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A_ϕ-INVARIANT SUBSPACES ON THE TORUS

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ABSTRACT. Generalizing the notion of invariant subspaces on the 2-dimensional torus T^2, we study the structure of A_ϕ-invariant subspaces of $L^2(T^2)$. A complete description is given of A_ϕ-invariant subspaces that satisfy conditions similar to those studied by Mandrekar, Nakazi, and Takahashi.

1. **Introduction.** Let $L^2(T^2)$ and $L^\infty(T^2)$ be the usual Lebesgue spaces on the 2-dimensional torus T^2. We use (z, w) or $(e^{i\theta}, e^{i\psi})$ as variables in T^2. Let Z and Z_+ be the sets of integers and non-negative integers respectively. A closed subspace M of $L^2(T^2)$ is called z-invariant if $zM \subset M$, and called invariant if $zM \subset M$ and $wM \subset M$. For a function f in $L^2(T^2)$, let

$$\hat{f}(n,k) = \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta}, e^{i\psi})e^{-(n\theta+k\psi)}\, d\theta\, d\psi/(2\pi)^2, \quad (n,k) \in Z^2,$$

where $d\theta\, d\psi/(2\pi)^2$ is normalized Lebesgue measure on T^2. The Hardy space $H^2(T^2)$ is the space of $f \in L^2(T^2)$ such that $\hat{f}(n,k) = 0$ for every $(n,k) \in Z^2 \setminus Z_+^2$. For $f, g \in L^2(T^2)$, we write $f \perp g$ if $\int_0^{2\pi}\int_0^{2\pi} f\overline{g}\, d\theta\, d\psi/(2\pi)^2 = 0$. Subsets E and F of $L^2(T^2)$ are called *mutually orthogonal* when $f \perp g$ for every $f \in E$ and $g \in F$, and in this case $E \oplus F$ denotes the direct sum of E and F. When $F \subset E \subset L^2(T^2)$, we denote by $E \ominus F$ the orthogonal complement of F in E.

The Beurling theorem says that every invariant subspace N on the unit circle T has the form $N = q(z)H^2(T)$ or $N = \chi_E L^2(T)$, where $q(z)$ is a unimodular function on T and χ_E is the characteristic function for a subset $E \subset T$. To avoid confusion, we use the notation T_z for the unit circle with the variable z. Hence every f in $L^2(T_z)$ is a z-variable function and $f = f(z)$. We may consider $L^2(T_z), H^2(T_z), L^2(T_w)$, and $H^2(T_w)$ as closed subspaces of $L^2(T^2)$ by the natural way. We note that $T^2 = T_z \times T_w$.

For a subset E of $L^2(T^2)$, we denote by $[E]$ the closed linear span of E in $L^2(T^2)$. Let $H_z^2(T^2) = \left[\bigcup\{z^{-n}H^2(T^2)\,;\, n \in Z_+\}\right]$. Then

$$H_z^2(T^2) = \sum_{j=-\infty}^{\infty} \oplus z^j H^2(T_w) = \sum_{j=0}^{\infty} \oplus w^j L^2(T_z).$$

Now we give notations and definitions to state our results. Our main purpose is to study generalized invariant subspaces. To define them, let

$$\phi\colon Z_+ \longrightarrow Z \cup \{-\infty\} \quad \text{and} \quad \phi(0) = 0,$$

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and let

$$A_\phi = \{z^i w^j \ ; \ i \ge \phi(j), j \in Z_+\}.$$

When $\phi(j) = -\infty$, we mean that $\{i \in Z \ ; \ i \ge \phi(j)\} = Z$. Moreover we assume that

$$(\sharp) \qquad A_\phi \text{ is a semigroup.}$$

Then, if $\phi(j) = -\infty$ then $\phi(i) = -\infty$ for every $i \ge j$. For each $n \in Z_+$, let $A_{\phi,n} = \{z^i w^k \ ; \ i \ge \phi(k), k \ge n\}$. A_ϕ is called *cyclic* if there exists $p \ge 1$ such that $\phi(p) \ne -\infty$ and $A_{\phi,p} = z^{\phi(p)} w^p A_\phi$. It is not difficult to see that A_ϕ is cyclic if and only if there exists $p \ge 1$ such that $\phi(p) \ne -\infty$ and $\phi(p) + \phi(j) = \phi(p + j)$ for every $j \in Z_+$. When A_ϕ is cyclic, we have $\phi(j) > -\infty$ for $j \in Z_+$.

A closed subspace M of $L^2(T^2)$ is called A_ϕ*-invariant* (see [7]) if

$$A_\phi M = \{fg \ ; \ f \in A_\phi, g \in M\} \subset M.$$

Moreover if A_ϕ is cyclic, M is called *cyclic* A_ϕ*-invariant*. Since $A_{\phi,n} \setminus A_{\phi,n+1} = \{z^i w^n \ ; \ i \ge \phi(n)\}$, $[A_{\phi,n} \setminus A_{\phi,n+1}] = w^n z^{\phi(n)} H^2(T_z)$, where we consider that $z^{\phi(n)} H^2(T_z) = L^2(T_z)$ if $\phi(n) = -\infty$. Then $[A_\phi] = \sum_{n=0}^\infty \oplus w^n z^{\phi(n)} H^2(T_z)$, and $[A_\phi]$ is an A_ϕ-invariant subspace. For a z-invariant subspace S of $L^2(T^2)$, let

$$z^{\phi(n)} S = \bigcup_{i > \phi(n)} z^i S \qquad \text{if } \phi(n) = -\infty.$$

In this paper, we study the structure of A_ϕ-invariant subspaces. Since $z \in A_\phi$, A_ϕ-invariant subspaces are z-invariant. When $\phi_0(j) = 0$ for every $j \in Z_+$, the family of A_{ϕ_0}-invariant subspaces coincides with the family of usual invariant subspaces. In [2], Curto, Muhly, Nakazi, and Yamamoto studied A_n-invariant subspaces for a positive integer n, where $A_n = \{z^i w^j \ ; \ i \in Z \text{ for } n \le j, i \in Z_+ \text{ for } 0 \le j < n\}$. Also Helson and Lowdenslager [4] studied invariant subspaces for A_1. When $\phi_1(j) = 0$ for $0 \le j < n$, and $\phi_1(j) = -\infty$ for $n \le j$, we have $A_{\phi_1} = A_n$. Hence the concept of A_ϕ-invariant subspaces is a generalization of invariant and A_n-invariant subspaces. We note that A_ϕ-invariant subspaces need not be invariant subspaces. For, let $\phi_2(j) = j$ for $j \in Z_+$; then $[A_{\phi_2}] = \sum_{j=0}^\infty \oplus (zw)^j H^2(T_z)$ is cyclic A_{ϕ_2}-invariant but not an invariant subspace. It is not difficult to see that for a given ϕ, every A_ϕ-invariant subspace is invariant if and only if $w \in A_\phi$.

In Section 2, we give *the basic procedure* to study A_ϕ-invariant subspaces which is used several times later.

In Section 3, we determine the A_ϕ-invariant subspaces M such that $M \ominus [A_{\phi,1} M]$ is a nonzero z-invariant subspace. This is a generalization of the work by Nakazi [10]. Also we give a characterization of closed subspaces of the form $\sum_{j=0}^\infty \oplus q_j(z) w^j H^2(T_z)$, where $q_j(z)$ is a unimodular function on T_z. These invariant subspaces are studied in [1].

In Sections 4, 5 and 6, we discuss the following special type of ϕ. Let $p \in Z_+ \setminus \{0\}$ and $k \in Z$. For each $n \in Z_+$, let $\phi(n)$ be the smallest integer such that $p\phi(n) - kn \ge 0$. Then $A_\phi = \{z^i w^j \ ; \ pi - kj \ge 0, (i, j) \in Z \times Z_+\}$. To have a one to one correspondence

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between A_ϕ and (p, k), we assume that p and $|k|$ are mutually prime if $k \neq 0$, and $p = 1$ if $k = 0$. In the case $k = 0$, the family of A_ϕ-invariant subspaces coincides with the family of usual invariant subspaces. We have $\phi(p) = k$ and $k + \phi(j) = \phi(p + j)$ for every $j \in Z_+$, so that A_ϕ is cyclic. In Section 4, we solve the following problem.

PROBLEM 1. Describe every A_ϕ-invariant subspace M such that $M = [A_{\phi,1}M]$ and $zM \neq M$.

Let M be an A_ϕ-invariant subspace. For $h \in A_\phi$, let $V_h : M \ni f \longrightarrow hf \in M$. Let P be the orthogonal projection of L^2 onto M. Then the adjoint operator V_h^* on M is given by $V_h^* f = P(\overline{h} f)$ for $f \in M$. In Section 5, we solve the following problem.

PROBLEM 2. Describe the A_ϕ-invariant subspaces M such that $V_{z^k w^p} V_z^* = V_z^* V_{z^k w^p}$.

The motivation of this problem comes from [9, 12], but obtained A_ϕ-invariant subspaces resemble the invariant subspaces given in [11, 13].

In Sections 6 and 7, we define (see Section 6) a homogeneous-type A_ϕ-invariant subspace. This definition is similar to the one given in [11, 13], and we study the following problem.

PROBLEM 3. Determine the homogeneous-type A_ϕ-invariant subspaces M with $z^k w^p M \subset zM$ and $z^k w^p M \neq zM$.

We cannot give the complete answer. It seems very complicated. In Section 7, we consider two special cases.

2. The Basic Procedure.

The following lemma follows from [2, Lemma 2.2].

LEMMA 2.1. *Let M be an invariant subspace of $L^2(T^2)$. Suppose that $M = zM$ and $M \neq wM$. Then M can be represented as follows*

$$M = \psi\Big(\chi_K(z) H_z^2(T^2) \oplus \chi_E L^2(T^2)\Big),$$

where ψ is a unimodular function on T^2, $K \subset T_z$, $d\theta/2\pi(K) > 0$, $E \subset T^2$, and $(K \times T_w) \cap E = \emptyset$. Moreover if $\bigcap_{k=0}^\infty w^k M = \{0\}$, we have $M = \psi \chi_K(z) H_z^2(T^2)$.

LEMMA 2.2. *Let M be an A_ϕ-invariant subspace. If $zM = M$, then M is an invariant subspace and $wM = [A_{\phi,1}M]$.*

PROOF. Since $A_{\phi,n} \setminus A_{\phi,n+1} = \{z^i w^n \ ; \ i \geq \phi(n)\}$, by our assumption we have $(A_{\phi,n} \setminus A_{\phi,n+1})M = w^n M$ for every $n \in Z_+$. Since M is A_ϕ-invariant, $wM \subset M$, so that M becomes an invariant subspace. Hence we get

$$[A_{\phi,1}M] = \Big[\bigcup_{n=1}^\infty (A_{\phi,n} \setminus A_{\phi,n+1})M\Big] = \Big[\bigcup_{n=1}^\infty w^n M\Big] = wM.$$

Let M be an A_ϕ-invariant subspace with $zM = M$. Moreover if $M = wM$ then $M = \chi_E L^2(T^2)$ for some $E \subset T^2$, and if $M \neq wM$ then the form of M is determined by Lemma 2.1. So that we are interested in the case of $M \neq zM$.

We use the following procedure (developed in the remainder of this section) several times in this paper.

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THE BASIC PROCEDURE. Let M be an A_ϕ-invariant subspace of $L^2(T^2)$ and let $p \geq 1$. Suppose that there exists a nonzero z-invariant subspace N such that

$$N \subset M \ominus [A_{\phi,p}M].$$

Let

$$\tilde{M} = \left[\bigcup\{z^n M \,;\, n \in Z\}\right].$$

Then $\tilde{M}$ is A_ϕ-invariant and $z\tilde{M} = \tilde{M}$. Hence by Lemma 2.2, $\tilde{M}$ is an invariant subspace and $M \subset \tilde{M}$. Since $N \perp [A_{\phi,p}M]$ and N is z-invariant, $z^n N \perp z^i w^p M$ for $n \in Z_+$ and $i \geq \phi(p)$. Hence

$$N \perp w^p \tilde{M}, \tag{2.1}$$

so that $\tilde{M} \neq w\tilde{M}$. Then by Lemma 2.1, $\tilde{M}$ has the following form

$$\tilde{M} = \psi\Big(\chi_K(z) H_z^2(T^2) \oplus \chi_E L^2(T^2)\Big),$$

where ψ is a unimodular function on T^2, $K \subset T_z$, $d\theta/2\pi(K) > 0$, $E \subset T^2$, and

$$(K \times T_w) \cap E = \emptyset. \tag{2.2}$$

For the sake of simplicity, we assume

$$\psi = 1,$$

so that $\tilde{M} = \chi_K(z) H_z^2(T^2) \oplus \chi_E L^2(T^2)$. Since $H_z^2(T^2) = \sum_{j=0}^\infty \oplus w^j L^2(T_z)$,

$$\tilde{M} = \left(\sum_{j=0}^{\infty} \oplus w^j \chi_K(z) L^2(T_z)\right) \oplus \chi_E L^2(T^2). \tag{2.3}$$

Since $M \subset \tilde{M}$, for each $f \in M$ we can write as

$$f = \left(\sum_{j=0}^{\infty} \oplus w^j \chi_K(z) f_j(z)\right) \oplus g,$$

where $f_j(z) \in L^2(T_z)$ and $g \in \chi_E L^2(T^2)$. Using the above representation of f, we set

$$S_j = \left\{\chi_K(z) f_j(z) \,;\, f \in M\right\} \subset \chi_K(z) L^2(T_z), \quad j \in Z_+. \tag{2.4}$$

Then S_j is a linear subspace of $L^2(T_z)$. Since $\tilde{M} \neq w\tilde{M}$, we have

$$S_j \neq \{0\} \text{ for every } j \in Z_+.$$

We note that S_j may not be closed. Since $zM \subset M$,

$$z S_j \subset S_j, \quad j \in Z_+. \tag{2.5}$$

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We have also that

$$M \subset \Big(\sum_{j=0}^{\infty} \oplus w^j S_j\Big) \oplus \chi_E L^2(T^2). \tag{2.6}$$

By (2.1), (2.3), (2.4), and (2.6)

$$N \subset \sum_{j=0}^{p-1} \oplus w^j S_j \subset \chi_K(z) \sum_{j=0}^{p-1} \oplus w^j L^2(T_z). \tag{2.7}$$

By (2.4) and (2.6),

$$[A_{\phi,n}M] \subset \Big(\sum_{j=n}^{\infty} \oplus w^j S_j\Big) \oplus \chi_E L^2(T^2) \quad \text{for } n \in Z_+. \tag{2.8}$$

Since $1 \in A_\phi$, $A_\phi M = M$, so that by (2.6) and the definition of S_n

$$S_n = \sum_{j=0}^{n} z^{\phi(n-j)} S_j = \bigcup_{j=0}^{n} z^{\phi(n-j)} S_j, \quad n \in Z_+, \tag{2.9}$$

here by (2.5),

$$z^{\phi(n-j)} S_j = \bigcup_{i \geq \phi(n-j)} z^i S_j.$$

By (2.7) and (2.9),

$$A_\phi N \subset \sum_{j=0}^{\infty} \oplus w^j S_j. \tag{2.10}$$

Here we have the following lemma for a cyclic A_ϕ.

