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AN EMBEDDING THEOREM FOR ORDERED GROUPS

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Definition. An *O^*-group* is a group wherein every partial order can be extended to some full order.

THEOREM. *Suppose the group G has a normal chain $G = G_1 \supseteq G_2 \supseteq \ldots$ such that*

$$\bigcap_{i=1}^{\infty} G_i = E \text{ (identity subgroup)}$$

and each G/G_i is locally nilpotent and torsion-free. Then G can be embedded in the complete direct product G' of divisible O^*-groups.

Proof. Consider this map f of G into the complete direct product H of the G/G_i:

$$f(g) = (G_1 g, G_2 g, \ldots) \text{ for every } g \text{ in } G.$$

Clearly f is a homomorphism and is one-to-one since $\bigcap_{i=1}^{\infty} G_i = E$; thus $G \cong f(G)$. Since each G/G_i is torsion-free and locally nilpotent, it can be embedded in a (unique minimal) divisible locally nilpotent torsion-free group D_i, where D_i consists of all roots of elements of G/G_i (cf. [**5**, p. 256]). Each D_i, being torsion-free and locally nilpotent, is O^* by [**6**, p. 174], and the theorem follows by taking G' as the complete direct product of the D_i. (The theorem is immediate if any $G_i = E$, since G itself is then a locally nilpotent torsion-free group.)

Definition. By a *central series* for a group G with identity subgroup E we mean a countably infinite normal chain (i.e., chain of subgroups normal in G):

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$$

such that $\bigcap_{i=1}^{\infty} G_i = E$ and G_{i-1}/G_i is in the centre of G/G_i for every i.

Remark. Clearly the theorem holds for such G if also the factors G/G_i are torsion-free.

Now if F is any free group and $F = F_1 \supseteq F_2 \supseteq \ldots$ is the lower central chain for F, then $\bigcap_{i=1}^{\infty} F_i = E$ by Magnus' theorem (cf. [**5**, p. 38]). Also, each F/F_i (the free nilpotent group of class i) is torsion-free; for if $a \notin F_i$ but $a^n \in F_i$ for nonzero n, then F_{s-1}/F_s is not torsion-free if $s \leqq i$ is chosen so that $a \in F_{s-1} - F_s$; however, Witt's theorem (cf. [**5**, p. 41]) states F_{s-1}/F_s

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is free abelian (and hence torsion-free; also, cf. [**7**]). Thus by the remark above, the theorem holds for F.

Necessary and sufficient conditions that a prescribed full order for a subgroup of a group extend to some full order for the group are unknown. However, if $G \subseteq G'$ are as in the theorem, and if no G_i is the identity E, then at least $2^{\aleph}$ full orders for G extend to full orders for G'. For each D_i ($i \geqq 2$) admits at least two full orders, and the resulting uncountably many lexicographic full orders on G' are distinct on $H \subseteq G'$ since each D_i consists only of roots of elements of G/G_i. Hence, if no G_i is E, these orders are distinct on $G \cong f(G) \subseteq H$ by definition of f. (Of course, these remarks hold for free groups of rank $\geqq 2$, since no F_i is E.)

Definitions. A partial order P for a group G is called *isolated* if, for all x in G and all $n > 0$, $x^n \geqq e$ implies $x \geqq e$, where e is the group identity. A group in which every partial order extends to an isolated partial order will be called an *I^*-group*, following [**2**]. Likewise a group in which every right partial order extends to an isolated right partial order (obvious meaning) will be called RI^*, and one in which every right partial order extends to a right full order will be called RO^*.

In [**2**, p. 468] Hollister shows that a free group of rank $\geqq 2$ is not I^*. Also, an RI^*-group is I^* (for if a partial order P is contained in the isolated right partial order Q, then $\bigcap_{x \in G} x^{-1}Qx$ is an isolated partial order containing P), and so a nonabelian free group is not RI^*; cf. [**1**, pp. 69–70]. These results and the isomorphism f applied to F (notation as above) yield two corollaries.

COROLLARY 1. *A subdirect product of O^* (hence I^*) groups need not be I^* (hence not RI^* nor RO^*).*

Finally, any right partial order for F is a right partial order for $H \supseteq F$, where $H \equiv \prod F/F_i$ (complete direct product as in the proof of the theorem); thus H is not RI^* or RO^*. Also, in [**3**] Kargapolov showed that H is not O^*. Thus we have:

COROLLARY 2. *The complete direct product of O^*-groups need be neither RI^* (hence not RO^*) nor O^*.*

Note. It is unknown whether the classes I^*, RO^*, RI^* are closed under direct products. (In [**3**] and [**4**] the writers show that the restricted direct product of O^*-groups is O^*, whereas in [**8**] it is shown that the direct product of V^* (respectively V) groups is not V; a V^* (respectively V) group is one wherein every partial (respectively full) order on every subgroup extends to some full order for the group.)

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