V)$, and $\alpha \in CI(V,\kappa)$ by Theorem 3.3. Hence, $\gamma\alpha \leq \gamma\beta$ on $L(V)$, and $\gamma\alpha \in CI(V,\kappa)$ since $CI(V,\kappa)$ is a left ideal of $L(V)$. Therefore, by Theorem 3.3, $\gamma\alpha \leq \gamma\beta$ on $K(V,\kappa)$.

(ii) Let B_1 be a basis of $\ker\gamma$ contained in a basis B of V. Let $u \in B_1$ and $w \in B\setminus B_1$. Define $\alpha,\beta \in K(V,\kappa)\cap CI(V,\kappa)$ by

$$\alpha = \begin{pmatrix} \{u,w\} & B\setminus\{u,w\} \\ w & 0 \end{pmatrix} \quad\text{and}\quad \beta = \begin{pmatrix} u & w & B\setminus\{u,w\} \\ w & u & 0 \end{pmatrix}.$$

By a similar argument to (i), $\alpha \leq \beta$ on $K(V,\kappa)$. Since

$$\alpha\gamma = \begin{pmatrix} \{u,w\} & B\setminus\{u,w\} \\ w\gamma & 0 \end{pmatrix} \quad\text{and}\quad \beta\gamma = \begin{pmatrix} u & B\setminus\{u\} \\ w\gamma & 0 \end{pmatrix},$$

$\ker\beta\gamma \nsubseteq \ker\alpha\gamma$. Hence, $V(\alpha\gamma)(\beta\gamma)^{-1} \neq E(\alpha\gamma,\beta\gamma)$. Therefore, $\alpha\gamma \nleq \beta\gamma$ on $K(V,\kappa)$ by Theorem 3.3. □

We investigate the left and the right compatible elements in $(CI(V,\kappa),\leq)$ in the following theorem.

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THEOREM 4.4. *Let $\gamma \in CI(V,\kappa)$ be nonzero. Then:*

(i) *γ is not left compatible on $(CI(V,\kappa), \leq)$;*
(ii) *γ is right compatible on $(CI(V,\kappa), \leq)$ if and only if γ is a monomorphism.*

PROOF. (i) Clearly, γ is not an epimorphism. Similar to the proof of the necessity of Theorem 4.3(i), by Theorem 3.5, we have that γ is not left compatible on $(CI(V,\kappa), \leq)$.

(ii) Suppose that γ is not a monomorphism. Similar to the proof of Theorem 4.3(ii), by Theorem 3.5, γ is not right compatible on $(CI(V,\kappa), \leq)$.

The sufficiency can be proved as in the converse proof of Theorem 4.3(i), applying Theorem 3.5 and Lemma 4.2(i). □

REMARK 4.5. Observing Theorems 4.3 and 4.4 and their proofs, we get the following results.

(i) The zero map is the unique compatible element in $(K(V,\kappa), \leq)$ $((CI(V,\kappa), \leq))$.
(ii) For each subsemigroup S of $L(V)$ containing $K(V,\kappa) \cap CI(V,\kappa)$, if γ is left (right) compatible on $(S, \leq)$, then γ is an epimorphism (a monomorphism).
(iii) Referring to α and β in the proof of the necessity of Theorem 4.3(i), if we choose $A_1 = C \setminus \{u, w\}$, $A_2 = \{u - w\}$ and $A_3 = \{w\}$, then $A_1 \cup A_2 \cup A_3$ is also a basis of V but $w\alpha = w \neq u = w\beta$. Hence, the converse of Proposition 2.3 is not true.

5. Minimal and maximal elements

In the rest of this paper, we describe minimal and maximal elements in $K(V,\kappa)$ and $CI(V,\kappa)$. Since 0 is the minimum element in $(L(V), \leq)$, it is interesting to find the minimal nonzero elements in subsemigroups of $(L(V), \leq)$.

THEOREM 5.1 [13]. *Let $\alpha \in L(V)$. Then:*

(i) *α is a minimal nonzero element in $(L(V), \leq)$ if and only if $\operatorname{rank} \alpha = 1$;*
(ii) *α is maximal in $(L(V), \leq)$ if and only if α is a monomorphism or an epimorphism.*

THEOREM 5.2. *Let $S(V,\kappa)$ be $K(V,\kappa)$ or $CI(V,\kappa)$ and let $\alpha \in S(V,\kappa)$. Then α is a minimal nonzero element in $(S(V,\kappa), \leq)$ if and only if $\operatorname{rank} \alpha = 1$.*

