PROJECTIVE SYSTEMS ON TREES AND VALUATION THEORY

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Introduction. It is our aim in this note to introduce methods from homological algebra in the study of some problems in valuation theory. In particular, we will use such methods to give a new, and, in some respect, simpler proof of a well-known theorem of Krull and Ribenboim; see (**2**). We shall also show that the same methods can be used to prove the Riemann-Roch theorem for algebraic curves and the Weierstrass product theorem.

In § 1 we study the functor $\varprojlim$ on the category of projective systems of modules on an ordered set V. If V is a tree, we show, (1.2), that

$$\varprojlim_V{}^{(p)} = 0 \quad \text{for } p \geqq 2$$

and we give an explicit formula for

$$\varprojlim_V{}^{(1)}.$$

If V is either a finite tree or the ordered set of the integers, we give conditions on the projective system F such that we have $\varprojlim^{(1)} F = \mathbf{0}$; see (1.4) and (1.8). In § 2 we specialize to the case where V is the ordered set of valuations of a field. It is known that V is a tree, and we may therefore use the results of § 1. Using (1.4), respectively (1.2), the Krull-Ribenboim approximation theorem and a weak form of the Riemann-Roch theorem for algebraic curves come out. The last section contains a proof of a "global" approximation theorem. As an example, we show that this generalizes the existence part of the Weierstrass product theorem.

1. Let L be an unitary ring and let V be an ordered set. If M is a subset of V and v an element of V, we put

$$\bar{M} = \{v' \in V | \, v' < v \in M\},$$
$$\bar{v} = \overline{\{v\}},$$
$$V_v = \{v' \in V | \, v' > v\}.$$

Let $\underline{c}$ be the abelian category of all projective systems of L-modules on V. An object F of $\underline{c}$ is then a family of L-modules $\{F_v\}_{v \in V}$, together with a family of homomorphisms $j_v{}^{v'}: F_{v'} \to F_v$, $v' > v$ such that, for $v'' > v' > v$,

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$$j_v^{v''} = j_v^{v'} \circ j_{v'}^{v''}.$$

For the definition and the main properties of the projective limit functor:

$$\varprojlim : \underline{c} \to \text{category of } L\text{-modules},$$

see (**1**). We denote by $\varprojlim^{(p)}$ the pth right derived functor of $\varprojlim$. By (**1**) we have

$$\operatorname{Ext}_{\underline{c}}^{(p)}(I, F) \simeq \operatorname{Hom}(L, \varprojlim^{(p)} F) \simeq \varprojlim^{(p)} F,$$

where I denotes the constant projective system on V associated with the L-module L. If for each $v \in V$ we are given an L-module $\bar{F}_v$, then we may construct a projective system F on V by defining

$$F_v = \coprod_{v' \in V_v} \bar{F}_{v'}.$$

If $v_1 > v_2$, then the homomorphism $j_{v_2}^{v_1}: F_{v_1} \to F_{v_2}$ is induced by the inclusion $V_{v_1} \subseteq V_{v_2}$. We shall call such projective systems elementary.

We easily prove that if all $\bar{F}_v$ are projective L-modules, then F is a projective object in c.

Definition 1.1. An ordered set V is called a tree if, for every $v \in V$,

(1) $\bar{v}$ is totally ordered,

(2) there exists a subset R_v of V such that

(a) if $v' \in R_v$, then $v' > v$ and $v' \neq v$,

(b) if $v'' > v$, $v'' \neq v$, then there exist a unique $v' \in R_v$ such that $v'' > v'$.

PROPOSITION 1.2. *Let V be a tree and suppose that for every $\bar{v} \in V$, $\bar{v}$ is finite, then*

(i) $\varprojlim_{V}^{(p)} = 0$ *for* $p \geqq 2$,

(ii) $\varprojlim_{V}^{(1)} F = \operatorname{coker} \phi$,

where

$$\phi: \prod_{v \in V} F_v \to \prod_{\substack{v \in V; \\ v' \in R_v}} F_{\min(v, v')}$$

is given by

$$\phi(\{f_v\})_{(v,v')} = fv - j_v^{v'} f_{v'}.$$

Proof. For every $v \in V$ let

$$\bar{p}^0{}_v = L \quad \text{and} \quad \bar{p}^1{}_v = \coprod_{v' \in R_v} L.$$

Denote by p^0 and p^1 the elementary objects of $\underline{c}$ generated by the families $\{\bar{p}^0{}_v\}_{v \in V}$ and $\{\bar{p}^1{}_v\}_{v \in V}$, respectively. Let $\epsilon: p^0 \to I$ be the morphism induced by the family of identity homomorphisms

$$\bar{p}^0{}_v \to I_v.$$

Now, as for every $v \in V$, V_v is the disjoint union

$$\bigcup_{v' \in V_v} R_{v'} \cup \{v\},$$

we have

$$p^1{}_v = \coprod_{v' \in V_v - \{v\}} L, \qquad p^0{}_v = \coprod_{v' \in V_v} L.$$

If $\{e_{v'}\}_{v' \in V_v}$ is a base for $p^0{}_v$, then $\{e_{v'}\}_{v' \in R_v}$ is a base for $\bar{p}^1{}_v$. Let $d: p^1 \to p^0$ be the morphism induced by the family of homomorphisms

$$i_v: \bar{p}^1{}_v \to p^0{}_v$$

given by

$$i_v\Big(\sum_{v' \in R_v} l_{v'} e_{v'}\Big) = \sum_{v' \in R_v} l_{v'}(e_{v'} - e_v).$$