LEMMA 2.3. *Suppose that A_ϕ is cyclic and $z^{\phi(p)} w^p A_\phi = A_{\phi,p}$. Let M be a cyclic A_ϕ-invariant subspace such that $N = M \ominus [A_{\phi,p}M]$ is nonzero and z-invariant. Then we have $w^{p-1} z^{\phi(p-1)} \overline{S_0} \subset N$ and $z^{\phi(1)+\phi(p-1)-\phi(p)} \overline{S_0} \subset N \cap S_0$, where $\overline{S_0}$ is the closure of S_0 in $L^2(T_z)$.*

PROOF. Since $N = M \ominus [A_{\phi,p}M]$, by (2.4), (2.6), (2.7) and (2.8) we obtain

$$S_j = \{\chi_K(z) f_j(z) \,;\, f \in N\}, \quad 0 \leq j \leq p-1. \tag{2.11}$$

Let $\zeta = z^{\phi(p)} w^p$. By our assumption, $\zeta M = \zeta[A_\phi M] = [A_{\phi,p}M]$ and $N = M \ominus \zeta M$. Hence we can write M as

$$M = \Big(\sum_{j=0}^{\infty} \oplus \zeta^j N\Big) \oplus \Big(\bigcap_{j=0}^{\infty} \zeta^j M\Big). \tag{2.12}$$

By (2.4) and (2.6), $\zeta^j M \subset \big(\sum_{i=jp}^{\infty} \oplus w^i \chi_K(z) L^2(T_z)\big) \oplus \chi_E L^2(T^2)$, so that

$$\bigcap_{j=0}^{\infty} \zeta^j M \subset \chi_E L^2(T^2). \tag{2.13}$$

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Since M is A_ϕ-invariant, by (2.10), (2.12), and (2.13),

$$A_\phi N \subset \sum_{j=0}^{\infty} \oplus \zeta^j N. \tag{2.14}$$

To prove our assertion, let $f \in N$. By (2.7) we can write f as

$$f = \sum_{j=0}^{p-1} \oplus w^j \chi_K(z) f_j(z), \quad f_j(z) \in L^2(T_z), \tag{2.15}$$

where $\chi_K(z)f_j(z) \in S_j$. By (2.14), $z^{\phi(p-1)}w^{p-1}f \in \sum_{j=0}^{\infty} \oplus \zeta^j N$. Moreover by (2.7) and (2.15),

$$z^{\phi(p-1)}w^{p-1}\chi_K(z)f_0(z) \oplus \left(\sum_{j=1}^{p-1} \oplus z^{\phi(p-1)} w^{p-1+j} \chi_K(z) f_j(z) \right) \in N \oplus \zeta N.$$

Therefore by (2.11), $z^{\phi(p-1)}w^{p-1}S_0 \subset N$. Since N is a closed subspace,

$$z^{\phi(p-1)}w^{p-1}\overline{S_0} \subset N. \tag{2.16}$$

Next we prove that

$$z^{\phi(1)+\phi(p-1)-\phi(p)}\overline{S_0} \subset N \cap S_0. \tag{2.17}$$

In the same way as in the first paragraph, we have $wz^{\phi(1)}N \subset N \oplus \zeta N$. Then by (2.16), $w^p z^{\phi(1)+\phi(p-1)}\overline{S_0} \subset z^{\phi(1)} wN \subset N \oplus \zeta N$. Since A_ϕ is a semigroup, by (2.5) and (2.7) it is easy to see that $w^p z^{\phi(1)+\phi(p-1)}\overline{S_0} \subset \zeta(N \cap S_0)$. Consequently we get (2.17).

Now we continue the basic procedure. We consider the following two cases separately; $zN = N$ and $zN \neq N$.

CASE 1. Suppose that $zN = N$. Then we have the following lemma.

LEMMA 2.4. *If $p = 1$ and $zN = N$, then M is an invariant subspace with $zM = M$ and $wM \neq M$.*

PROOF. Suppose that $zN = N$. By (2.7) for $p = 1$, $N \subset \chi_K(z)L^2(T_z)$. Hence by the Beurling theorem,

$$N = \chi_{K_0}(z)L^2(T_z), \tag{2.18}$$

where $K_0 \subset K$ and $d\theta/2\pi(K_0) > 0$. Since $A_{\phi,n} \setminus A_{\phi,n+1} = \{z^i w^n \ ; \ i \geq \phi(n)\}$, $w^n N = [(A_{\phi,n} \setminus A_{\phi,n+1})N]$. Since $N \subset M$ and $A_\phi M \subset M$,

$$\sum_{n=0}^{\infty} \oplus w^n N = [A_\phi N] \subset M. \tag{2.19}$$

Let $M_1 = M \ominus [A_\phi N]$. Then

$$M = [A_\phi N] \oplus M_1. \tag{2.20}$$

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Since $M_1 \subset \tilde{M}$, $w^j M_1 \subset w\tilde{M}$ for $j \geq 1$. By (2.1) for $p = 1$, $w^{-j}N \perp M_1$ for $j \geq 1$. Hence by (2.18), (2.19), and (2.20), we have $\chi_{K_0}(z)L^2(T^2) = \sum_{n=-\infty}^{\infty} \oplus w^n N \perp M_1$. Thus we get

$$\chi_{K_0^c}(z)M_1 = M_1. \tag{2.21}$$

Since $zM \subset M$, $zM_1 \subset M$. Since $zN = N$ and $M_1 \perp [A_\phi N]$, $zM_1 \perp [A_\phi N]$. Hence by the definition of M_1, $zM_1 \subset M_1$. We note that $\{f \in L^\infty(T_z) \,;\, fM_1 \subset M_1\}$ is a weak*-closed z-invariant subalgebra of $L^\infty(T_z)$. Since $d\theta/2\pi(K_0) > 0$, the Beurling theorem says that the weak*-closed invariant subspace $\left[\{z^n \chi_{K_0^c}(z) \,;\, n \in Z_+\}\right]_\infty$ of $L^\infty(T_z)$ generated by $\{z^n \chi_{K_0^c}(z) \,;\, n \in Z_+\}$ coincides with $\chi_{K_0^c} L^\infty(T_z)$. Since $zM_1 \subset M_1$, by (2.21) we have $zM_1 = M_1$. Therefore by (2.18), (2.19), and (2.20), $zM = M$. Hence by Lemma 2.2, M is an invariant subspace. By (2.18), (2.19), (2.20), and (2.21), $wM \neq M$.

CASE 2. Suppose that $zN \neq N$. To prove

$$K = T_z, \tag{2.22}$$

suppose that $K \neq T_z$. By (2.7), $\chi_K(z)N = N$. Then in the same way as in the last paragraph of Lemma 2.4, we have $zN = N$. This is a contradiction. Hence we get (2.22).

By (2.2) and (2.22), $E = \emptyset$. As a consequence, by (2.3), (2.4) and (2.6)

$$M \subset \sum_{j=0}^{\infty} \oplus w^j S_j \subset \tilde{M} = \sum_{j=0}^{\infty} \oplus w^j L^2(T_z). \tag{2.23}$$

By (2.7),

$$N \subset \sum_{j=0}^{p-1} \oplus w^j S_j \subset \sum_{j=0}^{p-1} \oplus w^j L^2(T_z). \tag{2.24}$$

By (2.8),

$$[A_{\phi,n}M] \subset \sum_{j=n}^{\infty} \oplus w^j S_j \subset \sum_{j=n}^{\infty} \oplus w^j L^2(T_z), \quad n \in Z_+. \tag{2.25}$$

This is the end of the basic procedure. In the rest of this paper, we use the same notations in the basic procedure.

3. **Simple A_ϕ-Invariant Subspaces.**

An A_ϕ-invariant subspace M of $L^2(T^2)$ is called simple if $z(M \ominus [A_{\phi,1}M]) \subset M \ominus [A_{\phi,1}M]$. The following theorem is a generalization of Nakazi's theorem [10].

THEOREM 3.1. *Let M be an A_ϕ-invariant subspace of $L^2(T^2)$ such that $M \ominus [A_{\phi,1}M]$ is a nonzero z-invariant subspace. Then*

(i) $z(M \ominus [A_{\phi,1}M]) = M \ominus [A_{\phi,1}M]$ if and only if M is an invariant subspace with $M = zM$ and $M \neq wM$.

(ii) $z(M \ominus [A_{\phi,1}M]) \neq M \ominus [A_{\phi,1}M]$ if and only if there exists a unimodular function ψ on T^2 such that $M = \psi[A_\phi]$.

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PROOF. Suppose that $M \ominus [A_{\phi,1}M]$ is a nonzero z-invariant subspace. Then we can use the basic procedure in Section 2 for $p = 1$ and $N = M \ominus [A_{\phi,1}M]$. Now we have

$$M = N \oplus [A_{\phi,1}M]. \tag{3.1}$$

By (2.7), $N \subset S_0 \subset \chi_K(z)L^2(T_z)$. Since $N = M \ominus [A_{\phi,1}M]$, (2.11) holds for $p = 1$, hence

$$N = S_0 \subset \chi_K(z)L^2(T_z). \tag{3.2}$$

(i) Suppose that $zN = N$. Then by Lemma 2.4, M is an invariant subspace with $M = zM$ and $M \neq wM$.

To prove the converse assertion, suppose that M is an invariant subspace with $M = zM$ and $M \neq wM$. Then we can use Lemma 2.1 to describe M, and it is not difficult to see that $z(M \ominus [A_{\phi,1}M]) = M \ominus [A_{\phi,1}M]$.

(ii) Suppose that $N \neq zN$. Then Case 2 in the basic procedure in Section 2 occurs. By (2.22) and (3.2), $S_0 = N \subset L^2(T_z)$. Since N is z-invariant and $N \neq zN$, by the Beurling theorem $S_0 = N = q(z)H^2(T_z)$, where $q(z)$ is a unimodular function on T_z. By induction, we shall prove

$$S_j = q(z)z^{\phi(j)}H^2(T_z) \quad \text{for } j \in Z_+, \tag{3.3}$$

where S_j is defined in (2.4). Suppose that $n \geq 1$ and

$$S_j = q(z)z^{\phi(j)}H^2(T_z) \quad \text{for } 0 \leq j \leq n-1. \tag{3.4}$$

By (3.1) and (3.2), $[A_{\phi,1}M] = M \ominus N = M \ominus S_0$. By (2.9), $\sum_{j=0}^{n-1} z^{\phi(n-j)}S_j \subset S_n \subset [\sum_{j=0}^{n-1} z^{\phi(n-j)}S_j]$ for $n \geq 1$. Hence by (3.4),

$$q(z)\sum_{j=0}^{n-1} z^{\phi(n-j)}z^{\phi(j)}H^2(T_z) \subset S_n \subset q(z)\Big[\sum_{j=0}^{n-1} z^{\phi(n-j)}z^{\phi(j)}H^2(T_z)\Big]. \tag{3.5}$$

Since A_ϕ is a semigroup, $\phi(n) \leq \phi(n-j) + \phi(j)$, so that $\sum_{j=0}^{n-1} z^{\phi(n-j)}z^{\phi(j)}H^2(T_z) = z^{\phi(n)}H^2(T_z)$. Hence by (3.5), $S_n = q(z)z^{\phi(n)}H^2(T_z)$. Therefore we obtain (3.3).

Since $q(z)H^2(T_z) = S_0 = N \subset M$, by (3.3) and $A_\phi M \subset M$ we have

$$w^jS_j = w^jq(z)z^{\phi(j)}H^2(T_z) \subset M \quad \text{for } j \in Z_+.$$

Hence by (2.23), $M \subset \sum_{j=0}^{\infty} \oplus w^jS_j \subset M$. As a consequence,

$$M = \sum_{j=0}^{\infty} \oplus w^jS_j = q(z)\sum_{j=0}^{\infty} \oplus w^iz^{\phi(j)}H^2(T_z) = q(z)[A_\phi].$$

To prove the converse assertion, let $M = \psi[A_\phi]$ for a unimodular function ψ on T^2. Since $A_{\phi,1}A_\phi = A_{\phi,1}$, $[A_{\phi,1}M] = \psi[A_{\phi,1}]$. Since $[A_\phi] \ominus [A_{\phi,1}] = [\{z^n ; n \in Z_+\}] = H^2(T_z)$, $M \ominus [A_{\phi,1}M] = \psi H^2(T_z)$. Of course, $\psi H^2(T_z)$ is z-invariant and $z\psi H^2(T_z) \neq \psi H^2(T_z)$. This completes the proof.

The following is a characterization of the invariant subspaces studied in [1].

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THEOREM 3.2. *Let M be an A_ϕ-invariant subspace of $L^2(T^2)$ with $M \neq zM$. For each $n \in Z_+$, let N_n be the largest z-invariant subspace which is contained in $M \ominus [A_{\phi,n+1}M]$. Then $N_0 \neq \{0\}$ and for each $n \in Z_+$*

$$\text{(a)} \qquad M \ominus \Big([A_{\phi,n+1}M] \oplus N_n\Big) \perp z^i N_n \quad \text{for every } i \in Z$$

if and only if M is represented as follows

$$\text{(b)} \qquad M = \psi\Big(\sum_{j=0}^{\infty} \oplus q_j(z) w^j H^2(T_z)\Big)$$

or there exists a positive integer l such that

$$\text{(c)} \qquad M = \psi\bigg(\Big(\sum l - 1_{j=0} \oplus q_j(z) w^j H^2(T_z)\Big) \oplus \Big(\sum_{j=l}^{\infty} \oplus w^j L^2(T_z)\Big)\bigg),$$

where ψ and $q_j(z), j \in Z_+$, are unimodular functions on T^2 and T_z, respectively, and

$$z^{\phi(i)} q_j(z) H^2(T_z) \subset q_{i+j}(z) H^2(T_z) \quad \text{for } (i,j) \in Z_+^2.$$

PROOF. First, suppose that M is represented by the form in (b). Since M is A_ϕ-invariant, by the form in (b) we have $\phi(i) > -\infty$ for $i \in Z_+$ and

$$z^{\phi(i)} w^i q_j(z) w^j H^2(T_z) \subset q_{i+j}(z) w^{i+j} H^2(T_z) \quad \text{for } i, j \in Z_+.$$

Then for each $t \in Z_+$, we have $\sum_{i=0}^{t} \oplus z^{\phi(t-i)} q_i(z) H^2(T_z) \subset q_t(z) H^2(T_z)$. Hence $M \ominus [A_{\phi,n+1}M]$ equals

$$\psi\bigg\{\Big(\sum_{j=0}^{n} \oplus q_j(z) w^j H^2(T_z)\Big) \oplus \Big(\sum_{j=n+1}^{\infty} \oplus w^j \Big(q_j(z) H^2(T_z) \ominus \Big[\sum_{i=0}^{j-n-1} \oplus z^{\phi(j-i)} q_i(z) H^2(T_z)\Big]\Big)\Big)\bigg\}.$$

Now it is easy to see that $N_n = \psi\big(\sum_{i=0}^{n} \oplus q_i(z) w^i H^2(T_z)\big)$, $N_0 \neq \{0\}$ and condition (a) is satisfied. In the same way, we can prove the same conclusion for M in (c).