PROOF. Assume that α is a minimal nonzero element in $(S(V,\kappa), \leq)$. Let B_1 be a basis of $\ker \alpha$. As is usual, we extend this to a basis B of V. Let $u \in B \setminus B_1$. Define $\beta \in K(V,\kappa) \cap CI(V,\kappa)$ by

$$\beta = \begin{pmatrix} B \setminus \{u\} & u \\ 0 & u\alpha \end{pmatrix}.$$

Then $\operatorname{im} \beta \subseteq \operatorname{im} \alpha$. We have $V\beta\alpha^{-1} = E(\beta, \alpha)$ by taking $A_1 = B_1$, $A_2 = B \setminus (B_1 \cup \{u\})$ and $A_3 = \{u\}$ in Proposition 2.3. Therefore, by the assumption and Theorems 3.3 and 3.5, $\beta = \alpha$. Hence, $\operatorname{rank} \alpha = 1$.

The converse is clear by Theorem 5.1(i). □

The next corollary follows from Theorems 5.1(i) and 5.2.

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Corollary 5.3. *Let $S(V,\kappa)$ be $K(V,\kappa)$ or $CI(V,\kappa)$ and let $\alpha \in S(V,\kappa)$. Then α is a minimal nonzero element in $(S(V,\kappa),\leq)$ if and only if α is a minimal nonzero element in $(L(V),\leq)$.*

From Theorems 3.3 and 3.5, we have the following result.

Lemma 5.4.

(i) *For each $\alpha \in K(V,\kappa)\setminus CI(V,\kappa)$, α is maximal in $(K(V,\kappa),\leq)$.*
(ii) *For each $\alpha \in CI(V,\kappa)\setminus K(V,\kappa)$, α is maximal in $(CI(V,\kappa),\leq)$.*

However, we can have elements in $K(V,\kappa)\cap CI(V,\kappa)$ which are maximal in $(K(V,\kappa),\leq)$ or in $(CI(V,\kappa),\leq)$.

Example 5.5. Let κ be a natural number and let $B = B_1 \cup B_2$ be a basis of V, where $\{B_1, B_2\}$ is a partition of B such that $|B| = |B_1| = |B_2|$. Choose $B_0 \subseteq B$ such that $|B_0| = \kappa$. Let ϕ be a bijection from $B\setminus B_0$ onto B_2.

(i) Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_0 & v \\ 0 & v\phi \end{pmatrix}_{v\in B\setminus B_0}.$$

Observe that $\dim(\ker\alpha) = |B_0| = \kappa$ and $\dim(V/\operatorname{im}\alpha) = |B_1| > \kappa$, so $\alpha \in K(V,\kappa)\cap CI(V,\kappa)$. To show that α is maximal in $(K(V,\kappa),\leq)$, we assume that $\alpha \leq \beta$ on $K(V,\kappa)$ for some $\beta \in K(V,\kappa)$. Then, by Theorem 3.3, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$. Moreover, $\ker\beta \subseteq \ker\alpha$. Hence, $\kappa \leq \dim(\ker\beta) \leq \dim(\ker\alpha) = \kappa$. This implies that $\dim(\ker\beta) = \dim(\ker\alpha) = \kappa$. Since κ is finite, $\ker\alpha = \ker\beta$. Therefore, $\alpha = \beta$ by Lemma 2.5(ii).

(ii) Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_1 & v \\ 0 & v\phi^{-1} \end{pmatrix}_{v\in B_2}.$$

Since $\dim(\ker\alpha) > \kappa$ and $\dim(V/\operatorname{im}\alpha) = \kappa$, $\alpha \in K(V,\kappa)\cap CI(V,\kappa)$. To see that α is a maximal element in $(CI(V,\kappa),\leq)$, we assume that $\alpha \leq \beta$ on $CI(V,\kappa)$ for some $\beta \in CI(V,\kappa)$. Then, by Theorem 3.5, $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $V\alpha\beta^{-1} = E(\alpha,\beta)$. Notice that $\kappa \leq \dim(V/\operatorname{im}\beta) \leq \dim(V/\operatorname{im}\alpha) = \kappa$, so $\dim(V/\operatorname{im}\beta) = \dim(V/\operatorname{im}\alpha) = \kappa$. As κ is finite and $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$, $\operatorname{im}\alpha = \operatorname{im}\beta$. By Lemma 2.5(i), we get $\alpha = \beta$.

The next lemma is a generalisation of Example 5.5, and we omit the proof as it is similar to the example.

Lemma 5.6.