Obviously, $\epsilon \circ d = 0$ so that d defines a morphism $d^*: p^1 \to \ker \epsilon$. We shall show that d^* is an isomorphism. Let $x \in p^1{}_v$ and suppose that

$$x = \sum_{v' \in V_v - \{v\}} l_{v'} e_{v'} \neq 0.$$

If v'_0 is maximal among those v' for which $l_{v'} \neq 0$, then we may write

$$d(x) = l_{v'_0} e_{v'_0} + \sum_{v'' \neq v'_0} l''_{v''} e_{v''}$$

so that $d(x) \neq 0$. Therefore, d^* is monomorphic. Let $y \in \ker \epsilon_v$, then

$$y = \sum_{v' \in V_v} l_{v'} e_{v'} \quad \text{with} \quad \sum_{v' \in V_v} l_{v'} = 0.$$

For every $v' \in V_v$ we know, since $\bar{v}'$ is finite, that there exists a finite maximal sequence

$$v = v_0 \lneqq v_1 \lneqq \ldots \lneqq v_n = v'$$

such that $v_{i+1} \in R_{v_i}$ for $i = 0, 1, \ldots, n-1$. Then

$$e_{v'} - e_v = \sum_{i=0}^{n-1} (e_{v_{i+1}} - e_{v_i})$$

and

$$y = \sum_{v' \in V_v} l_{v'} e_{v'} = \sum_{v' \in V_v} l_{v'}(e_{v'} - e_v) = \sum_{\substack{v'' \in R_{v'};\\ v' \in V_v}} l_{v'', v'}(e_{v''} - e_{v'})$$

so that $y \in \operatorname{im} d^*$. Therefore d^* is epimorphic, and we then know that

$$0 \to p^1 \xrightarrow{d} p^0 \xrightarrow{\epsilon} I \to 0$$

is an exact sequence of objects in $\underline{c}$. As p^0 and p^1 are projectives, we may calculate $\varprojlim^{(p)}$ by using the complex $\operatorname{Hom}_{\underline{c}}(p\cdot, -)$. In particular, we find:

$$\varprojlim_{V}{}^{(p)} = 0$$

for $p \geqq 2$, and

$$\varprojlim_V{}^{(1)} F = \operatorname{coker}\left\{\operatorname{Hom}_{\underline{c}}(p^0, F) \xrightarrow{\operatorname{Hom}(d, \mathrm{id}_F)} \operatorname{Hom}(p^1, F)\right\}.$$

Now

$$\operatorname{Hom}_{\underline{c}}(p^0, F) \simeq \prod_{v \in V} F_v \quad \text{and} \quad \operatorname{Hom}_{\underline{c}}(p^1, F) \simeq \prod_{\substack{v' \in Rv;\\ v \in V}} F \min(v, v'),$$

and $\phi = \operatorname{Hom}(d, id_F)$ is given by $\phi(\{f_v\}_{v \in V})_{(v,v')} = f_v - j^{v'}{}_v f_{v'}$.

Suppose M is a subset of the ordered set V, and suppose F is a projective system on V, then there is a canonical homomorphism

$$F(V, M)\colon \varprojlim_V F \to \varprojlim_M F.$$

We shall use the following lemma.

LEMMA 1.3. *Let M_1, M_2, and N be subsets of the ordered set V such that $\bar{M}_i = M_i$ for $i = 1, 2$, $V = M_1 \cup M_2$ and $N = M_1 \cap M_2$. Then we have an exact sequence*

$$0 \to \varprojlim_N F/\operatorname{im} F(M_1, N) + \operatorname{im} F(M_2, N) \to \varprojlim_V{}^{(1)} F \to \varprojlim_{M_1}{}^{(1)} F \underset{\varprojlim_N^{(1)} F}{\times} \varprojlim_{M_2}{}^{(1)} F \to 0$$

Proof. Let $W = \{0, a, b\}$ be the ordered set with the only non-trivial relations $0 < a$, $0 < b$. Let

$$\kappa\colon W \to PV$$

be the κ-functor given by $\kappa(a) = M_1$, $\kappa(b) = M_2$ and $\kappa(0) = N$. If G is a projective system on W, we find

$$\varprojlim_W G = G_a \underset{G_0}{\times} G_b, \qquad \varprojlim_W{}^{(1)} G = G_0/\operatorname{im}\alpha + \operatorname{im}\beta,$$

where $\alpha\colon G_a \to G_0$ and $\beta\colon G_b \to G_0$ are the obvious homomorphisms. The lemma now follows from (1.3.1) of (**1**).

LEMMA 1.4. *Suppose V is a finite tree, then the following statements are equivalent.*

(i) $\varprojlim_V{}^{(1)} F = 0$.