Next, suppose that $N_0 \neq \{0\}$ and M satisfies condition (a). Then we can use the basic procedure in Section 2. For the space N_0, we can apply the case $p = 1$. If $zN_0 = N_0$, then by Lemma 2.4 we have $zM = M$. Hence by our assumption, $zN_0 \neq N_0$. By (2.22), $K = T_z$. Then by (2.24) for $p = 1$ and the Beurling theorem,

$$N_0 = q(z) H^2(T_z) \tag{3.6}$$

for a unimodular function $q(z)$ on T_z. By (2.23),

$$M \subset \sum_{j=0}^{\infty} \oplus w^j S_j, \quad S_j \subset L^2(T_z), \tag{3.7}$$

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By (2.25),

$$[A_{\phi,n+1}M] \subset \sum_{j=n+1}^{\infty} \oplus w^j L^2(T_z). \tag{3.8}$$

Also for the space N_n, we can apply the basic procedure for the case $p = n+1$. Since $zN_0 \neq N_0$, by (3.8) we have $zN_n \neq N_n$. Then by (2.24),

$$N_n \subset \sum_{j=0}^{n} \oplus w^j S_j \subset \sum_{j=0}^{n} \oplus w^j L^2(T_z). \tag{3.9}$$

Since $N_0 \subset M$, $w^j z^{\phi(j)} N_0 \subset M$ for $j \in Z_+$. By (3.9), $N_0 \subset S_0$, so that by (2.9) we have $\sum_{j=0}^{n} \oplus w^j z^{\phi(j)} N_0 \subset M \cap (\sum_{j=0}^{n} \oplus w^j S_j)$. Then by (3.6), (3.8) and the definition of N_n, we obtain

$$q(z) \sum_{j=0}^{n} \oplus w^j z^{\phi(j)} H^2(T_z) \subset N_n. \tag{3.10}$$

Here we shall use condition (a). Then by (a) and (3.10),

$$M \ominus \big([A_{\phi,n+1}M] \oplus N_n\big) \perp \sum_{j=0}^{n} \oplus w^j L^2(T_z).$$

Then by (3.7), (3.8) and (3.9), we have $M \subset N_n \oplus \big(\sum_{j=n+1}^{\infty} \oplus w^j L^2(T_z)\big)$ for $n \in Z_+$. By this fact and the definition of S_j,

$$\sum_{j=0}^{n} \oplus w^j S_j = N_n \subset M. \tag{3.11}$$

Hence $\sum_{j=0}^{\infty} \oplus w^j S_j \subset M$. Therefore by (3.7),

$$M = \sum_{j=0}^{\infty} \oplus w^j S_j. \tag{3.12}$$

By (3.11), $w^j S_j = N_j \ominus N_{j-1}$ for $j \geq 1$ and $S_0 = N_0$, so that S_j is a closed z-invariant subspace of $L^2(T_z)$ for every $j \in Z_+$. By the Beurling theorem,

$$S_j = q_j(z) H^2(T_z) \tag{3.13}$$

or

$$S_j = \chi_{E_j} L^2(T_z), \tag{3.14}$$

where $q_j(z)$ is a unimodular function on T_z and $E_j \subset T_z$. If (3.13) happens for every $j \in Z_+$, by (3.12) M has the form of (b). Suppose that (3.14) happens for some $j \in Z_+$. Let l be the smallest integer in Z_+ such that $S_l = \chi_{E_l} L^2(T_z)$. Then $S_j = q_j(z) H^2(T_z)$ for $0 \leq j < l$. Since $S_0 = N_0$, by (3.6) we have $l \geq 1$. By (2.9),

$$q(z) z^{\phi(l+j)} H^2(T_z) + z^{\phi(j)} \chi_{E_l} L^2(T_z) = z^{\phi(l+j)} S_0 + z^{\phi(j)} S_l \subset S_{l+j}, \quad j \in Z_+.$$

Hence $S_{l+j} = L^2(T_z)$ for $j \in Z_+$. Therefore, in this case, M has the form (c). This completes the proof.

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4. **A Semi-Double Type of A_ϕ-Invariant Subspace.** In this section, we study an A_ϕ-invariant subspace M with $M = [A_{\phi,1}M]$ which is called of semi-double type. A closed subspace M of $L^2(T^2)$ is called doubly invariant if $zM = wM = M$. In this case $M = \chi_E L^2(T^2)$ for some $E \subset T^2$. First we prove the following.

PROPOSITION 4.1. *Suppose that there exists a sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ such that $k_n \longrightarrow \infty$ and $z^{-k_n}(A_{\phi,1})^n \cup w^{-k_n}(A_{\phi,1})^n \subset A_{\phi,1}$. If M is an A_ϕ-invariant subspace with $M = [A_{\phi,1}M]$, then M is doubly invariant.*

PROOF. Suppose that $M = [A_{\phi,1}M]$. Then $M = [(A_{\phi,1})^j M]$ for every $j \in Z_+$. Hence by our condition, for $n \geq 1$ we have

$$z^{-k_n}A_{\phi,1}M = z^{-k_n}A_{\phi,1}\left[(A_{\phi,1})^{n-1}M\right] \subset \left[z^{-k_n}(A_{\phi,1})^n M\right] \subset [A_{\phi,1}M] = M.$$

In the same way, $w^{-k_n}A_{\phi,1}M \subset M$. We note that $\{f \in L^\infty(T^2) \ ; \ fM \subset M\}$ is a semigroup. Since the semigroup generated by $\{z^{-k_n}A_{\phi,1} \cup w^{-k_n}A_{\phi,1} \ ; \ n \geq 1\}$ coincides with $\{z^i w^j \ ; \ i, j \in Z\}$, by the above two inclusions M becomes doubly invariant.

EXAMPLE 4.1. Let $\phi(0) = 0$ and $\phi(j) = 1$ for $j \geq 1$. Then ϕ satisfies the condition of Proposition 4.1.

EXAMPLE 4.2. Let $n \geq 1$. Let $\phi_n(j) = 0$ for $0 \leq j \leq n-1$ and $\phi_n(j) = -\infty$ for $j \geq n$. Then ϕ_n satisfies the condition of Proposition 4.1.

As mentioned in Section 1, in the rest of this paper we consider the following special ϕ. Let $p \in Z_+ \setminus \{0\}$, $k \in Z$, and assume that $p, |k|$ are mutually prime if $k \neq 0$, and $p = 1$ if $k = 0$. For each $n \in Z_+$, let $\phi(n)$ be the smallest integer such that $p\phi(n) - kn \geq 0$. Then

$$A_\phi = \left\{z^i w^j \ ; \ pi - kj \geq 0, (i, j) \in Z \times Z_+\right\}.$$

It is trivial that A_ϕ is a semigroup. In this section, we solve the following problem.

PROBLEM 1. Describe every A_ϕ-invariant subspace M such that $M = [A_{\phi,1}M]$ and $zM \neq M$.

By our definition of ϕ, $\phi(p) = k, \phi(p) + \phi(j) = \phi(p + j)$ for $j \in Z_+$, and hence A_ϕ is cyclic, that is,

$$A_{\phi,p} = z^{\phi(p)}w^p A_\phi = z^k w^p A_\phi. \tag{4.1}$$

Since p and $|k|$ are mutually prime (when $k \neq 0$),

$$p\phi(j) - kj \neq p\phi(i) - ki \quad \text{for } 0 \leq i, j \leq p-1, i \neq j,$$

and $p\phi(j) - kj > 0$ for $1 \leq j \leq p-1$. Rearranging the order, let $\{j_0, j_1, \ldots, j_{p-1}\} = \{0, 1, \ldots, p-1\}$ such that

$$p\phi(j_i) - kj_i < p\phi(j_{i+1}) - kj_{i+1}, \quad 0 \leq i \leq p-2.$$

We note that $j_0 = 0$ and

$$p\phi(j_i) - kj_i = i, \quad 0 \leq i \leq p-1. \tag{4.2}$$

When $p = 1$ and $k = 0$, we do not need the above argument. Also we have the following lemma.

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LEMMA 4.1.
(i) $\phi(p) = k$.
(ii) $\phi(j) + \phi(p-j) = k+1$ *for* $1 \le j \le p-1$.
(iii) $j_1 + j_{p-1} = p$.
(iv) *If* $j_1 + j_i < p, 0 \le i \le p-1$, *then* $j_1 + j_i = j_{i+1}$ *and* $\phi(j_1) + \phi(j_i) = \phi(j_{i+1})$.
(v) *If* $j_1 + j_i > p, 0 \le i \le p-1$, *then* $j_1 + j_i = p + j_{i+1}$ *and* $\phi(j_1) + \phi(j_i) = k + \phi(j_{i+1})$.

PROOF. (i) is already mentioned.

(ii) Let $1 \le j \le p-1$. Then $1 \le p-j$, so that by the definition of ϕ we have $p\big(\phi(j)-1\big)-kj < 0 < p\phi(j)-kj$ and $p\big(\phi(p-j)-1\big)-k(p-j) < 0 < p\phi(p-j)-k(p-j)$. Hence

$$p\big(\phi(j) + \phi(p-j) - 2\big) - kp < 0 = pk - kp < p\big(\phi(j) + \phi(p-j)\big) - kp.$$

This means that $\phi(j) + \phi(p-j) - 2 < k < \phi(j) + \phi(p-j)$. Therefore we get (ii).

(iii) Since p and $|k|$ are mutually prime, (4.2) gives (iii).

(iv) Suppose that $0 \le i \le p-1$ and $j_1 + j_i < p$. By (4.2), $p\phi(j_i) - kj_i = i$. Then $p\big(\phi(j_1) + \phi(j_i)\big) - k(j_1 + j_i) = i + 1$. Since $j_1 + j_i < p$, (4.2) implies that $j_1 + j_i = j_{i+1}$ and $\phi(j_1) + \phi(j_i) = \phi(j_{i+1})$.

(v) Suppose that $j_1 + j_i > p$. By (4.2), $p\big(\phi(j_1) + \phi(j_i) - k\big) - k(j_1 + j_i - p) = i + 1$. Since $j_1 + j_i - p < p$, by (4.2) again we get $j_1 + j_i - p = j_{i+1}$ and $\phi(j_1) + \phi(j_i) - k = \phi(j_{i+1})$. Thus we get (v).

The following lemma follows from the Beurling theorem (see the proof of [11, Theorem 3]).

LEMMA 4.2. *Let S be a closed subspace of $L^2(T^2)$ such that $z^k w^p S = S$. Moreover suppose that $S \perp z^i w^j S$ for $(i,j) \notin \{(nk, np)\ ;\ n \in Z\}$. Then there exist a unimodular function ψ on T^2 and $E_0 \subset T^2$ such that $S = \psi\chi_{E_0}[\{(z^k w^p)^n\ ;\ n \in Z\}]$ and $\chi_{E_0} \in [\{(z^k w^p)^n\ ;\ n \in Z\}]$.*

Let

$$H_{p,k} = \left\{z^i w^j\ ;\ pi - kj \ge 0, (i,j) \in Z^2\right\}.$$

Then $A_\phi \subset H_{p,k}$ and

$$H_{p,k} = \bigcup\left\{(z^k w^p)^n A_\phi\ ;\ n \in Z\right\} = \bigcup\left\{(z^{\phi(p)} w^p)^n A_\phi\ ;\ n \in Z\right\}. \tag{4.3}$$

Now we solve Problem 1.

THEOREM 4.1. *Let M be an A_ϕ-invariant subspace such that $M = [A_{\phi,1}M]$ and $zM \ne M$. Then*

$$M = \psi\chi_{E_0}[H_{p,k}] \oplus \chi_E L^2(T^2)$$

for a unimodular function ψ on T^2, $\chi_{E_0} \in [\{(z^k w^p)^n\ ;\ n \in Z\}], E \subset T^2$, and $E_0 \cap E = \emptyset$. Moreover
(i) if $\bigcap_{n=0}^{\infty} z^n M = \{0\}$, then $M = \psi\chi_{E_0}[H_{p,k}]$;
(ii) if $\bigcap_{n=0}^{\infty} z^n M = \{0\}$ and there exists $h \in M$ such that $|h| > 0$ a.e. on T^2, then $M = \psi[H_{p,k}]$.

It is not difficult to prove our theorem for the case $p = 1$ and $k = 0$ (see Lemma 2.1).