(i) *Any element α in $K(V,\kappa)$ with* $\dim(\ker\alpha) = \kappa < \infty$ *is a maximal element in $(K(V,\kappa),\leq)$.*
(ii) *Any element α in $CI(V,\kappa)$ with* $\dim(V/\operatorname{im}\alpha) = \kappa < \infty$ *is a maximal element in $(CI(V,\kappa),\leq)$.*

The characterisations of the maximality in $(K(V,\kappa),\leq)$ and $(CI(V,\kappa),\leq)$ are shown in the following theorem. The sufficient conditions follow from Lemmas 5.4 and 5.6. We therefore only show the necessity of the conditions via the contrapositive.

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THEOREM 5.7.

(i) *For each* $\alpha \in K(V,\kappa)$*,* α *is maximal in* $(K(V,\kappa), \leq)$ *if and only if* $\alpha \notin CI(V,\kappa)$ *or* $\dim(\ker\alpha) = \kappa < \infty$.

(ii) *For each* $\alpha \in CI(V,\kappa)$*,* α *is maximal in* $(CI(V,\kappa), \leq)$ *if and only if* $\alpha \notin K(V,\kappa)$ *or* $\dim(V/\operatorname{im}\alpha) = \kappa < \infty$.

PROOF. To deal with (i) and (ii), we first provide common results needed in our proof. Let $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$ and let $w \in V \setminus \operatorname{im}\alpha$. Suppose that B_1 is a basis of $\ker\alpha$ containing a nonzero element u. Then there exists a basis of V containing B_1, say B. It is known that $(B \setminus B_1)\alpha$ is a basis of $\operatorname{im}\alpha$. Define $\beta \in L(V)$, as in [13, Theorem 4.3], by

$$\beta = \begin{pmatrix} v & u \\ v\alpha & w \end{pmatrix}_{v \in B\setminus\{u\}}.$$

Clearly, $\operatorname{im}\alpha \subsetneq \operatorname{im}\beta$, and $V\alpha\beta^{-1} = E(\alpha,\beta)$ by substituting $A_1 = B_1 \setminus \{u\}$, $A_2 = \{u\}$ and $A_3 = B \setminus B_1$ in Proposition 2.3.

(i) Assume that $\dim(\ker\alpha) > \kappa$ or κ is infinite. Then

$$\dim(\ker\beta) = |B_1 \setminus \{u\}| = |B_1| - 1 = \dim(\ker\alpha) - 1 \geq \kappa,$$

so $\beta \in K(V,\kappa)$. Hence, $\alpha < \beta$ on $K(V,\kappa)$, by Theorem 3.3.

(ii) Assume that $\dim(V/\operatorname{im}\alpha) > \kappa$ or κ is infinite. Note that $\operatorname{im}\beta = \langle \{w\} \cup \operatorname{im}\alpha \rangle$. By assumption,

$$\dim(V/\operatorname{im}\beta) = \dim(V/\operatorname{im}\alpha) - 1 \geq \kappa.$$

This implies that $\beta \in CI(V,\kappa)$. Hence, $\alpha < \beta$ on $CI(V,\kappa)$, by Theorem 3.5. □

Consequently, we have the following interesting results.

COROLLARY 5.8.

(i) $K(V,\kappa) \setminus CI(V,\kappa)$ *is the set of all maximal elements in* $(K(V,\kappa), \leq)$*, where* κ *is infinite.*

(ii) $CI(V,\kappa) \setminus K(V,\kappa)$ *is the set of all maximal elements in* $(CI(V,\kappa), \leq)$*, where* κ *is infinite.*

(iii) *There are no* $\alpha, \beta \in K(V,\kappa) \setminus CI(V,\kappa)$ *such that* $\alpha < \beta$ *on* $K(V,\kappa)$*.*

(iv) *There are no* $\alpha, \beta \in CI(V,\kappa) \setminus K(V,\kappa)$ *such that* $\alpha < \beta$ *on* $CI(V,\kappa)$*.*

Finally, we construct maximal elements in $(K(V,\kappa), \leq)$ and $(CI(V,\kappa), \leq)$.

EXAMPLE 5.9. Let κ be a natural number and let B and C be bases of V. There exist $B_0 \subseteq B$ and $C_0 \subseteq C$ such that $|B_0| = \kappa = |C_0|$. Moreover, we have a bijection $\phi : B \setminus B_0 \to C \setminus C_0$. Define $\alpha \in L(V)$ by

$$\alpha = \begin{pmatrix} B_0 & v \\ 0 & v\phi \end{pmatrix}_{v \in B\setminus B_0}.$$

Then $\dim(\ker\alpha) = \kappa = \dim(V/\operatorname{im}\alpha)$, so $\alpha \in K(V,\kappa) \cap CI(V,\kappa)$. Hence, α is maximal in $(K(V,\kappa), \leq)$ and in $(CI(V,\kappa), \leq)$ by Theorem 5.7.

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