(ii) *If $v \in V$ and $Rv = \{v_1, \ldots, v_r\}$, put*

$$M_j^{\,1} = \bigcup_{i=1}^{j} \bar{V}_{vi}, \qquad M_j^{\,2} = \bar{V}_{vj+1},$$

then, for every $j = 1, \ldots, r - 1$, we have

$$F_v = \text{im } F(M_j^1, \bar{v}) + \text{im } F(M_j^2, \bar{v}).$$

Proof. Define the function h: $V \to Z^+$ by $h(v) = \max\{s|$ there exists in V a sequence $v_s \lneqq v_{s-1} \lneqq \ldots \lneqq v_0 = v\}$. Suppose

$$\varprojlim_V{}^{(1)} F = 0.$$

By induction on $h(v)$ we shall prove that for every $v \in V$,

$$\varprojlim_{V_v}{}^{(1)} F = 0.$$

If $h(v) = 0$, then V_v is a connected component of V, and therefore

$$\varprojlim_{V_v}{}^{(1)} F = \varprojlim_V{}^{(1)} F = 0.$$

Suppose now that

$$\varprojlim_{V_v}{}^{(1)} F = 0$$

for all v with $h(v) < n$, and let v_r be such that $h(v_r) = n$. Since we may suppose $h(v_r) \geqq 1$, there exists a unique v such that $R_v = \{v_1, \ldots, v_r\}$. Then

$$M_{r-1}^2 = \bar{V}_{v_r}, \quad \bar{V}_v = M_{r-1}^1 \cup M_{r-1}^2 \quad \text{and} \quad N = M_{r-1}^1 \cap M_{r-1}^2 = \bar{v}.$$

As

$$\varprojlim_{\bar{v}}{}^{(1)} F = 0,$$

we have

$$\frac{\varprojlim_{M_{r-1}^1}{}^{(1)} F \times \varprojlim_{M_{r-1}^2}{}^{(1)} F}{\varprojlim_N{}^{(1)}} \simeq \varprojlim_{M_{r-1}^1}{}^{(1)} F \times \varprojlim_{M_{r-1}^2}{}^{(1)} F$$

By the induction hypothesis, we have

$$\varprojlim_{V_v}{}^{(1)} F = 0,$$

thus Lemma 1.3 implies

$$\varprojlim_{V_{v_r}}{}^{(1)} F = \varprojlim_{M_{r-1}^2}{}^{(1)} F = 0, \quad \text{and} \quad \varprojlim_{M_{r-1}^1}{}^{(1)} F = 0.$$

Therefore

$$\varprojlim_{V_v}{}^{(1)} F = 0 \quad \text{for every } v \in V.$$

Now, since

$$M_{j+1}^1 = M_j^1 \cup M_j^2 \quad \text{and} \quad M_j^1 \cap M_j^2 = \bar{v}_0 \quad \text{for every } j = 1, \ldots, r-1,$$

we may use the same method to prove that for every $v \in V$ and every $j = 1, \ldots, r - 1$, we have that

$$\varprojlim_{M^1_{j+1}}{}^{(1)} F = 0.$$

But (ii) is then an immediate consequence of Lemma 1.3. Reversing everything, we prove that (ii) implies (i).

COROLLARY 1.5. *Suppose that V is a finite tree and that*

$$\varprojlim_{V}{}^{(1)} F = 0.$$

If M is a subset of V, then

$$\varprojlim_{M}{}^{(1)} F = 0.$$

Proof. If V satisfies condition (ii) of Lemma 1.4, then so will M.

Suppose V is the ordered set of the positive integers Z^+. Then, given a projective system F, we define the completion $\hat{F}$ of F by

$$\hat{F}_n = \varprojlim_{n' > n} F_n/\mathrm{im}\, j_n^{n'}.$$

There is an obvious morphism

$$g \colon F \to \hat{F}.$$

Now we have the following result.

THEOREM 1.6. *If*

$$\varprojlim_{Z^+}{}^{(1)} F = 0,$$

then g is epimorphic. If F is monomorphic and g is epimorphic, then

$$\varprojlim_{Z^+}{}^{(1)} F = 0.$$

Proof. If

$$\varprojlim_{Z^+}{}^{(1)} F = 0,$$

then, using the fact that

$$\varprojlim_{Z^+}{}^{(p)} = 0 \quad \text{for } p \geqq 2$$

and applying $\varprojlim_{Z^+}$ to the exact sequence

$$0 \to \mathrm{im}\, j_n^{n'} \to F_n \to F_n/\mathrm{im}\, j_n^{n'} \to 0,$$

we easily find that

$$g_n \colon F_n \to \varprojlim_{n' > n} F_n/\mathrm{im}\, j_n^{n'}$$

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is onto, so that g is epimorphic. Now, if F is monomorphic, then

$$(\ker g)_n = \bigcap_{n'>n} \operatorname{im} j_n^{n'}$$

so that $\ker g$ is constant.

If, in addition, g is epimorphic, we have an exact sequence

$$0 \longrightarrow \ker g \longrightarrow F \longrightarrow \hat{F} \longrightarrow 0.$$

As $\ker g$ is constant,

$$\varprojlim_{Z^+}{}^{(1)} \ker g = 0$$

so that

$$\varprojlim_{Z^+}{}^{(1)} F = \varprojlim_{Z^+}{}^{(1)} \hat{F}.$$

But we always have

$$\varprojlim_{Z^+}{}^{(p)} \hat{F} = 0 \quad \text{for } p \geqq 0.$$

In fact, consider the projective system H on $Z^+ \times Z^+$ defined by

$$H_{m,n} = F_{\min(m,n)}/\operatorname{im} j_{\min(m,n)}^{\max(m,n)}.$$

We have $H_{m,m} = 0$ for $m \in Z^+$ and therefore

$$\varprojlim_{Z^+ \times Z^+}{}^{(p)} H = 0 \quad \text{for } p \geqq 0.$$

However, by (1.3.2) of (**1**), there exists a spectral sequence converging to

$$\varprojlim_{Z^+ \times Z^+}{}^{(\cdot)} H \quad \text{with} \quad E_{p,q}^2 = \varprojlim_{m \in Z^+}{}^{(p)} \varprojlim_{n \in Z^+}{}^{(q)} H_{m,n}.$$