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PROOF OF THEOREM 4.1. Let $D = M \ominus zM$. Since $zM \neq M$, $D \neq \{0\}$. Since M is z-invariant,

$$M = D \oplus zM = \Big(\sum_{n=0}^{\infty} \oplus z^n D\Big) \oplus D_\infty \quad \text{and} \quad D_\infty = \bigcap_{n=0}^{\infty} z^n M. \tag{4.4}$$

Then D_∞ is A_ϕ-invariant and $zD_\infty = D_\infty$. By Lemma 2.2, D_∞ is an invariant subspace. Since $M = [A_{\phi,1}M]$, $M = [(A_{\phi,1})^p M]$. Since $(A_{\phi,1})^p \subset A_{\phi,p}$, $M = [A_{\phi,p}M]$. Then by (4.1),

$$M = [A_{\phi,p}M] = z^k w^p [A_\phi M] = z^k w^p M. \tag{4.5}$$

By (4.4) and (4.5),

$$w^p D_\infty = \bigcap_{n=0}^{\infty} z^n w^p M = \bigcap_{j=-k}^{\infty} z^j (z^k w^p M) = \bigcap_{j=-k}^{\infty} z^j M = D_\infty.$$

Since D_∞ is an invariant subspace, $wD_\infty = D_\infty$. Therefore D_∞ is a doubly invariant subspace and

$$D_\infty = \chi_E L^2(T^2), \quad E \subset T^2. \tag{4.6}$$

By (4.3), (4.5) and $M = [A_\phi M]$, we have $M = [H_{p,k}M]$. Hence by (4.4),

$$M = D \oplus z[H_{p,k}M]. \tag{4.7}$$

Let $\{j_0, j_1, \dots, j_{p-1}\} = \{0, 1, \dots, p-1\}$ such that (see above Lemma 4.1) $p\phi(j_i) - kj_i < p\phi(j_{i+1}) - kj_{i+1}$, $0 \le i \le p-2$. Let

$$L_p = zH_{p,k} \quad \text{and} \quad L_i = z^{\phi(j_i)} w^{j_i} H_{p,k} \quad \text{for } 0 \le i \le p-1. \tag{4.8}$$

Since $j_0 = 0$, $L_0 = H_{p,k}$. Then $H_{p,k} = L_0 \supset L_i \supset L_{i+1} \supset L_p = zH_{p,k}$ for $0 \le i \le p-1$. By the definition of $H_{p,k}$,

$$z^k w^p H_{p,k} = H_{p,k}. \tag{4.9}$$

Hence by Lemma 4.1, $z^{\phi(j_1)} w^{j_1} L_i = L_{i+1}$, and then

$$L_{i+1} = z^{\phi(j_i)} w^{j_i} L_1. \tag{4.10}$$

Let $D_i = [L_i M] \ominus [L_{i+1}M]$. Then by (4.7),

$$D = \sum_{i=0}^{p-1} \oplus D_i. \tag{4.11}$$

Here we have

$$\begin{aligned} D_i &= z^{\phi(j_i)} w^{j_i} \big(z^{-\phi(j_i)} w^{-j_i} [L_i M] \ominus [L_1 M]\big) && \text{by (4.10)} \\ &= z^{\phi(j_i)} w^{j_i} \big([H_{p,k}M] \ominus [L_1 M]\big) && \text{by (4.8)} \\ &= z^{\phi(j_i)} w^{j_i} D_0. \end{aligned}$$

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Thus we get

$$D_i = z^{\phi(j_i)} w^{j_i} D_0, \quad 0 \le i \le p-1. \tag{4.12}$$

By (4.8) and (4.9), $z^k w^p L_i = L_i$. Hence $z^k w^p D_i = D_i$, so that by (4.11) and (4.12), $z^k w^p D_0 = D_0$, and $D_0 \perp z^t w^s D_0$ for $(t,s) \in Z^2$ and $pt - ks \neq 0$. Then by Lemma 4.2, there exists a unimodular function ψ on T^2 and $E_0 \subset T^2$ such that

$$D_0 = \psi\chi_{E_0}\Big[\big\{(z^k w^p)^n \,;\, n \in Z\big\}\Big] \quad \text{and} \quad \chi_{E_0} \in \Big[\big\{(z^k w^p)^n \,;\, n \in Z\big\}\Big]. \tag{4.13}$$

Therefore by (4.3), (4.4), (4.6), (4.11), (4.12) and (4.13),

$$\begin{aligned} M &= \left(\sum_{n=0}^{\infty} \oplus z^n \left(\sum_{i=0}^{p-1} \oplus D_i\right)\right) \oplus \chi_E L^2(T^2) \\ &= \left(\sum_{n=0}^{\infty} \oplus z^n \left(\sum_{i=0}^{p-1} \oplus z^{\phi(j_i)} w^{j_i} D_0\right)\right) \oplus \chi_E L^2(T^2) \\ &= \left(\psi\chi_{E_0}[H_{p,k}]\right) \oplus \chi_E L^2(T^2). \end{aligned}$$

The rest is easy to prove. This completes the proof.

5. **Commuting Operators and A_ϕ-Invariant Subspaces.** In this section, we discuss a special type of ϕ which is studied in Section 4. Let $p \in Z_+ \setminus \{0\}$ and $k \in Z$ such that p and $|k|$ are mutually prime if $k \neq 0$, and $p = 1$ if $k = 0$. For each $n \in Z_+$, let $\phi(n)$ be the smallest integer which satisfies $p\phi(n) - kn \ge 0$. We note that $\phi(p) = k$. Let $A_\phi = \{z^i w^j \,;\, pi - kj \ge 0, (i,j) \in Z \times Z_+\}$. Rearranging the order, let $\{j_0, j_1, \dots, j_{p-1}\} = \{0, 1, \dots, p-1\}$ such that $p\phi(j_i) - kj_i < p\phi(j_{i+1}) - kj_{i+1}$ for $0 \le i \le p-2$. We note that $j_0 = 0$. When $p = 1$ and $k = 0$, we do not need the above argument.

Let M be an A_ϕ-invariant subspace. For $h \in A_\phi$, let

$$V_h \colon M \ni f \longrightarrow hf \in M.$$

Let P be the orthogonal projection of L^2 onto M. Then the adjoint operator V_h^* on M satisfies

$$V_h^* f = P(\overline{h} f) \quad \text{for } f \in M.$$

Hence we have that

$$\operatorname{Ker} V_{z^n}^* = M \ominus z^n M \quad \text{for } n \ge 1\,; \tag{5.1}$$

$$\operatorname{Ker} V_{z^k w^p}^* = M \ominus z^k w^p M. \tag{5.2}$$

We study the following problem (see [9, 12]).

PROBLEM 2. Describe A_ϕ-invariant subspaces M such that $V_{z^k w^p}^* V_z = V_z V_{z^k w^p}^*$.

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PROPOSITION 5.1. *Let M be an A_ϕ-invariant subspace. Then the following three conditions are equivalent.*

(i) $V^*_{z^k w^p} V_z = V_z V^*_{z^k w^p}$.
(ii) $V^*_{z^k w^p} V_{z^n} = V_{z^n} V^*_{z^k w^p}$ *for every* $n \geq 1$.
(iii) $V^*_{z^k w^p} V_{z^n} = V_{z^n} V^*_{z^k w^p}$ *for some* $n \geq 1$.

PROOF. It is easy to prove that (i) $\iff$ (ii) and (ii) $\Rightarrow$ (iii). So we only have to prove that (iii) $\Rightarrow$ (i). Suppose that $V^*_{z^k w^p} V_{z^n} = V_{z^n} V^*_{z^k w^p}$ for $n \geq 2$. Then

$$V^*_{z^n} V_{z^k w^p} = V_{z^k w^p} V^*_{z^n}. \tag{5.3}$$

By (5.1), $\operatorname{Ker} V^*_{z^n} = M \ominus z^n M$. Hence by (5.3),

$$z^k w^p (M \ominus z^n M) \subset M \ominus z^n M. \tag{5.4}$$

To prove $V^*_{z^k w^p} V_z = V_z V^*_{z^k w^p}$, we need to prove that

$$z^k w^p (M \ominus zM) \subset M \ominus zM. \tag{5.5}$$

We note that $zM \subset M$. If $zM = M$, there is nothing to prove. Suppose that $zM \neq M$. Then

$$M = \left(\sum_{j=0}^{n-1} \oplus z^j (M \ominus zM) \right) \oplus z^n M. \tag{5.6}$$

To prove (5.5), suppose not. Then there exists an f in $M \ominus zM$ such that

$$z^k w^p f = f_1 \oplus z f_2 \in (M \ominus zM) \oplus zM, \quad f_2 \neq 0. \tag{5.7}$$

Then

$$z^k w^p z^{n-1} f = z^{n-1} f_1 \oplus z^n f_2 \in \left(\sum_{j=0}^{n-1} \oplus z^j (M \ominus zM) \right) \oplus z^n M. \tag{5.8}$$

Since $f \in M \ominus zM$, $z^{n-1} f \in M \ominus z^n M$, so that by (5.4) we have $z^k w^p z^{n-1} f \in M \ominus z^n M$. But by (5.6), (5.7) and (5.8), $z^k w^p z^{n-1} f \notin M \ominus z^n M$. This is a contradiction. Hence we get (5.5).

Then by (5.1) and (5.5), $V_{z^k w^p} V^*_z = V^*_z V_{z^k w^p} = 0$ on $M \ominus zM$. Also we have $V_{z^k w^p} V^*_z = V^*_z V_{z^k w^p}$ on zM. Hence $V_{z^k w^p} V^*_z = V^*_z V_{z^k w^p}$ on $M = (M \ominus zM) \oplus zM$. Therefore $V^*_{z^k w^p} V_z = V_z V^*_{z^k w^p}$.

In the same way as in the proof of Proposition 5.1, we can prove the following.

LEMMA 5.1. *Let M be an A_ϕ-invariant subspace. Then $V^*_{z^k w^p} V_z = V_z V^*_{z^k w^p}$ if and only if $z(M \ominus z^k w^p M) \subset M \ominus z^k w^p M$.*

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THEOREM 5.1. *Let M be an A_ϕ-invariant subspace with $[A_{\phi,1}M] \neq M$. Then $V^*_{z^k w^p} V_z = V_z V^*_{z^k w^p}$ if and only if one of the following happens.*

(i) There exists a unimodular function ψ on T^2 and a positive integer n such that $1 \le n \le p$ and

$$M = \psi \sum_{j=0}^{\infty} \oplus (z^k w^p)^j \left\{ \left(\sum_{i=0}^{n-1} \oplus z^{\phi(j_i)} w^{j_i} H^2(T_z) \right) \oplus \left(\sum_{i=n}^{p-1} \oplus z^{\phi(j_i)-1} w^{j_i} H^2(T_z) \right) \right\}.$$

(ii) M is an invariant subspace with $zM = M$ and $wM \neq M$.

The case $p = 1$ and $k = 0$ of this theorem is proved in [9, 12].

PROOF OF THEOREM 5.1. Let

$$\zeta = z^k w^p. \tag{5.9}$$

Suppose that

$$V^*_\zeta V_z = V_z V^*_\zeta. \tag{5.10}$$

Let $N = M \ominus \zeta M$. By (4.1) and (5.9), $\zeta A_\phi = A_{\phi,p}$. Since $[A_\phi M] = M$, $\zeta M = [A_{\phi,p}M]$. Then $N = M \ominus [A_{\phi,p}M]$. Since $A_{\phi,p} \subset A_{\phi,1}$, $\zeta M \subset [A_{\phi,1}M]$. Hence by our assumption, $N \neq \{0\}$. Then we have the following decomposition

$$M = \left(\sum_{j=0}^{\infty} \oplus \zeta^j N \right) \oplus \bigcap_{j=0}^{\infty} \zeta^j M. \tag{5.11}$$

By (5.10) and Lemma 5.1, $zN \subset N$. Therefore we can use the basic procedure in Section 2. Using it, we shall study the structures of N and M. As in Section 2, let $\tilde{M} = [\bigcup\{z^l M \,;\, l \in Z\}]$. Then by (5.9), $\zeta\tilde{M} = z^k w^p \tilde{M} = w^p \tilde{M}$, and by (2.1), $N \perp w^p \tilde{M}$. By (2.4) and (2.7),

$$N \subset \sum_{j=0}^{p-1} \oplus w^j S_j \subset \chi_K(z) \left(\sum_{j=0}^{p-1} \oplus w^j L^2(T_z) \right) \quad \text{and} \quad S_j \subset \chi_K(z) L^2(T_z). \tag{5.12}$$

By (2.3),

$$M \subset \tilde{M} = \chi_K(z) \left(\sum_{j=0}^{\infty} \oplus w^j L^2(T_z) \right) \oplus \chi_E L^2(T^2). \tag{5.13}$$

Then we have

$$\bigcap_{j=0}^{\infty} \zeta^j M \subset \bigcap_{j=0}^{\infty} w^{jp} \tilde{M} = \chi_E L^2(T^2). \tag{5.14}$$

By Lemma 4.1 (ii), $\phi(1) + \phi(p-1) - k = 1$. Since $\phi(p) = k$, by Lemma 2.3 we have

$$z^{\phi(p-1)} w^{p-1} \overline{S_0} \subset N\,; \tag{5.15}$$

$$z\overline{S_0} \subset N \cap S_0. \tag{5.16}$$

Now we separate the proof into two cases; $z\overline{S_0} \neq \overline{S_0}$ and $z\overline{S_0} = \overline{S_0}$.

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CASE 1. Suppose that $z\overline{S}_0 \neq \overline{S}_0$. Then by (2.5) and the Beurling theorem,

$$\overline{S}_0 = q(z)H^2(T_z) \tag{5.17}$$

for a unimodular function $q(z)$ on T_z. By (5.12), $\overline{S}_0 \subset \chi_K(z)L^2(T_z)$. Hence in this case, we have $K = T_z$, and by (2.2), $E = \emptyset$. Hence by (5.11)–(5.14),

$$M = \sum_{j=0}^{\infty} \oplus \zeta^j N \subset \sum_{j=0}^{\infty} \oplus \Big(\sum_{i=0}^{p-1} \oplus z^{jk} w^{jp+i} S_i \Big) \subset \tilde{M} = \sum_{t=0}^{\infty} \oplus w^t L^2(T_z). \tag{5.18}$$

We note that for each pair of i and j there corresponds a unique t such that $z^{jk}w^{jp+i}S_i \subset w^t L^2(T_z)$ and $t = jp+i$. By (5.16), $z\overline{S}_0 \subset S_0 \subset \overline{S}_0$, hence by (5.17) we have $q(z)zH^2(T_z) \subset S_0 \subset qH^2(T_z)$. Since $\dim\big(H^2(T_z) \ominus zH^2(T_z)\big) = 1$, S_0 becomes a closed subspace, and

$$S_0 = \overline{S}_0 = q(z)H^2(T_z). \tag{5.19}$$

Since S_0 is a closed subspace, by (5.16) we have

$$zS_0 \subset N \cap S_0 \subset S_0. \tag{5.20}$$

Here we want to prove

$$S_0 \subset N. \tag{5.21}$$

To prove this, suppose not. Then by (5.19) and (5.20),

$$N \cap S_0 = zS_0. \tag{5.22}$$

For $f \in N$, by (5.12) we can write f as $f = \sum_{j=0}^{p-1} \oplus w^j f_j(z)$, $f_j \in S_j$. By (5.18), using the above representation of $f \in N$ we have

$$S_i = \{ f_i \, ; f \in N \} \quad \text{for } 0 \leq i \leq p-1. \tag{5.23}$$

Then $z^{\phi(1)}wf = \sum_{j=0}^{p-1} \oplus z^{\phi(1)}ww^j f_j(z) \in M$. Since $M = N \oplus \zeta N \oplus \zeta^2 M$, by (5.12) and (5.22) we have $z^{\phi(1)}ww^{p-1}S_{p-1} \subset \zeta(N \cap S_0) = \zeta z S_0$. Therefore by (5.15) and Lemma 4.1 (ii),

$$w^{p-1}S_{p-1} \subset z^{-\phi(1)}w^{-1}\zeta z S_0 = z^{\phi(p-1)}w^{p-1}S_0 \subset N. \tag{5.24}$$