It follows that $E_{p,q}^2 = 0$ for all $p, q \geqq 0$ and, in particular,

$$\varprojlim_{Z^+}{}^{(p)} \hat{F} = E_{p,0}^2 = 0 \quad \text{for } p \geqq 0.$$

THEOREM 1.7. *Let F be a projective system of topological abelian groups and continuous homomorphisms on the ordered set Z^+. Suppose that*

(i) *for all $n \in Z^+$, F_n is complete metrizable,*

(ii) *for all $n' > n$, $\operatorname{im} j_n^{n'}$ is dense in F_n,*

then we may conclude that $g\colon F \longrightarrow \hat{F}$ is onto.

Proof. We fix an $n \in Z^+$ and we shall prove that $g_n\colon F_n \longrightarrow \hat{F}_n$ is onto. Let

$$u_m\colon F_n \longrightarrow F_n/\operatorname{im} j_n^{m+n}$$

be the canonical homomorphism, and fix an element $\hat{f} = \{\hat{f}_m\} \in \hat{F}_n$. For each $m \in Z^+$, let $f_m \in F_n$ be such that $u_m(f_m) = \hat{f}_m$. We then have:

$$f_{m+1} - f_m \in \operatorname{im} j_n^{m+n} \quad \text{for all } m \in Z^+.$$

Let δ_m be the metric on F_{n+m}, and put $f_0 = 0$. Since $\operatorname{im} j_n^{n+1}$ is dense in F_n, we may find an element $g_{10} \in F_{n+1}$ such that if $g_{01} = j_n^{n+1}(g_{10})$, then

$$\delta_0(f_1 - f_0 + g_{01}, 0) < \frac{1}{2^0} = 1.$$

Put $\bar{f}_1 = f_1 + g_{01}$; then we have:

$$u_1(\bar{f}_1) = u_1(f_1) = f_1, \qquad f_2 - \bar{f}_1 \in \operatorname{im} j_n^{n+1}.$$

Thus we may find an element $h_{10} \in F_{n+1}$ such that $h_{01} = j_n^{n+1}(h_{10}) = f_2 - \bar{f}_1$. Since all $j_n^{n'}$ are continuous, we may, using (ii), find an element $g_{20} \in F_{n+2}$ such that if

$$g_{11} = j_{n+1}^{n+2}(g_{20}), \qquad g_{02} = j_n^{n+2}(g_{20}),$$

then

$$\delta_0(f_2 - \bar{f}_1 + g_{02}, 0) < \tfrac{1}{2},$$

$$\delta_1(h_{10} + g_{11}, 0) < \frac{1}{2^0} = 1.$$

Put $\bar{f}_2 = f_2 + g_{02}$, then

$$u_2(\bar{f}_2) = u_2(f_2) = \hat{f}_2, \qquad f_3 - \bar{f}_2 \in \operatorname{im} j_n^{n+2}.$$

Continuing this process, we construct elements $\bar{f}_m \in F_n$, $h_{ij} \in F_{n+i}$, $i + j = m$, $g_{rs} \in F_{n+r}$, $r + s = m + 1$, such that:

$$\delta_r(h_{r,s-1} + g_{r,s}, 0) < \tfrac{1}{2}r \quad \text{for} \quad r + s = m,$$

$$j_{n+i-1}^{n+i}(h_{ij}) = h_{i-1,j+1}, \qquad j_n^{n+m}(h_{m,0}) = f_{m+1} - \bar{f}_m,$$

$$j_{n+r-1}^{n+r}(g_{r,s}) = g_{r-1,s+1}, \quad u_m(\bar{f}_m) = u_m(f_m) = \hat{f}_m, \quad f_{m+1} - \bar{f}_m \in \operatorname{im} j_n^{n+m}.$$

By construction,

$$\bar{f} = \sum_{i=0}^{\infty} (\bar{f}_{i+1} - \bar{f}_i) = \sum_{i=0}^{\infty} (h_{0,i} + g_{0,i+1})$$

exists. Further, by construction,

$$h_j = \sum_{i=0}^{\infty} (h_{ji} + g_{j,i+1})$$

exists, $h_j \in F_{n+j}$, and we have

$$\bar{f} - j_n^{n+j}(h_j) = \sum_{i=0}^{j-1} \bar{f}_{i+1} - \bar{f}_i = \bar{f}_j,$$

therefore,we have

$$u_j(\bar{f}) = u_j(\bar{f}_j) = \hat{f}_j \quad \text{for all } j \in Z^+.$$

This means that $g_n(\bar{f}) = \hat{f}$ and the proof is complete.

COROLLARY 1.8. *Under the hypotheses of Theorem 1.7, if F is monomorphic, then*

$$\varprojlim_{Z}{}^{(1)} F = 0.$$

2. Let K be a field and let V be the ordered set of all non-archimedean valuations of K.

If $v \in V$, we denote by m_v the maximal ideal of the valuation ring $\mathcal{O}_v$, and by Γ_v the linearly ordered value group of v.

For every $v \in V$, we then have an exact sequence of abelian groups

$$(1) \qquad \{1\} \to U_v \to K^* \xrightarrow{v} \Gamma_v \to 0,$$

where U_v is the multiplicative group of units in $\mathcal{O}_v$.

Obviously, the families $\{\mathcal{O}_v\}_{v \in V}$, $\{U_v\}_{v \in V}$, and $\{\Gamma_v\}_{v \in V}$ define projective systems of abelian groups on V. We shall denote these by $\mathcal{O}_K$, U_K, and Γ_K, respectively. Then we have an exact sequence of projective systems of abelian groups:

$$(2) \qquad \{1\} \to U_K \to K^* \to \Gamma_K \to 0.$$

Suppose that the subset N of V has a least element, then we have

$$\varprojlim_N{}^{(1)} K^* = 0.$$

Applying the functor $\varprojlim_N$ to the exact sequence (2) we then get the exact sequence

$$(3) \qquad \{1\} \to \varprojlim_N U_K \to K^* \xrightarrow{v(N)} \varprojlim_N \Gamma_K \to \varprojlim_N{}^{(1)} U_K \to 0.$$

Definition 2.1. Let N be a subset of V, then we shall call N an A-set (approximation set) if for every

$$\gamma \in \varprojlim_N \Gamma_K$$

there exists an element $x \in K^*$ such that $v(N)(x) = \gamma$.

LEMMA 2.2. *Suppose N contains a least element, then N is an* A-*set if and only if*

$$\varprojlim_N{}^{(1)} U_K = 0.$$

Proof. This follows immediately from the definition and from the exact sequence (3).

Let $V_0 = \bar{V}_0$ be a subset of V, then V_0 has an induced order. If N is a subset of V_0, let $D(N, V_0)$ denote the subset of V_0 consisting of all v' such that $\bar{v}' \cap \bar{N} = \{*\}$, where $*$ is the trivial valuation.

Definition 2.3. Let $N \subseteq V_0$ be subsets of V. We shall say that N is a GA-set (global approximation set) with respect to V_0, if for every

$$\gamma \in \varprojlim_{V_0/D(N, V_0)} \Gamma_K$$

there exists an element $x \in K^*$ such that $v(N)(x) = \gamma$.

LEMMA 2.4. *Let $N \subseteq V_0$, then N is a* GA-*set with respect to V_0 if and only if the canonical homomorphism*

$$\varprojlim_{V_0}{}^{(1)} U_K \to \varprojlim_{D(N, V_0)}{}^{(1)} U_K$$

is monomorphic.

Proof. We may assume that $D(N, V_0) \neq V_0$. Applying the functors

$$\varprojlim_{V_0} \quad \text{and} \quad \varprojlim_{V_0/D(N, V_0)}$$

to the exact sequence (2) we get a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} \{1\} \to \varprojlim_{V_0} U_K \to K^* \to & \varprojlim_{V_0} \Gamma_K & \xrightarrow{\quad j \quad} & \varprojlim_{V_0}{}^{(1)} U_K \to 0 \\ & \uparrow i & & \uparrow t \\ \{1\} \to & \varprojlim_{V_0/D(N, V_0)} \Gamma_K & \xrightarrow{s} & \varprojlim_{V/D(N, V_0)}{}^{(1)} U_K \to 0. \end{array}$$

Now N is a GA-set with respect to V_0 if and only if $t \circ s = j \circ i = 0$. s being an isomorphism, this is equivalent to $t = 0$. From the exact sequence

$$\ldots \varprojlim_{D(N, V_0)} U_K \to \varprojlim_{V_0/D(N, V_0)}{}^{(1)} U_K \xrightarrow{t} \varprojlim_{V_0}{}^{(1)} U_K \to \varprojlim_{D(N, V_0)}{}^{(1)} U_K \to \ldots,$$

it follows that $t = 0$ if and only if

$$\varprojlim_{V_0}{}^{(1)} U_K \to \varprojlim_{D(N, V_0)} U_K$$

is monomorphic.

LEMMA 2.5. *Suppose that N is a finite subset of V, containing a least element v_0. Consider the field $k = \mathcal{O}_{v_0}/m_{v_0}$ and let $M = \{v/v_0 | v \in N\}$ be the set of valuations on k associated with N. Then*

$$\varprojlim_{N}{}^{(1)} U_K \simeq \varprojlim_{M}{}^{(1)} U_k.$$

Proof. Let $U_0 = \{x | x \in K, 1 - x \in m_{v_0}\}$, then $U_0 \subseteq U_v$ for all $v \in N$ and $U_{k,(v/v_0)} \simeq U_{K,v}/U_0$. The family of exact sequences

$$\{1\} \to U_0 \to U_{K,v} \to U_{k,(v/v_0)} \to \{1\}$$

defines an exact sequence of projective systems on the ordered set $N \simeq M$.

Since

$$\varprojlim_{N}{}^{(p)} U_0 = 0 \quad \text{for } p \geqq 1,$$

it follows that

$$\varprojlim_{N}{}^{(1)} U_K \simeq \varprojlim_{M}{}^{(1)} U_k.$$

LEMMA 2.6. *Suppose that N is a finite subset of V, and let $v \in N$. If $R_v = \{v_1, \ldots, v_r\}$ and*

$$M_j^1 = \bigcup_{i=1}^{j} \bar{N}_{v_i}, \qquad M_j^2 = \bar{N}_{v_{j+1}},$$

then for every $j = 1, \ldots, r-1$,

$$U_{K,v} = \bigcap_{v' \in M_j^1} U_{K,v'} \cdot \bigcap_{v' \in M_j^2} U_{K,v'}.$$