Next we shall prove

$$w^{p-2}S_{p-2} \subset N. \tag{5.25}$$

Since $z^{\phi(2)}w^2N \subset M$ and $f = \sum_{j=0}^{p-1} \oplus w^j f_j(z) \in N$, we have $\sum_{j=0}^{p-1} \oplus z^{\phi(2)}w^2w^j f_j(z) \in M$. Then by (5.18), $z^{\phi(2)}w^2w^{p-1}f_{p-1}(z) + z^{\phi(2)}w^2w^{p-2}f_{p-2}(z) \in \zeta N$. By (5.18) and (5.24), $z^{\phi(2)}w^2w^{p-1}f_{p-1}(z) \in \zeta N$, so that $z^{\phi(2)}w^2w^{p-2}f_{p-2}(z) \in \zeta(N \cap S_0)$. Therefore by (5.22),

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$z^{\phi(2)}w^2w^{p-2}S_{p-2} \subset \zeta(N \cap S_0) = \zeta zS_0$. Since $z^{\phi(2)}w^2z^{\phi(p-2)}w^{p-2} = \zeta z$ by Lemma 4.1 (ii), we obtain

$$(5.26) \qquad w^{p-2}S_{p-2} \subset z^{\phi(p-2)}w^{p-2}S_0.$$

Since $z^{\phi(p-2)}w^{p-2}f = \sum_{j=0}^{p-1} \oplus z^{\phi(p-2)}w^{p-2}w^j f_j(z) \in M$, we have

$$z^{\phi(p-2)}w^{p-2}f_0(z) \oplus z^{\phi(p-2)}w^{p-1}f_1(z) \in N.$$

Then $z^{\phi(p-2)}w^{p-1}f_1(z) \in w^{p-1}S_{p-1}$, so that by (5.24) we have $z^{\phi(p-2)}w^{p-2}S_0 \subset N$. Therefore by (5.26), we obtain (5.25). In the same way, we can prove by induction that $w^{p-i}S_{p-i} \subset N$ for $1 \le i \le p-1$. Since $f = \sum_{j=0}^{p-1} \oplus w^j f_j(z) \in N$ and $f_j(z) \in S_j$, by the above we have $f_0(z) \in N$. By (5.23), $S_0 \subset N$ and this contradicts (5.22). Thus we get (5.21).

Now we shall prove that

$$(5.27) \qquad w^jS_j \subset N \quad \text{for } 0 \le j \le p-1.$$

The reader may think that (5.27) is already proved in the last paragraph. But these arguments are done under the assumption $N \cap S_0 = zS_0$. Here we want to prove (5.27) under the assumption $N \cap S_0 = S_0$. By (5.21), (5.27) is true for $j = 0$. By induction we prove (5.27). Suppose that

$$(5.28) \qquad w^jS_j \subset N \quad \text{for } 0 \le j \le n-1$$

for n with $1 \le n \le p-1$. We prove that $w^nS_n \subset N$. When $n = p-1$, by (5.12), (5.23) and (5.28) we have $w^nS_n = w^{p-1}S_{p-1} \subset N$ easily. Hence we assume $n < p-1$. For $f = \sum_{j=0}^{p-1} \oplus w^j f_j(z) \in N$, $z^{\phi(p-n-1)}w^{p-n-1}f \in M$. Then

$$\left(\sum_{j=0}^{n} \oplus z^{\phi(p-n-1)}w^{p+j-n-1}f_j\right) \oplus \left(\sum_{j=n+1}^{p-1} \oplus z^{\phi(p-n-1)}w^{p+j-n-1}f_j\right) \in N \oplus \zeta N.$$

Hence by our assumption (5.28), $z^{\phi(p-n-1)}w^{p-1}f_n \in N$. By (5.23),

$$(5.29) \qquad z^{\phi(p-n-1)}w^{p-1}\overline{S_n} \subset N.$$

This implies that $z^{\phi(n+1)}w^{n+1}z^{\phi(p-n-1)}w^{p-1}\overline{S_n} \subset \zeta(N \cap w^nS_n)$. Since $\phi(p) = k$, by Lemma 4.1 we have

$$(5.30) \qquad zw^n\overline{S_n} \subset N \cap w^nS_n \subset w^n\overline{S_n}.$$

We note that (5.29) and (5.30) correspond to (5.15) and (5.16) respectively. By the same argument used to prove (5.21), we can prove $w^nS_n \subset N$. Here we only give an outline of this proof. If $z\overline{S_n} = \overline{S_n}$, (5.30) immediately gives $w^nS_n \subset N$. Next suppose that $z\overline{S_n} \neq \overline{S_n}$.

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Then S_n becomes a closed subspace of $L^2(T_z)$. To prove $w^n S_n \subset N$, suppose not. Then by (5.30),

$$N \cap w^n S_n = z w^n S_n. \tag{5.31}$$

By the fact $z^{\phi(n+1)} w^{n+1} N \subset N \oplus \zeta N$ and (5.31), we have $w^{p-1} S_{p-1} \subset N$. By induction, we can prove that $w^n S_n \subset N$. As a consequence, we get (5.27).

Therefore by (5.12) and (5.27), we obtain

$$N = \sum_{j=0}^{p-1} \oplus w^j S_j. \tag{5.32}$$

Here we note that $z^{\phi(p-j)} w^{p-j} w^j S_j \subset \zeta S_0$ for $0 \leq j \leq p-1$. By Lemma 4.1 (ii), $\phi(p-j) + \phi(j) = \phi(p) + 1$, so that by (5.19) we have

$$S_j \subset q(z) z^{\phi(j)-1} H^2(T_z), \quad 0 \leq j \leq p-1. \tag{5.33}$$

Now we shall prove that there exists an integer n such that $1 \leq n \leq p$ and

$$N = q(z)\Big(\Big(\sum_{i=0}^{n-1} \oplus z^{\phi(j_i)} w^{j_i} H^2(T_z)\Big) \oplus \Big(\sum_{i=n}^{p-1} \oplus z^{\phi(j_i)-1} w^{j_i} H^2(T_z)\Big)\Big). \tag{5.34}$$

By (5.19) and (5.21),

$$q(z)\Big(\sum_{j=0}^{p-1} \oplus z^{\phi(j)} w^j H^2(T_z)\Big) \subset N. \tag{5.35}$$

If $q(z)\big(\sum_{j=0}^{p-1} \oplus z^{\phi(j)} w^j H^2(T_z)\big) = N$, then N has the desired form and in this case we have $n = p$. Suppose that $q(z)\big(\sum_{j=0}^{p-1} \oplus z^{\phi(j)} w^j H^2(T_z)\big) \neq N$. Then there is a positive integer n such that

$$w^{j_n} S_{j_n} \neq q(z) z^{\phi(j_n)} w^{j_n} H^2(T_z), \quad 1 \leq n \leq p-1. \tag{5.36}$$

Here we may assume that n is the smallest integer which satisfies (5.36). Then

$$w^{j_i} S_{j_i} = q(z) z^{\phi(j_i)} w^{j_i} H^2(T_z), \quad 0 \leq i < n. \tag{5.37}$$

By (5.32) and (5.35), $w^{j_n} S_{j_n} \supset z^{\phi(j_n)} w^{j_n} S_0 = q(z) z^{\phi(j_n)} w^{j_n} H^2(T_z)$. Then by (5.33) and (5.36),

$$w^{j_n} S_{j_n} = q(z) z^{\phi(j_n)-1} w^{j_n} H^2(T_z). \tag{5.38}$$

When $n = p-1$, N has the desired form in (5.34), so that we may assume $n < p-1$.

We shall prove that

$$w^{j_i} S_{j_i} = q(z) z^{\phi(j_i)-1} w^{j_i} H^2(T_z) \quad \text{for } n < i \leq p-1. \tag{5.39}$$

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By (5.32) and (5.38),

$$z^{\phi(j_1)}w^{j_1}w^{j_n}S_{j_n} = q(z)z^{\phi(j_1)+\phi(j_n)-1}w^{j_1+j_n}H^2(T_z) \subset M. \tag{5.40}$$

We note that $p \neq j_1 + j_n$, because $n < p - 1$. Hence it happens $j_1 + j_n < p$ or $p < j_1 + j_n$.

First, suppose that $j_1 + j_n < p$. Then by Lemma 4.1 (iv), $\phi(j_1) + \phi(j_n) = \phi(j_1 + j_n)$ and $j_1 + j_n = j_{n+1}$. Hence by (5.40), $q(z)z^{\phi(j_{n+1})-1}w^{j_{n+1}}H^2(T_z) \subset M$. Since $j_{n+1} < p$, by (5.32) we have $q(z)z^{\phi(j_{n+1})-1}w^{j_{n+1}}H^2(T_z) \subset w^{j_{n+1}}S_{j_{n+1}}$. Then by (5.33), $S_{j_{n+1}} = q(z)z^{\phi(j_{n+1})-1}H^2(T_z)$.

Next, suppose that $p < j_1 + j_n$. Then by Lemma 4.1 (v), $j_1 + j_n = p + j_{n+1}$ and $\phi(j_1) + \phi(j_n) = k + \phi(j_{n+1})$, so that by (5.40), $q(z)\zeta z^{\phi(j_{n+1})-1}w^{j_{n+1}}H^2(T_z) \subset M$. By (5.18), $q(z)\zeta z^{\phi(j_{n+1})-1}w^{j_{n+1}}H^2(T_z) \subset \zeta N$. Hence $q(z)z^{\phi(j_{n+1})-1}w^{j_{n+1}}H^2(T_z) \subset w^{j_{n+1}}S_{j_{n+1}}$. By (5.33), we get $S_{j_{n+1}} = q(z)z^{\phi(j_{n+1})-1}w^{j_{n+1}}H^2(T_z)$. Therefore by induction, we can prove (5.39). By (5.32), (5.37) and (5.39), we get (5.34), so that by (5.18) M is of the form (i).

CASE 2. Suppose that $z\overline{S_0} = \overline{S_0}$. By (5.16), $z\overline{S_0} \subset N \cap S_0 \subset \overline{S_0}$. Hence S_0 is a closed subspace of $L^2(T_z)$ and $zS_0 = S_0 \subset N$. By (5.8), $S_0 \subset M \ominus [A_{\phi,1}M]$, so that S_0 plays the role of N in the basic procedure in Section 2 for $p = 1$. Since $zS_0 = S_0$, Case 1 happens in the basic procedure. Then by Lemma 2.4, M is an invariant subspace with $zM = M$ and $wM \neq M$. Therefore M satisfies the condition (ii).

By Lemma 5.1, it is not difficult to prove the converse assertion.

6. **Homogeneous-Type A_ϕ-Invariant Subspaces.** We discuss the same ϕ which is studied in Section 4. Let $p \in Z_+ \setminus \{0\}$ and $k \in Z$ such that p and $|k|$ are mutually prime if $k \neq 0$, and $p = 1$ if $k = 0$. For each $n \in Z_+$, let $\phi(n)$ be the smallest integer such that $p\phi(n) - kn \geq 0$.

Let M be an A_ϕ-invariant subspace. For $n \in Z_+$, let

$$M_n = \Big[\sum_{j=0}^{n}(z^k w^p)^j z^{n-j}M\Big]. \tag{6.1}$$

Then M_n is A_ϕ-invariant and $M = M_0 \supset M_1 \supset M_2 \supset \cdots$. Let $X_n = M_n \ominus M_{n+1}$ for $n \in Z_+$. Then we have the following decomposition

$$M = \Big(\sum_{n=0}^{\infty} \oplus X_n\Big) \oplus M_\infty, \tag{6.2}$$

where $M_\infty = \bigcap_{n=0}^{\infty} M_n$. Here we call M a *homogeneous-type* A_ϕ-invariant subspace if

$$zX_n \subset X_{n+1} \quad \text{and} \quad z^k w^p X_n \subset X_{n+1} \quad \text{for } n \in Z_+ \tag{6.3}$$

and

$$M_\infty = \{0\}. \tag{6.4}$$

In this section, we study the following problem (see [11, 13]).

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PROBLEM 3. Determine the homogeneous-type A_ϕ-invariant subspaces M with $z^k w^p M \subset zM$ and $z^k w^p M \neq zM$.

In [11], Nakazi gave an answer for the case $p = 1$ and $k = 0$.

LEMMA 6.1. *Let M be A_ϕ-invariant. Then M is of homogeneous-type if and only if there is a closed subspace E of $L^2(T^2)$ such that $M = \sum_{n=0}^{\infty} \oplus[\sum_{j=0}^{n}(z^k w^p)^j z^{n-j} E]$.*

PROOF. Suppose that M is of homogeneous-type. Then by (6.2) and (6.4),

$$M = \sum_{n=0}^{\infty} \oplus X_n. \tag{6.5}$$

We shall show that

$$X_n = \Big[\sum_{j=0}^{n}(z^k w^p)^j z^{n-j} X_0\Big] \quad \text{for } n \in Z_+. \tag{6.6}$$

By (6.3), $[z^k w^p X_n + zX_n] \subset X_{n+1}$. Then by (6.1) and (6.5), $M_1 = \sum_{n=0}^{\infty} \oplus[z^k w^p X_n + zX_n]$, so that $X_0 = M \ominus M_1 = X_0 \oplus \big(\sum_{n=1}^{\infty} \oplus(X_n \ominus [z^k w^p X_{n-1} + zX_{n-1}])\big)$. Thus $X_n = [z^k w^p X_{n-1} + zX_{n-1}]$ for $n \geq 1$. Hence we have (6.6). Set $E = X_0$; then M has the desired form.

Next, suppose that there exists a closed subspace E of $L^2(T^2)$ such that

$$M = \sum_{n=0}^{\infty} \oplus \Big[\sum_{j=0}^{n}(z^k w^p)^j z^{n-j} E\Big].$$

Then we have

$$M_i = \sum_{n=i}^{\infty} \oplus \Big[\sum_{j=0}^{n}(z^k w^p)^j z^{n-j} E\Big].$$

Hence

$$X_n = \Big[\sum_{j=0}^{n}(z^k w^p)^j z^{n-j} E\Big] \quad \text{and} \quad M_\infty = \{0\}.$$

Now it is easy to see that X_n satisfies (6.3), so that M is of homogeneous-type.