Proof. We must prove that for every $x \in U_{K,v}$ there exist

$$x_1 \in \bigcap_{v' \in M_j^1} U_{K,v'} \quad \text{and} \quad x_2 \in \bigcap_{v' \in M_j^2} U_{K,v'}$$

such that $x = x_1 \cdot x_2$. As N is finite, the function $h: N \to Z^+$ has a maximum n_0 (see the proof of Lemma 1.4). If $h(v) = n_0$, then $R_v = \emptyset$ and there is nothing to prove. Suppose that the lemma has been proved for all $v \in N$ such that $h(v) \geqq m + 1$ and let $v \in N$ be such that $h(v) = m$. By Lemma 1.4 we know that the conclusion of the lemma is equivalent to

$$\varprojlim_{\bar{N}_v}{}^{(1)} U_K = 0.$$

By Lemma 2.5 we may therefore suppose that v is the trivial valuation. Let u_s, $s = 1, \ldots, k$, be the maximal elements of M_j^2 and let w_t, $t = 1, \ldots, l$, be the maximal elements of M_j^1. For each $i = 1, \ldots, j$, let v_i' be an element of V such that $v < v_i' < v_i$ and such that v_i' is of rank 1. By (**4**, Lemma 1, Chapter VI, § 7) we may find elements y_s, $s = 1, \ldots, k$, and z_t, $t = 1, \ldots, l$, in K such that

$$u_s(y_s) = 0, \quad u_s(y_{s'}) > 0 \qquad \text{for } s \neq s'$$

and

$$v_i'(y_s) > 0 \quad \text{for } i = 1, \ldots, j, \ s = 1, \ldots, k;$$

$$w_t(z_t) = 0, \quad w_t(z_{t'}) > 0 \qquad \text{for } t \neq t',$$

and

$$u_s(z_t) > 0 \quad \text{for } s = 1, \ldots, k, \ t = 1, \ldots, l.$$

We may suppose that $v_{j+1}(x) \leqq 0$. If this is not the case, we might consider x^{-1}. Since v_i', $i = 1, \ldots, j$, are of rank 1 we may, by taking high enough powers of the y_s, assume that $v_i'(y_s) > -v_i'(x)$ for $i = 1, \ldots, j$. This implies that $w_t(y_s) > -w_t(x)$ for $t = 1, \ldots, l$. Since $v_{j+1}(x) \leqq 0$ we have $u_s(x) \leqq 0$ for $s = 1, \ldots, k$. Put

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$$x_1 = \sum_{t=1}^{l} z_t + \sum_{s=1}^{k} y_s \cdot x.$$

We then have $w_t(x_1) = 0$ for $t = 1, \ldots, l$ so that

$$x_1 \in \bigcap_{v' \in M_j^1} U_{K,v'},$$

and

$$u_s(x_1) = u_s(x) \quad \text{for } s = 1, \ldots, k.$$

Let $x_2 = x/x_1$, then the last relations imply that $u_s(x_2) = 0$ for $s = 1, \ldots, k$ so that

$$x_2 \in \bigcap_{v' \in M_j^2} U_{K,v'}.$$

It follows that the conclusion of the lemma is true for all $v \in N$.

From this lemma we easily deduce the following well-known theorem.

THEOREM 2.7 (Krull-Ribenboim). *If v_i, $i = 1, \ldots, r$, are valuations of a field K, and if for every $i = 1, \ldots, r$, γ_i is an element of Γ_{v_i} such that for each couple (i, j) the image of γ_i and γ_j in the value group of $v_i \wedge v_j$ coincides, then there exists an element $x \in K^*$ such that $v_i(x) = \gamma_i$ for all $i = 1, \ldots, r$.*

Proof. Let N be a finite subset of V containing all v_i, $i = 1, \ldots, r$, and being closed under the operation $\wedge$. The conclusion of the theorem is by Definition 2.1 and Lemma 2.2 equivalent to

$$\varprojlim_{N}{}^{(1)} U_K = 0.$$

But this follows from Lemmas 1.4 and 2.6.

We now let V' be a subset of V consisting of discrete valuations of rank 1. Let $\underline{D}(V')$ denote the free group generated by V'. If

$$D = \sum_{v \in V'} n_v v,$$

we put

$$d(D) = \sum_{v \in V'} n_v, \qquad v(D) = n_v.$$

Let $V^* = \bar{V}'$ and let $\underline{L}(D)$ be the projective system of abelian groups on V^* given by

$$\underline{L}(D)_v = \{x \in K \mid v(x) \geqq -v(D)\} \quad \text{if } v \in V',$$

$$\underline{L}(D)_* = K.$$

Note that $V^* = V' \cup \{*\}$, where $*$ is the trivial valuation. If D_1 and D_2 are elements of $\underline{D}(V')$, then, by definition, $D_1 \leqq D_2$ if for every $v \in V'$, $v(D_1) \leqq v(D_2)$. Suppose that $D_1 \leqq D_2$, then there is an exact sequence of projective systems on V^*

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$$0 \to \underline{L}(D_1) \to \underline{L}(D_2) \to \underline{P} \to 0,$$

where $\underline{P}$ is given by:

$$\underline{P}_v \simeq m_v^{-v(D_1)}/m_v^{-v(D_2)}.$$

If we put

$$L(D) = \varprojlim_{V^*} \underline{L}(D), \qquad I'(D) = \varprojlim_{V^*}{}^{(1)} \underline{L}(D),$$

then the above exact sequence induces an exact sequence

$$0 \to L(D_1) \to L(D_2) \to \prod_{v \in V'} m_v^{-v(D_1)}/m_v^{-v(D_2)} \xrightarrow{\partial} I'(D_1) \to I'(D_2) \to 0.$$

By Proposition 1.2 we have

$$I'(D) = \operatorname{coker}\left\{ \prod_{v \in V^*} \underline{L}(D)_v \to \prod_{\substack{v \in V^*; \\ v' \in R_v}} \underline{L}(D)_{\min(v,v')} \right\}$$

$$\simeq \prod_{v \in V'} K \Big/ \left\{ K + \prod_{v \in V'} \underline{L}(D)_v \right\}.$$

Let $I(D)$ denote the subgroup of $I'(D)$ consisting of those elements x with representatives

$$\{x_v\}_{v \in V'} \in \prod_{v \in V'} K$$

such that for all but a finite number of the v's, $x_v \in \mathcal{O}_v$. Then we easily find that $\operatorname{im}\partial \subseteq I(D_1)$ and that $I(D_1) \to I(D_2)$ is epimorphic. It follows that we have the exact sequence

$$0 \to L(D_1) \to L(D_2) \to \prod m_v^{-v(D_1)}/m_v^{-v(D_2)} \to I(D_1) \to I(D_2) \to 0.$$

Suppose that K contains a subfield k such that all valuations of V' are trivial on k. Suppose further that:

(i) $\dim_k L(0) < \infty$,

(ii) $\dim_k I(0) < \infty$,

(iii) for every $v \in V'$, $e_v = \dim_k(\mathcal{O}_v/m_v) < \infty$.

Then

$$\dim_k \prod_{v \in V'} m_v^{-v(D_1)}/m_v^{-v(D_2)} = \sum_{v \in V'} (v(D_2) - v(D_1))e_v$$

and for every $D \in \underline{D}(V')$

$$l(D) = \dim_k L(D) < \infty, \qquad i(D) = \dim_k I(D) < \infty,$$

and

$$l(D_2) - i(D_2) = l(D_1) - i(D_1) + \sum_{v \in V'} (v(D_2) - v(D_1))e_v.$$

From this, the "weak" form of the Riemann-Roch theorem for non-singular algebraic curves follows easily; see (**3**, Chapter 2).

THEOREM 2.8. *If K is an algebraic function field over the algebraically closed field k, then, if D is a divisor, we have that*

$$l(D) - i(D) = 1 - i(0) + d(D),$$

where $i(D) = \dim_k I(D)$, $I(D) \simeq R/K + R(D)$, *and* R *is the* k*-algebra of repartitions.*

3. Suppose we are given a sequence of fields

$$K_0 \supseteq K_1 \supseteq \ldots \supseteq K_i \supseteq \ldots \supseteq K = \bigcap_{i=1}^{m} K_i.$$

Let V_i and V be the ordered set of all non-archimedean valuations of K_i, respectively K. Then there is a sequence of epimorphisms of ordered sets:

$$V_0 \xrightarrow{s_0} V_1 \xrightarrow{s_1} \ldots \to V_i \xrightarrow{s_i} \ldots \to V.$$

Let $V^0{}_i$ be a subset of V such that $\bar{V}^0{}_i = V^0{}_i$, and such that $s_i(V^0{}_i) \subseteq V^0_{i+1}$. We put

$$V^0 = \bigcup_{i=1}^{\infty} \operatorname{im}(V^0{}_i \to V).$$

Denote by t_i the map $V^0{}_i \to V^0$ and let $\kappa_i\colon V^0 \to PV^0{}_i$ be the κ-functor defined by $\kappa_i(v) = \{v_i \in V^0{}_i | \, t_i(v') \leqq v\}$. Then there are natural homomorphisms:

$$\varprojlim_{\kappa_{i+1}(v)} U_{K_{i+1}} \to \varprojlim_{\kappa_i(v)} U_{K_i},$$

$$\varprojlim_{V_{i+1}} U_{K_{i+1}} \to \varprojlim_{V_i} U_{K_i}.$$

If N is a subset of V and $N_i = t_i^{-1}(N)$, then there are also natural homomorphisms

$$\varprojlim_{D(N_{i+1}, V_{i+1})} U_{K_{i+1}} \to \varprojlim_{D(N_i, V_i)} U_{K_i}.$$

THEOREM 3.1. *If for every* $i \in Z^+$, N_i *is a* GA-*set with respect to* $V^0{}_i$, *then* N *is a* GA-*set with respect to* V^0 *if the natural homomorphism*

$$\varprojlim_{Z^+}{}^{(1)} \varprojlim_{V^0{}_i} U_{K_i} \to \varprojlim_{Z^+}{}^{(1)} \varprojlim_{D(N_i, V^0{}_i)} U_{K_i}$$

is monomorphic.

Proof. By Lemma 2.4 we know that, for every $i \in Z^+$,

$$j_i\colon \varprojlim_{V^0{}_i}{}^{(1)} U_{K_i} \to \varprojlim_{D(N_i, V^0{}_i)}{}^{(1)} U_{K_i}$$

is monomorphic, and we have to prove that

$$t\colon \varprojlim_{V^0}{}^{(1)} U_K \to \varprojlim_{D(N, V^0)}{}^{(1)} U_K$$

is monomorphic.