LEMMA 6.2. *Let M be an A_ϕ-invariant subspace with $z^k w^p M \subset zM$ and $M \neq \{0\}$. Suppose that M is of homogeneous-type. Let E be the closed subspace of $L^2(T^2)$ which is given in Lemma 6.1. Then $M = \sum_{n=0}^{\infty} \oplus z^n E$ and $z^{k-1} w^p E \subset E$.*

PROOF. Let $\zeta = z^k w^p$. Suppose that M is of homogeneous-type. Then by Lemma 6.1, there is a nonzero closed subspace E of $L^2(T^2)$ such that

$$M = \sum_{n=0}^{\infty} \oplus X_n, \quad X_n = \Big[\sum_{j=0}^{n} \zeta^j z^{n-j} E\Big]. \tag{6.7}$$

By our assumption, $\zeta M \subset zM$, so that $\zeta M = \sum_{n=0}^{\infty} \oplus \zeta X_n \subset \sum_{n=0}^{\infty} \oplus zX_n$. Since $\zeta X_n \cup zX_n \subset X_{n+1}$, by the above inclusion we have $\zeta X_n \subset zX_n$. Hence

$$\Big[\sum_{j=0}^{n} \zeta^{j+1} z^{n-j} E\Big] \subset \Big[\sum_{j=0}^{n} \zeta^j z^{n+1-j} E\Big], \quad n \in Z_+.$$

When $n = 0$, $\zeta E \subset zE$. Hence $X_n = [\sum_{j=0}^{n} \zeta^j z^{n-j} E] \subset z^n E \subset X_n$, so that we get $X_n = z^n E$. Therefore by (6.7), $M = \sum_{n=0}^{\infty} \oplus z^n E$.

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THEOREM 6.1. *Let M be an A_ϕ-invariant subspace with $z^k w^p M \subset zM$ and $z^k w^p M \neq zM$. Suppose that M is of homogeneous-type. Then M has one of the following forms.*

(i)
$$M = \psi \sum_{j=0}^{\infty} \oplus (z^{k-1}w^p)^j \left(G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)} w^i H^2(T_z) \right) \right),$$

where ψ is a unimodular function on T^2 and G is a closed subspace such that

$$G \subset \left[\{ z^{\phi(i)-1} w^i, z^{\phi(i)-2} w^i \ ; \ 1 \le i \le p-1 \} \right].$$

(ii)
$$M = \psi \sum_{j=0}^{\infty} \oplus (z^{k-1}w^p)^j \left(G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)+1} w^i H^2(T_z) \right) \right),$$

where ψ is a unimodular function on T^2 and G is a closed subspace such that

$$G \subset \left[\{ 1, z^{\phi(i)} w^i, z^{\phi(i)-1} w^i, z^{\phi(i)-2} w^i \ ; \ 1 \le i \le p-1 \} \right].$$

The structure of G is in general too complicated to describe more explicitly. In Section 7, we determine G for two special kinds of ϕ.

PROOF OF THEOREM 6.1. Let

$$\zeta = z^k w^p. \tag{6.8}$$

Since M is of homogeneous-type, by Lemmas 6.1 and 6.2 there is a nonzero closed subspace E of $L^2(T^2)$ such that

$$M = \sum_{n=0}^{\infty} \oplus \left[\sum_{j=0}^{n} \zeta^j z^{n-j} E \right] = \sum_{n=0}^{\infty} \oplus z^n E, \quad \zeta z^{-1} E \subset E. \tag{6.9}$$

If $\zeta z^{-1} E = E$, then by (6.9), $\zeta M = zM$. This contradicts our assumption. Therefore $\zeta z^{-1} E \neq E$. Let $Y = E \ominus \zeta z^{-1} E \neq \{0\}$. Then

$$E = Y \oplus \zeta z^{-1} E. \tag{6.10}$$

By (6.9), $z^i Y \perp z^j Y$ for $i, j \in Z_+, i \neq j$. Let

$$N = \sum_{i=0}^{\infty} \oplus z^i Y. \tag{6.11}$$

Then by (6.9), (6.10) and (6.11),

$$M = N \oplus \zeta z^{-1} M. \tag{6.12}$$

Here let B be the semigroup in $\{z^i w^j \ ; \ i, j \in Z\}$ generated by ζz^{-1} and A_ϕ. For each $n \in Z_+$, we put $\mu(n) = \min\{i \in Z \ ; \ z^i w^n \in B\}$. Then $\mu(0) = 0$ and $A_\mu = B$. By (6.8) and the definition of ϕ,

$$\mu(ip + j) = \phi(ip + j) - i \quad \text{for } i \in Z_+, 0 \le j \le p-1 \ ; \tag{6.13}$$

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$$\mu(p) = k - 1; \tag{6.14}$$

$$\zeta z^{-1} A_\mu = A_{\mu,p}. \tag{6.15}$$

Hence A_μ is cyclic. By our assumption, $\zeta z^{-1} M \subset M$, so that M is A_μ-invariant. Then by (6.15), $[A_{\mu,p} M] = \zeta z^{-1} M$. Hence (6.11) and (6.12) imply that N is a nonzero z-invariant subspace, $zN \neq N$ and

$$N = M \ominus [A_{\mu,p} M]. \tag{6.16}$$

Now we can use Case 2 of the basic procedure in Section 2 for $\mu(n)$ instead of $\phi(n)$. Then by (2.23), there is a nonzero subspace S_j of $L^2(T_z)$ (perhaps not closed) such that

$$M \subset \sum_{j=0}^{\infty} \oplus w^j S_j \subset \tilde{M} = \sum_{j=0}^{\infty} \oplus w^j L^2(T_z), \tag{6.17}$$

and by (2.24),

$$N \subset \sum_{j=0}^{p-1} \oplus w^j S_j \subset \sum_{j=0}^{p-1} \oplus w^j L^2(T_z). \tag{6.18}$$

By (6.12) and the definition of S_j (see (2.4)),

$$\zeta z^{-1} S_j = w^p S_{j+p}, \quad j \in Z_+. \tag{6.19}$$

By (2.25),

$$[A_{\mu,n} M] \subset \sum_{j=n}^{\infty} \oplus w^j S_j \subset \sum_{j=n}^{\infty} \oplus w^j L^2(T_z), \quad n \in Z_+. \tag{6.20}$$

By (2.9),

$$\sum_{j=0}^{n} z^{\mu(n-j)} S_j \subset S_n \subset \left[\sum_{j=0}^{n} z^{\mu(n-j)} S_j \right], \quad n \in Z_+. \tag{6.21}$$

By (6.13), (6.14) and Lemma 4.1 (ii), $\mu(1)+\mu(p-1)-\mu(p) = 2$. Hence by Lemma 2.3,

$$z^2 \overline{S_0} \subset N \cap S_0. \tag{6.22}$$

By (2.5), $\overline{S_0}$ is a z-invariant subspace of $L^2(T_z)$, so that by the Beurling theorem $\overline{S_0} = q(z) H^2(T_z)$ or $\overline{S_0} = \chi_F(z) L^2(T_z)$, where $q(z)$ is a unimodular function on T_z and $F \subset T_z$. By (6.22), $z^2 \overline{S_0} \subset S_0 \subset \overline{S_0}$. Then for both cases, S_0 becomes a closed subspace and $S_0 = q(z) H^2(T_z)$ or $S_0 = \chi_F(z) L^2(T_z)$. Moreover by (6.22),

$$z^2 S_0 \subset N. \tag{6.23}$$

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Here we note that $S_0 \neq \chi_F(z)L^2(T_z)$. For, suppose that $S_0 = \chi_F(z)L^2(T_z)$. By (6.20), $S_0 \perp [A_{\mu,1}M]$. Then by Lemma 2.4, M is an invariant subspace with $zM = M$ and $wM \neq M$. But by (6.9), M satisfies $zM \neq M$. This is a contradiction. Therefore $S_0 = q(z)H^2(T_z)$.

For the sake of simplicity we assume that

$$S_0 = H^2(T_z). \tag{6.24}$$

Now recall the proof of (5.21) in the proof of Theorem 5.1. In the same way, from (6.23) we can prove $zS_0 \subset N$. Since $z^2S_0 \subset N \cap S_0 \subset S_0$, by the above inclusion we have

$$N \cap S_0 = S_0 \quad \text{or} \quad N \cap S_0 = zS_0. \tag{6.25}$$

By (6.19) for $j = 0$, $\zeta z^{-1}S_0 = w^pS_p$. Then by (6.21),

$$z^{\mu(p-j)}w^{p-j}w^jS_j \subset \zeta z^{-1}S_0, \quad 0 \leq j \leq p-1. \tag{6.26}$$

By (6.13), (6.14) and Lemma 4.1 (ii), $\mu(p)-\mu(p-j) = \mu(j)-2$. Hence by (6.8) and (6.26), $S_j \subset z^{\mu(j)-2}S_0$. On the other hand, by (6.21) we have $z^{\mu(j)}S_0 \subset S_j$, $0 \leq j \leq p-1$. Hence $z^{\mu(j)}S_0 \subset S_j \subset z^{\mu(j)-2}S_0$ for $0 \leq j \leq p-1$. Then by (6.24), S_j is a closed subspace of $L^2(T_z)$ and

$$S_j = z^{\mu(j)-\epsilon(j)}S_0 = z^{\mu(j)-\epsilon(j)}H^2(T_z) \quad \text{for some } \epsilon(j) = 0, 1, 2. \tag{6.27}$$

Since $\mu(0) = 0$,

$$\epsilon(0) = 0. \tag{6.28}$$

By (6.16), (6.18), (6.20), and the A_μ-invariantness of M,

$$\sum_{j=0}^{p-1} \oplus z^{\mu(j)}w^j(N \cap S_0) \subset N. \tag{6.29}$$

By (6.18) and (6.27),

$$N \subset \sum_{j=0}^{p-1} \oplus w^jS_j = \sum_{j=0}^{p-1} \oplus z^{\mu(j)-\epsilon(j)}w^jH^2(T_z). \tag{6.30}$$

By (6.29), we can define

$$G = N \ominus \left(\sum_{j=0}^{p-1} \oplus z^{\mu(j)}w^j(N \cap S_0)\right). \tag{6.31}$$

We consider the following two cases separately (see (6.25)); $N \cap S_0 = S_0$ and $N \cap S_0 = zS_0$.

When $N \cap S_0 = S_0$, by (6.24) and (6.31) we have

$$N = G \oplus \sum_{j=0}^{p-1} \oplus z^{\mu(j)}w^jH^2(T_z).$$

By (6.12) and (6.17), $M = \sum_{j=0}^{\infty} \oplus (\zeta z^{-1})^jN$. Hence, in this case, M has the form given by (i). By (6.28), (6.30), and (6.31), it is not difficult to see that G satisfies the desired condition.

In the same way, when $N \cap S_0 = zS_0$, M has the form given by (ii).

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7**. Examples of Homogeneous-Type A_ϕ-Invariant Subspaces.** This section is a continuation of Section 6. Let $\{j_0, j_1, \ldots, j_{p-1}\} = \{0, 1, \ldots, p-1\}$ such that $p\phi(j_i) - kj_i < p\phi(j_{i+1}) - kj_{i+1}$ for $0 \le i \le p-2$. We note that $j_0 = 0$ (see for detail Section 4), and the structure of $\{j_i\}_{i=0}^{p-1}$ depends strongly on the given p and k. We study in Theorem 7.1 the case $j_i = i$, $1 \le i \le p-1$, and in Theorem 7.2 the case $j_i = p - i$, $1 \le i \le p-1$. Comparing these theorems, we find that the structures of G are completely different. For general cases, it is natural to expect that G has the mixed structures of G in Theorems 7.1 and 7.2.

THEOREM 7.1. *Suppose that $j_i = i$ for $0 \le i \le p-1$ for given p and k. Let M be an A_ϕ-invariant subspace with $z^k w^p M \subset zM$ and $z^k w^p M \neq zM$. Then M is of homogeneous-type if and only if*

$$M = \psi \sum_{j=0}^{\infty} \oplus (z^{k-1}w^p)^j \left(G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)} w^i H^2(T_z) \right) \right),$$

where ψ is a unimodular function on T^2 and G has one of the following forms.

(i) $$G = \{0\} \quad \textit{or} \quad G = \left[\{ z^{\phi(s)-1} w^s \ ; s_1 \le s \le p-1 \} \right]$$

for some s_1 with $1 \le s_1 \le p-1$.

(ii) $$G = \left[\{ z^{\phi(i)-1} w^i, z^{\phi(j)-2} w^j \ ; s_1 \le i \le p-1, s_2 \le j \le p-1 \} \right]$$

for some s_1 and s_2 with $1 \le s_1 \le s_2 \le p-1$.

(iii) $$G = G_1 \oplus \left[\{ z^{\phi(i)-1} w^i, z^{\phi(j)-2} w^j \ ; t_1 \le i \le p-1, t_2 \le j \le p-1 \} \right]$$

where

$$G_1 = \left[\left\{ (z^{\phi(1)} w)^j \left(\sum_{i=0}^{t_1 - s_1 - j - 1} (\alpha_i z^{\phi(s_2+i)-2} w^{s_2+i} + \beta_i z^{\phi(s_1+i)-1} w^{s_1+i}) \right) ; 0 \le j \le t_1 - s_1 - 1 \right\} \right]$$

for some complex numbers $\{\alpha_i, \beta_i\}_{i=0}^{t_1-s_1-1}$ with $\alpha_0 \neq 0$ and $\beta_0 \neq 0$, and for some s_1, s_2, t_1, t_2 with $0 \le s_1 < t_1 \le s_2 < t_2 \le p$ and $t_2 - s_2 = t_1 - s_1$.

We note that for a given $p \in Z_+ \setminus \{0\}$, a pair (p, k) satisfies the assumption of Theorem 7.1 if and only if $k = lp - 1$ and $lp \neq 1$ for some $l \in Z$.

PROOF OF THEOREM 7.1. Suppose that M is of homogeneous-type. We use the same notations as in the proof of Theorem 6.1 and we continue our proof from the end of the proof of Theorem 6.1. Since $j_i = i$ for $0 \le i \le p-1$,

(7.1) $$z^{\phi(j)} w^j = (z^{\phi(1)} w)^j, \quad 0 \le j \le p-1.$$

This is the key point of our assumption.

First suppose that

(7.2) $$N \cap S_0 = S_0 = H^2(T_z).$$

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Then by the end of the proof of Theorem 6.1, we may consider

$$M=\sum_{j=0}^{\infty}\oplus(z^{k-1}w^p)^j\left(G\oplus\left(\sum_{i=0}^{p-1}\oplus z^{\phi(i)}w^iH^2(T_z)\right)\right), \tag{7.3}$$

where G is a closed subspace with

$$G\subset\left[\{z^{\phi(j)-1}w^j, z^{\phi(j)-2}w^j\ ;\ 1\le j\le p-1\}\right]. \tag{7.4}$$

Using the property that $A_\phi M\subset M$, we will describe G.

For $i=1$ or 2, we define positive integers t_i and s_i. When $z^{\phi(t)-i}w^t\in G$ for some $1\le t\le p-1$, let t_i be the smallest integer t satisfying the above condition. For convenience, let $t_i=p$ when $z^{\phi(t)-i}w^t\notin G$ for every $1\le t\le p-1$. When $\hat{f}\big(\phi(s)-i,s\big)\neq 0$ for some $f\in G$ and for some s with $1\le s\le p-1$, let s_i be the smallest integer s satisfying the above condition. Then $\hat{f}\big(\phi(s)-i,s\big)=0$ for every $f\in G$ and $1\le s<s_i$ and $\hat{g}\big(\phi(s_i)-i,s_i\big)\neq 0$ for some $g\in G$. We note that s_1 and s_2 may not exist. If s_i exists, by the definitons we have $s_i\le t_i$. In the following, we shall see that the structure of G depends on the data of s_i and t_i. To study the structure of G, we separate into several cases. The following follows from (7.4).

(a) If both s_1 and s_2 do not exist, $G=\{0\}$.

(b) If s_1 exists and s_2 does not, then $s_1=t_1$ and

$$G=\left[\{z^{\phi(s)-1}w^s\ ;\ s_1\le s\le p-1\}\right],\quad 1\le s_1\le p-1.$$

For, by our assumptions and the definitions of s_1 and s_2,

$$G\subset\left[\{z^{\phi(s)-1}w^s\ ;\ s_1\le s\le p-1\}\right], \tag{7.5}$$

and there exists $f\in G$ such that

$$f=\sum_{s=s_1}^{p-1}a_sz^{\phi(s)-1}w^s,\quad a_{s_1}\neq 0. \tag{7.6}$$

Since $z^{\phi(p-s_1-1)}w^{p-s_1-1}G\subset z^{\phi(p-s_1-1)}w^{p-s_1-1}M\subset M$,

$$\sum_{s=s_1}^{p-1}a_sz^{\phi(p-s_1-1)+\phi(s)-1}w^{p+s-s_1-1}\in M.$$

Then by (6.16), (6.18) and (6.20), $a_{s_1}z^{\phi(p-s_1-1)+\phi(s_1)-1}w^{p-1}\in N$. Since $a_{s_1}\neq 0$, by (7.1) we have $z^{\phi(p-1)-1}w^{p-1}\in N$. By (6.13), (6.31), and (7.2), we have $z^{\phi(p-1)-1}w^{p-1}\in G$, so that by (7.6) we get $\sum_{s=s_1}^{p-2}a_sz^{\phi(s)-1}w^s\in G$. In the same way, using $z^{\phi(p-s_1-2)}w^{p-s_1-2}G\subset M$, we have $z^{\phi(p-2)-1}w^{p-2}\in G$. By induction, we can prove that $z^{\phi(s)-1}w^s\in G, s_1\le s\le p-1$. By (7.5), we see that G has the desired form.

The above proof also shows the following two facts (c) and (d).

(c) If $t_i\le p-1$, then $z^{\phi(t)-i}w^t\in G$ for every $t_i\le t\le p-1$.

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(d) If $\sum_{s=l}^{p-1} a_s z^{\phi(s)-i} w^s \in G$ and $a_l \neq 0$, then $t_i \leq l$.
(e) If s_2 exists, then s_1 exists and $s_1 \leq t_1 \leq s_2$.

For, suppose that s_2 exists. Then by (7.4) there exists $f \in G$ such that

$$f = \sum_{s=s_2}^{p-1} a_s z^{\phi(s)-2} w^s + \sum_{j=s_1}^{p-1} b_j z^{\phi(j)-1} w^j, \quad a_{s_2} \neq 0. \tag{7.7}$$

When s_1 does not exist, we consider $b_j = 0$ for $s_1 \leq j \leq p-1$. Since $zf \in zG \subset zN \subset N$, by (6.31) and (7.2) we have $\sum_{s=s_2}^{p-1} a_s z^{\phi(s)-1} w^s \in G, a_{s_2} \neq 0$. Then by (d), $t_1 \leq s_2$. The inequality $s_1 \leq t_1$ follows from the definitions of s_1 and t_1.

(f) If s_2 exists and $s_1 = t_1$, then $s_2 = t_2$ and

$$G = \left[\{z^{\phi(i)-1} w^i, z^{\phi(j)-2} w^j \,;\, s_1 \leq i \leq p-1, s_2 \leq j \leq p-1\}\right], \quad 1 \leq s_1 \leq s_2 \leq p-1. \tag{7.8}$$

For, suppose that s_2 exists and $s_1 = t_1$. Then there exists $f \in G$ satisfying (7.7). Since $s_1 = t_1$, by (c) we have $\sum_{s=s_2}^{p-1} a_s z^{\phi(s)-2} w^s \in G, s_{s_2} \neq 0$. By (d), $t_2 \leq s_2$. The opposite inequality follows from the definitions of s_2 and t_2, so that $s_1 \leq s_2 = t_2$. Then (c) gives (7.8).

Finally, suppose that s_2 exists and $s_1 < t_1$. We first prove that

$$t_2 - s_2 = t_1 - s_1. \tag{7.9}$$

Let $f \in G$ such that

$$f = \sum_{s=s_2}^{p-1} a_s z^{\phi(s)-2} w^s + \sum_{j=s_1}^{p-1} b_j z^{\phi(j)-1} w^j, \quad b_{s_1} \neq 0.$$

Since $z^{\phi(t_2-s_2)} w^{t_2-s_2} f \in M$, by (7.3) and (7.4)

$$\sum_{s=s_2}^{p+s_2-t_2-1} a_s z^{\phi(t_2-s_2)+\phi(s)-2} w^{t_2+s-s_2} + \sum_{j=s_1}^{p+s_2-t_2-1} b_j z^{\phi(t_2-s_2)+\phi(j)-1} w^{t_2+j-s_2} \in G.$$

By (7.1),

$$\sum_{s=s_2}^{p+s_2-t_2-1} a_s z^{\phi(t_2+s-s_2)-2} w^{t_2+s-s_2} + \sum_{j=s_1}^{p+s_2-t_2-1} b_j z^{\phi(t_2+j-s_2)-1} w^{t_2+j-s_2} \in G.$$

Since $t_2 + s - s_2 \geq t_2$ for $s \geq s_2$, by (c) we have $\sum_{j=s_1}^{p+s_2-t_2-1} b_j z^{\phi(t_2+j-s_2)-1} w^{t_2+j-s_2} \in G$. Since $b_{s_1} \neq 0$, by (d) we have $t_1 \leq t_2 + s_1 - s_2$. Hence $t_1 - s_1 \leq t_2 - s_2$.

Let $g \in G$ such that

$$g = \sum_{s=s_2}^{p-1} c_s z^{\phi(s)-2} w^s + \sum_{j=s_1}^{p-1} d_j z^{\phi(j)-1} w^j, \quad c_{s_2} \neq 0.$$

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Since $z^{\phi(t_1-s_1)}w^{t_1-s_1}g \in M$, in the same way as above we have

$$\sum_{s=s_2}^{p+s_1-t_1-1} c_s z^{\phi(t_1+s-s_1)-2} w^{t_1+s-s_1} \in G.$$

Since $c_{s_2} \neq 0$, by (d) we get $t_2 \leq t_1 + s_2 - s_1$, so that $t_2 - s_2 \leq t_1 - s_1$. Therefore we get (7.9).

Consequently there exist t_1, t_2, s_1, and s_2 such that $s_1 < t_1 \leq s_2 < t_2$, $t_2 - t_1 = s_2 - s_1$, and

$$G = G_1 \oplus \left[\{z^{\phi(i)-1}w^i, z^{\phi(j)-2}w^j \ ; t_1 \leq i \leq p-1, t_2 \leq j \leq p-1\}\right], \tag{7.10}$$

where

$$G_1 \subset \left[\{z^{\phi(i)-1}w^i, z^{\phi(j)-2}w^j \ ; \ s_1 \leq i < t_1, s_2 \leq j < t_2\}\right] \tag{7.11}$$

and

$$z^i w^j \notin G_1 \quad \text{for every } (i,j) \in Z^2. \tag{7.12}$$

To describe G_1, fix $f_0 \in G_1$ such that $\hat{f}_0\big(\phi(s_2) - 2, s_2\big) \neq 0$. Then we have

$$\hat{f}_0\big(\phi(s_1) - 1, s_1\big) \neq 0.$$

For, write f_0 as

$$f_0 = \sum_{s=s_2}^{t_2-1} a_s z^{\phi(s)-2} w^s + \sum_{j=s_1}^{t_1-1} b_j z^{\phi(j)-1} w^j, \quad a_{s_2} \neq 0.$$

For the sake of simplicity, let $a_s = 0$ for $t_2 \leq s \leq p-1$, and $b_j = 0$ for $t_1 \leq j \leq p-1$. Then

$$f_0 = \sum_{s=s_2}^{p-1} a_s z^{\phi(s)-2} w^s + \sum_{j=s_1}^{p-1} b_j z^{\phi(j)-1} w^j, \quad a_{s_2} \neq 0.$$

To show $b_{s_1} \neq 0$, suppose that $b_{s_1} = 0$. By our assumption, $t_1 - s_1 > 0$, so that $z^{\phi(t_1-s_1-1)}w^{t_1-s_1-1}f_0 \in M$. Hence

$$\sum_{s=s_2}^{p+s_1-t_1} a_s z^{\phi(t_1-s_1-1)+\phi(s)-2} w^{t_1+s-s_1-1} + \sum_{j=s_1+1}^{p+s_1-t_1} b_j z^{\phi(t_1-s_1-1)+\phi(j)-1} w^{t_1+j-s_1-1} \in G.$$

By (7.1),

$$\sum_{s=s_2}^{p+s_1-t_1} a_s z^{\phi(t_1+s-s_1-1)-2} w^{t_1+s-s_1-1} + \sum_{j=s_1+1}^{p+s_1-t_1} b_j z^{\phi(t_1+j-s_1-1)-1} w^{t_1+j-s_1-1} \in G.$$

Since $t_1 + j - s_1 - 1 \geq t_1$ for $j \geq s_1 + 1$, by the definition of t_1 and (c) we have

$$\sum_{s=s_2}^{p+s_1-t_1} a_s z^{\phi(t_1+s-s_1-1)-2} w^{t_1+s-s_1-1} \in G.$$

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Since $a_{s_2} \neq 0$, by (d) we have $t_2 \leq t_1 + s_2 - s_1 - 1$. This contradicts (7.9), so that $\hat{f}_0\big(\phi(s_1) - 1, s_1\big) = b_{s_1} \neq 0$.

By (7.9) and the above fact, we can rewrite f_0 as

$$f_0 = \sum_{i=0}^{p-1}(\alpha_i z^{\phi(s_2+i)-2} w^{s_2+i} + \beta_i z^{\phi(s_1+i)-1} w^{s_1+i}), \tag{7.13}$$

$$\alpha_0 \neq 0, \beta_0 \neq 0 \quad \text{and} \quad \alpha_i = \beta_i = 0 \quad \text{for } t_1 - s_1 \leq i \leq p-1.$$

Now we shall prove that

$$G_1 = \left[\left\{(z^{\phi(1)}w)^j \left(\sum_{i=0}^{t_1-s_1-j-1}(\alpha_i z^{\phi(s_2+i)-2} w^{s_2+i} + \beta_i z^{\phi(s_1+i)-1} w^{s_1+i})\right) ;\, 0 \leq j \leq t_1 - s_1 - 1\right\}\right],$$

where $1 \leq s_1 < t_1 \leq s_2 < t_2 \leq p$.

Let $0 \leq j \leq t_1 - s_1 - 1$. Since $z^{\phi(j)} w^j f_0 \in M$,

$$\sum_{i=0}^{p-s_2-j-1} \alpha_i z^{\phi(j)+\phi(s_2+i)-2} w^{j+s_2+i} + \sum_{i=0}^{p-s_1-j-1} \beta_i z^{\phi(j)+\phi(s_1+i)-1} w^{j+s_1+i} \in G.$$

By (7.1),

$$\sum_{i=0}^{p-s_2-j-1} \alpha_i z^{\phi(j+s_2+i)-2} w^{j+s_2+i} + \sum_{i=0}^{p-s_1-j-1} \beta_i z^{\phi(j+s_1+i)-1} w^{j+s_1+i} \in G.$$

By (7.10) and (7.11),

$$\sum_{i=0}^{t_2-s_2-j-1} \alpha_i z^{\phi(j+s_2+i)-2} w^{j+s_2+i} + \sum_{i=0}^{t_1-s_1-j-1} \beta_i z^{\phi(j+s_1+i)-1} w^{j+s_1+i} \in G_1.$$

By (7.1) and (7.9),

$$(z^{\phi(1)}w)^j \sum_{i=0}^{t_1-s_1-j-1}(\alpha_i z^{\phi(s_2+i)-2} w^{s_2+i} + \beta_i z^{\phi(s_1+i)-1} w^{s_1+i}) \in G_1. \tag{7.14}$$

For convenience, put

$$f_j = (z^{\phi(1)}w)^j \sum_{i=0}^{t_1-s_1-j-1}(\alpha_i z^{\phi(s_2+i)-2} w^{s_2+i} + \beta_i z^{\phi(s_1+i)-1} w^{s_1+i}) \tag{7.15}$$

for $0 \leq j \leq t_1 - s_1 - 1$. Therefore by (7.14),

$$\big[\{ f_j \,;\, 0 \leq j \leq t_1 - s_1 - 1\}\big] \subset G_1. \tag{7.16}$$

To show the converse inclusion, let take $f \in G_1$, $f \neq 0$, arbitrary. We can write f as

$$f = \sum_{i=0}^{t_1-s_1-1}(a_i z^{\phi(s_2+i)-2} w^{s_2+i} + b_i z^{\phi(s_1+i)-1} w^{s_1+i}).$$

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By the same reasoning as in the paragraph before (7.13), there exists an integer m, $0 \le m \le t_1 - s_1 - 1$, such that

$$f = \sum_{i=m}^{t_1-s_1-1} (a_i z^{\phi(s_2+i)-2} w^{s_2+i} + b_i z^{\phi(s_1+i)-1} w^{s_1+i}), \quad a_m \neq 0, b_m \neq 0. \tag{7.17}$$

Here we have

$$\frac{\alpha_0}{\beta_0} = \frac{a_m}{b_m}. \tag{7.18}$$

For, suppose not, that is, $\alpha_0/\beta_0 \neq a_m/b_m$. By multiplying $z^{\phi(t_1-s_1-1)} w^{t_1-s_1-1}$ with f_0, by (7.9), (7.10), (7.11), and (7.13) we have

$$\alpha_0 z^{\phi(t_2-1)-2} w^{t_2-1} + \beta_0 z^{\phi(t_1-1)-1} w^{t_1-1} \in G_1. \tag{7.19}$$

By multiplying $z^{\phi(t_1-s_1-1-m)} w^{t_1-s_1-1-m}$ with f, by (7.17) we can also get

$$a_m z^{\phi(t_2-1)-2} w^{t_2-1} + b_m z^{\phi(t_1-1)-1} w^{t_1-1} \in G_1. \tag{7.20}$$

Since $\alpha_0/\beta_0 \neq a_m/b_m$, by (7.19) and (7.20) we have $z^{\phi(t_2-1)-2} w^{t_2-1}, z^{\phi(t_1-1)-1} w^{t_1-1} \in G_1$. This contradicts (7.12). Hence we get (7.18).