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By (1.3.1) of (**1**) we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 \to & \varprojlim^{(1)}_{V^0} \varprojlim_{\kappa_i} U_{K_i} & \to & \varprojlim^{(1)}_{V^0_i} U_{K_i} & \to & \varprojlim_{V^0} \varprojlim^{(1)}_{\kappa_i} U_{K_i} & \to 0 \\
& \downarrow{\scriptstyle l_i} & & \downarrow{\scriptstyle j_i} & & \downarrow & \\
0 \to & \varprojlim_{D(N,V^0)}{}^{(1)} \varprojlim_{\kappa_i} U_{K_i} & \to & \varprojlim_{D(N_i,V^0_i)}{}^{(1)} U_{K_i} & \to & \varprojlim_{D(N,V^0)} \varprojlim_{\kappa_i} U_{K_i} & \to 0.
\end{array}$$

As j_i is monomorphic, l_i is monomorphic, and therefore, so is the homomorphism

$$\iota\colon \varprojlim_{Z^+} \varprojlim_{V^0}{}^{(1)} \varprojlim_{\kappa_i} U_{K_i} \to \varprojlim_{Z^+} \varprojlim_{D(N,V^0)}{}^{(1)} \varprojlim_{i} U_{K_i}.$$

Using the same spectral-sequence argument as above (**1**, (1.3.2)), we find abelian groups G and H and a homomorphism $\phi\colon G \to H$ such that the following diagrams are commutative.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\varprojlim^{(1)}_{Z^+} \varprojlim_{V^0} \varprojlim_{\kappa_i} U_{K_i} & \xrightarrow{s} & \varprojlim^{(1)}_{Z^+} \varprojlim_{D(N,V^0)} \varprojlim_{\kappa_i} U_{K_i} \\
\downarrow & & \downarrow \\
G & \xrightarrow{\phi} & H \\
\downarrow & & \downarrow \\
\varprojlim_{Z^+} \varprojlim^{(1)}_{V^0} \varprojlim_{\kappa_i} U_{K_i} & \xrightarrow{l} & \varprojlim_{Z^+} \varprojlim_{D(N,V^0)}{}^{(1)} \varprojlim_{\kappa_i} U_{K_i}, \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\varprojlim^{(1)}_{V^0} \cdot \varprojlim_{Z^+} \varprojlim_{\kappa_i} U_{K_i} & \xrightarrow{t} & \varprojlim_{D(N,V^0)}{}^{(1)} \varprojlim_{Z^+} \varprojlim_{\kappa_i} U_{K_i} \\
\downarrow & & \downarrow \\
G & \xrightarrow{\phi} & H \\
\downarrow & & \downarrow \\
\varprojlim_{V^0} \varprojlim^{(1)}_{Z^+} \varprojlim_{\kappa_i} U_{K_i} & \to & \varprojlim_{D(N,V^0)} \varprojlim^{(1)}_{Z^+} \varprojlim_{\kappa_i} U_{K_i}. \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Now we can easily find:

$$\varprojlim_{V^0} \varprojlim_{K_i} U_{K_i} \simeq \varprojlim_{V^0_i} U_{K_i},$$

$$\varprojlim_{D(N,\, V^0)} \varprojlim_{K_i} U_{K_i} \simeq \varprojlim_{D(N_i,\, V^0_i)} U_{K_i},$$

$$\varprojlim_{Z^+} \varprojlim_{K_i(v)} U_{K_i} \simeq U_{K,v} \quad \text{for} \quad v \in V^0.$$

As l is monomorphic and as, by assumption, s is monomorphic, the first diagram shows that ϕ is monomorphic. Therefore, the second diagram shows that t is monomorphic.

COROLLARY 3.2. *If for every $i \in Z^+$, N_i is a* GA-*set with respect to $V^0{}_i$, then N is a* GA-*set with respect to V^0 if*

$$\varprojlim_{Z^+}{}^{(1)} \varprojlim_{V^0_i} U_{K_i} = 0.$$

As an example, we prove the product theorem of Weierstrass. Let K be the field of meromorphic functions on an open and connected subset D of the complex plane. Let D_i, $i \in Z^+$, be relatively compact open connected subsets of D such that

$$D = \bigcup_{i \in Z^+} D_i, \quad \bar{D}_i \subseteq D_{i+1}, \qquad i \in Z^+.$$

Let K_i be the field of meromorphic functions on D_i, $i \in Z^+$, then we have a sequence of fields

$$K_0 \supseteq \ldots \supseteq K_i \supseteq \ldots \supseteq K = \bigcap_{i \in Z^+} K_i.$$

Let $V^0{}_i = D_i$ be the set of valuations on K_i corresponding to the points in D_i, and put $V^0 = D$. Let N be a subset of V^0 such that $N_i = N \cap V^0{}_i$ is finite for every $i \in Z^+$. Then we know that N_i is a GA-set with respect to $V^0{}_i$ (this is the obvious rational case), and, therefore, a condition for N to be a GA-set with respect to V^0 is:

$$\varprojlim_{Z^+}{}^{(1)} \varprojlim_{V^0_i} U_{K_i} = 0.$$

But,

$$\varprojlim_{V^0{}_i} U_{K_i}$$

is the multiplicative group of units U_i in the complete metrizable algebra A_i of all holomorphic functions on D_i, with the topology of uniform convergence on compact subsets. Now, A_{i+1} is a dense subset of A_i, $i \in Z^+$. It can then be seen that U_i are all complete metrizable and that U_{i+1} is a dense subset of U_i, thus, by Corollary 1.8,

$$\varprojlim_{Z^+}{}^{(1)} U_i = 0$$

and this implies the existence part of the Weierstrass product theorem.

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