By (7.1), (7.15), (7.16), (7.17), and (7.18),

$$G_1 \ni f - \frac{a_m}{\alpha_0} f_m = \sum_{i=m+1}^{t_1-s_1-1} (c_i z^{\phi(s_2+i)-2} w^{s_2+i} + d_i z^{\phi(s_1+i)-1} w^{s_1+i}), \quad \text{say.}$$

We note that the number of terms in the above sum is less than in (7.17). Repeating these arguments, we can prove that there exist complex numbers $\{c_m, c_{m+1}, \dots, c_{t_1-s_1-1}\}$ such that $f = \sum_{i=m}^{t_1-s_1-1} c_i f_i$. Hence $G_1 \subset [\{ f_j ; 0 \le j \le t_1 - s_1 - 1\}]$. By (7.16), we get the desired equality. This completes the proof for the case $N \cap S_0 = S_0 = H^2(T_z)$, and in this case, one of (i), (ii) and (iii) with $s_1 \ge 1$ happens.

Next we study the case

$$N \cap S_0 = zS_0 = zH^2(T_z). \tag{7.21}$$

By Theorem 6.1 (and its proof), we may assume

$$M = \sum_{j=0}^{\infty} \oplus (z^{k-1} w^p)^j \left(G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)+1} w^i H^2(T_z) \right) \right),$$

and

$$N = G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)+1} w^i H^2(T_z) \right), \tag{7.22}$$

where G is a closed subspace such that

$$G \subset \left[\{1, z^{\phi(i)} w^i, z^{\phi(i)-1} w^i, z^{\phi(i)-2} w^i ; 1 \le i \le p-1\} \right].$$

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In this case,

$$G \subset \left[\{z^{\phi(i)}w^i, z^{\phi(j)-1}w^j \; ; 0 \le i \le p-1, 1 \le j \le p-1\}\right]. \tag{7.23}$$

To prove this, suppose that there exists $h \in G$ such that $\hat{h}\big(\phi(i)-2, i\big) \neq 0$ for some $1 \le i \le p-1$. Then we can write h as

$$h = \sum_{i=1}^{t} a_i z^{\phi(i)-2} w^i + \sum_{j=1}^{p-1} b_j z^{\phi(j)-1} w^j + \sum_{m=0}^{p-1} c_m z^{\phi(m)} w^m, \quad a_t \neq 0$$

for some t with $1 \le t \le p-1$. Since $z^{\phi(p-t)}w^{p-t}h \in M$, by (6.9)–(6.20) we have

$$a_t z^{\phi(p-t)+\phi(t)-2} w^p + \sum_{j=t}^{p-1} \{b_j z^{\phi(p-t)+\phi(j)-1} w^{p+j-t} + c_j z^{\phi(p-t)+\phi(j)} w^{p+j-t}\} \in \zeta\, \overline{z} N.$$

Then

$$a_t z^{\phi(p-t)+\phi(t)-k-1} + \sum_{j=t}^{p-1} \{b_j z^{\phi(p-t)+\phi(j)-k} w^{j-t} + c_j z^{\phi(p-t)+\phi(j)-k+1} w^{j-t}\} \in N.$$

Hence by Lemma 4.1 and (7.1),

$$a_t + \sum_{j=t}^{p-1} \{b_j z^{\phi(j-t)+1} w^{j-t} + c_j z^{\phi(j-t)+2} w^{j-t}\} \in N.$$

Therefore by (7.22), $1 \in G \subset N$. By the definition of S_0 (see (2.4)), $1 \in S_0$, so that $1 \in N \cap S_0$. This contradicts (7.21). Hence we get (7.23).

Since M can be written as

$$M = z \sum_{j=0}^{\infty} \oplus (z^{k-1}w^p)^j \left(z^{-1}G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)} w^i H^2(T_z) \right) \right),$$

we can proceed in the same way as in the case $N \cap S_0 = S_0$. By (6.24), $S_0 = H^2(T_z)$. By the definition of S_0, we note that (5.23) holds. Then there exists $h \in N$ such that $\hat{h}(0,0) \neq 0$. By (7.22), there exists g in G such that $\hat{g}(0,0) \neq 0$. By (7.21), $1 \notin N$. Hence $1 \notin G$, so that $z^{-1} \notin z^{-1}G$. Therefore in this case only (iii) happens and $s_1 = 0$.

The converse assertion is not difficult to prove. This completes the proof.

THEOREM 7.2. *Suppose that $j_i = p - i$ for $1 \le i \le p-1$ for a given ϕ. Let M be an A_ϕ-invariant subspace with $z^k w^p M \subset zM$ and $z^k w^p M \neq zM$. Then M is of homogeneous-type if and only if*

$$M = \psi \sum_{j=0}^{\infty} \oplus (z^{k-1}w^p)^j \left(G \oplus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)} w^i H^2(T_z) \right) \right),$$

where ψ is a unimodular function on T^2 and G has one of the following forms.

$$G = G_1 \oplus \left[\{z^{\phi(i)-1}w^i \; ; 1 \le i \le p-1\}\right], \tag{i}$$

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where G_1 is a nonzero closed subspace of $[\{z^{\phi(j)-2}w^j\ ;\ 1 \le j \le p-1\}]$.

(ii) *G is a closed subspace with* $G \subset [\{z^{\phi(i)-1}w^i\ ;\ 1 \le i \le p-1\}]$.

(iii) $$G = G_1 \oplus \left[\{z^{\phi(i)-1}w^i\ ;\ 1 \le i \le p-1\}\right],$$

where G_1 is a closed subspace of $[\{z^{-1}, z^{\phi(j)-2}w^j\ ;\ 1 \le j \le p-1\}]$ *and there exists a function g in G_1 such that* $\hat{g}(-1,0) \ne 0$.

We note that for a given $p \in Z_+ \setminus \{0\}$, a pair (p,k) satisfies the assumption of Theorem 7.2 if and only if $k = lp + 1$ and $lp \ne -1$ for some $l \in Z$.

PROOF. We use the same notations as in the proof of Theorem 6.1 and we continue our proof from the end of the proof of Theorem 6.1. By our assumption, we have

$$\text{if } 1 \le s,t \le p-1 \text{ and } s+t \le p, \text{ then } \phi(s) + \phi(t) = \phi(s+t) + 1, \tag{7.24}$$

$$\text{if } 1 \le s,t \le p-1 \text{ and } s+t > p, \text{ then } \phi(s) + \phi(t) = \phi(s+t-p) + k. \tag{7.25}$$

We separate the proof into two cases; $N \cap S_0 = S_0 = H^2(T_z)$ and $N \cap S_0 = zH^2(T_z)$.

First suppose that $N \cap S_0 = H^2(T_z)$. Then by Section 6,

$$G = N \ominus \left(\sum_{i=0}^{p-1} \oplus z^{\phi(i)} w^i H^2(T_z)\right) \tag{7.26}$$

and

$$G \subset \left[\{z^{\phi(i)-1}w^i, z^{\phi(i)-2}w^i\ ;\ 1 \le i \le p-1\}\right].$$

Suppose that there exists f in G such that $\hat{f}\big(\phi(i)-2, i\big) \ne 0$ for some $1 \le i \le p-1$. Then f can be written as

$$f = \sum_{j=m}^{t} a_j z^{\phi(j)-2} w^j + \sum_{i=1}^{p-1} b_i z^{\phi(i)-1} w^i, \quad a_m \ne 0, a_t \ne 0$$

where $1 \le m \le t \le p-1$. Since $z^{\phi(p-m-1)} w^{p-m-1} f \in M$,

$$a_m z^{\phi(p-m-1)+\phi(m)-2} w^{p-1} + \sum_{i=1}^{m} b_i z^{\phi(p-m-1)+\phi(i)-1} w^{p+i-m-1} \in N.$$

By (7.24),

$$a_m z^{\phi(p-1)-1} w^{p-1} + \sum_{i=1}^{m} b_i z^{\phi(p+i-m-1)} w^{p+i-m-1} \in N.$$

By (7.26) and $a_m \ne 0$,

$$z^{\phi(p-1)-1} w^{p-1} \in G. \tag{7.27}$$

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Then using $z^{\phi(p-m-2)}w^{p-m-2}f \in M$, we get $z^{\phi(p-2)-1}w^{p-2} \in G$. For, $z^{\phi(p-m-2)}w^{p-m-2}f \in M$ implies that

$$\sum_{j=m}^{m+1} a_j z^{\phi(p-m-2)+\phi(j)-2}w^{p+j-m-2} + \sum_{i=1}^{m+1} b_i z^{\phi(p-m-2)+\phi(i)-1}w^{p+i-m-2} \in N.$$

By (7.24), (7.25) and (7.26), we have $a_m z^{\phi(p-2)-1}w^{p-2} + a_{m+1}z^{\phi(p-1)-1}w^{p-1} \in G$. By (7.27) and $a_m \neq 0$, $z^{\phi(p-2)-1}w^{p-2} \in G$. Repeating this argument, we have

$$z^{\phi(i)-1}w^i \in G, \quad m \leq i \leq p-1. \tag{7.28}$$

Next we show

$$z^{\phi(i)-1}w^i \in G, \quad 1 \leq i \leq t-1. \tag{7.29}$$

Since $z^{\phi(p+1-t)}w^{p+1-t}f \in M$,

$$\sum_{j=t-1}^{t} a_j z^{\phi(p+1-t)+\phi(j)-2}w^{p+j+1-t} + \sum_{i=t-1}^{p-1} b_i z^{\phi(p+1-t)+\phi(i)-1}w^{p+i+1-t} \in \zeta\overline{z}N.$$

By (7.24) and (7.25),

$$a_t z^{\phi(1)+k-2}w^{p+1} + a_{t-1}z^{k-1}w^p + b_{t-1}z^k w^p + \sum_{i=t}^{p-1} b_i z^{\phi(i+1-t)+k-1}w^{p+i+1-t} \in \zeta\overline{z}N.$$

Then by (7.26) and $a_t \neq 0$, $z^{\phi(1)-1}w \in G$. Then using $z^{\phi(p+2-t)}w^{p+2-t}f \in M$, we get $z^{\phi(2)-1}w^2 \in G$. Repeating this argument, we obtain (7.29).

Since $m \leq t$, by (7.28) and (7.29) we have $z^{\phi(i)-1}w^i \in G$ for every i with $1 \leq i \leq p-1$. Hence in this case G has the form in (i).

When $\hat{f}\big(\phi(i)-2, i\big) = 0$ for every $f \in G$ and $1 \leq i \leq p-1$, G has the form in (ii).

Next suppose that

$$N \cap S_0 = zH^2(T_z). \tag{7.30}$$

Then by Section 6,

$$G = N \ominus \Big(\sum_{i=0}^{p-1} \oplus z^{\phi(i)+1}w^i H^2(T_z)\Big) \tag{7.31}$$

and

$$G \subset \Big[\big\{1, z^{\phi(i)}w^i, z^{\phi(i)-1}w^i, z^{\phi(i)-2}w^i \ ; \ 1 \leq i \leq p-1\big\}\Big].$$

In this case, we prove

$$G \subset \Big[\big\{1, z^{\phi(i)}w^i, z^{\phi(i)-1}w^i \ ; \ 1 \leq i \leq p-1\big\}\Big]. \tag{7.32}$$

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To prove (7.32), suppose not. Then there exists g in G such that $\hat{g}\big(\phi(i)-2, i\big) \neq 0$ for some $1 \leq i \leq p-1$. Write g as

$$g = \sum_{j=m}^{s} a_j z^{\phi(j)-2} w^j + \sum_{i=0}^{p-1} (b_i z^{\phi(i)-1} w^i + c_i z^{\phi(i)} w^i),$$

where $1 \leq s \leq p-1$, $a_s \neq 0$ and $b_0 = 0$. Since $z^{\phi(p-s)} w^{p-s} g \in M$,

$$a_s z^{\phi(p-s)+\phi(s)-2} w^p + \sum_{i=s}^{p-1} z^{\phi(p-s)} w^{p-s} (b_i z^{\phi(i)-1} w^i + c_i z^{\phi(i)} w^i) \in \zeta \bar{z} N.$$

Then by (7.24) and (7.25),

$$a_s + b_s z + c_s z^2 + \sum_{i=s+1}^{p-1} (b_i z^{\phi(i-s)} w^{i-s} + c_i z^{\phi(i-s)+1} w^{i-s}) \in N.$$

By (7.31),

$$a_s + \sum_{i=s+1}^{p-1} b_i z^{\phi(i-s)} w^{i-s} \in G.$$

This fact gives us that $z^{\phi(i)} w^i \in G$ for $1 \leq i \leq p-1$, which is proved in the same way as in the proof of (7.28). Since $a_s \neq 0$, we therefore have $1 \in G$. This means that $1 \in N \cap S_0$ and $N \cap S_0 = H^2(T_z)$. This contradicts (7.30). Thus we get (7.32).

Since $S_0 = H^2(T_z)$, there exists h in N such that $\hat{h}(0,0) \neq 0$. By (7.31), we may assume $h \in G$. Then in the same way as in the proof of (7.28), we can prove that $z^{\phi(i)} w^i \in G$ for $1 \leq i \leq p-1$. Let $G_1 = G \ominus [\{z^{\phi(i)} w^i\ ;\ 1 \leq i \leq p-1\}]$, $G' = z^{-1} G$ and $G_1' = z^{-1} G_1$. Then G' and G_1' have the desired forms (iii) in place of G and G_1 respectively.

The converse assertion is not difficult to prove.

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