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On the Maximum and Minimum Modulus of Rational Functions

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Abstract. We show that if $m, n \geq 0$, $\lambda > 1$, and R is a rational function with numerator, denominator of degree $\leq m, n$, respectively, then there exists a set $\mathcal{S} \subset [0,1]$ of linear measure $\geq \frac{1}{4}\exp(-\frac{13}{\log\lambda})$ such that for $r \in \mathcal{S}$,

$$\max_{|z|=r} |R(z)| / \min_{|z|=r} |R(z)| \leq \lambda^{m+n}.$$

Here, one may not replace $\frac{1}{4}\exp(-\frac{13}{\log\lambda})$ by $\exp(-\frac{2-\varepsilon}{\log\lambda})$, for any $\varepsilon > 0$. As our motivating application, we prove a convergence result for diagonal Padé approximants for functions meromorphic in the unit ball.

1 Introduction and Results

The ratio of maximum modulus to minimum modulus,

$$\max_{|z|=r} |f(z)| / \min_{|z|=r} |f(z)|, \quad r > 0,$$

for entire or meromorphic functions f, plays a role in complex function theory in topics ranging from distribution of zeros and deficient values, to gap series. More specifically, the ratio of maximum modulus to minimum modulus for polynomials P plays a crucial role in several questions in rational approximation [1], [4], [10].

The usual way to estimate this ratio in the case of polynomials is to apply a classic lemma of Cartan [1, 11]: if P is a monic polynomial of degree n, then the lemniscate $\{z : |P(z)| \leq \varepsilon^n\}$ can be covered by at most n balls, the sum of whose diameters is at most $4e\varepsilon$. By grouping the zeros of P into sets "near to" and "far from" the circle $|z| = r$, one can prove that given $s > 0$,

$$\max_{|z|=r} |P(z)| / \min_{|z|=r} |P(z)| \leq (24e)^n, \quad r \in \mathcal{S} \subset [0, s] \tag{1.1}$$

where $\operatorname{meas}(\mathcal{S}) \geq s/2$. Here meas denotes linear Lebesgue measure. See, for example, [10, Lemma 2.1, p. 3152]. While estimates of this type are sufficient for some purposes, the drawback is that no matter how small is s, we still obtain a geometric factor $(24e)^n$ in (1.1). In contrast, for an individual polynomial P, the left-hand side of (1.1) approaches 1 as $r \to 0+$.

We prove in this paper a more appropriate inequality, not only for polynomials, but also for rational functions. Recall that a rational function is of type (m, n), if its numerator and

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denominator have degree $\leq m, n$ respectively, and of course, its denominator should not be identically 0.

Theorem 1 *Let $\lambda > 1$ and $m, n \geq 0$. Then for rational functions R of type (m, n),*

$$\max_{|z|=r} |R(z)| / \min_{|z|=r} |R(z)| \leq \lambda^{m+n}, \quad r \in \mathcal{S}, \tag{1.2}$$

where $\mathcal{S} \subset [0, 1]$ satisfies

$$\operatorname{meas}(\mathcal{S}) \geq \frac{1}{4} \exp\Big(-\frac{13}{\log \lambda}\Big). \tag{1.3}$$

This is sharp in form in the following sense: let $0 < \varepsilon < 1$. Then for λ close enough to 1 and m large enough, there exists a polynomial R of degree m for which the set $\mathcal{S} \subset [0, 1]$ on which (1.2) holds, satisfies

$$\operatorname{meas}(\mathcal{S}) \leq \exp\Big(-\frac{2-\varepsilon}{\log \lambda}\Big). \tag{1.4}$$

Remarks (a) Let $\rho > 0$. By replacing $R(z)$ by $R(\rho z)$, we deduce that (1.2) holds on a set $\mathcal{S} \subset [0, \rho]$ with

$$\operatorname{meas}(\mathcal{S}) \geq \frac{\rho}{4} \exp\Big(-\frac{13}{\log \lambda}\Big). \tag{1.5}$$

(b) Initially the above seems related to the Zolotarev numbers studied in [7], [14]. However, there the maxima and minima are taken over disjoint sets.

(c) The following Cartan type lemma is an essential ingredient of Theorem 1:

Theorem 2 *Let $n \geq 1$, $a_1, a_2, \dots, a_n \in \mathbb{C}$ and $0 < \varepsilon \leq \frac{1}{6}$. Let*

$$\mathcal{E} := \Big\{x \in [0, \infty) : \Big|\prod_{j=1}^{n} \Big(\frac{x - a_j}{x + a_j}\Big)\Big| \leq \varepsilon^n\Big\}. \tag{1.6}$$

Then

$$\int_{\mathcal{E}} \frac{dx}{x} \leq 37\varepsilon. \tag{1.7}$$

In particular, given $s > 0$,

$$\operatorname{meas}(\mathcal{E} \cap [0, s]) \leq 37\varepsilon s. \tag{1.8}$$

The inequality (1.7) is sharp in form in the sense that one may not replace 37 by any constant smaller than $4\sqrt{2}$.

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Remarks (a) The author's initial (unsuccessful) attempt at proving Theorem 2 involved the ideas behind Loomis' Lemmas [2, p. 129], [3, p. 345]. The Cartan type method used here can yield estimates for more general measures than linear measure.

(b) Fryntov and Rossi have studied the size of sets on which finite Blaschke products in the unit disk are small [5]. By a conformal map of the right-half plane onto the unit ball, one can deduce from their result that if

$$\mathcal{E} := \Big\{ z : \operatorname{Re} z > 0 \text{ and } \prod_{j=1}^{n} \Big| \frac{z - a_j}{z + \overline{a_j}} \Big| \le \varepsilon^n \Big\},$$

then

$$\iint_{\mathcal{E}} \frac{dm(z)}{(\operatorname{Re} z)^2} \le \frac{4\pi\varepsilon^2}{1 - \varepsilon^2},$$

with equality iff all $a_j = a$. Here dm denotes planar Lebesgue measure. Is there an analogue of this elegant result in the context of Theorem 2? If so, the extremal rational functions minimising the measure above will (at first, surprisingly) not have all $a_j = a$, see Theorem 7 below.

The motivation for Theorem 1 lies in the convergence theory of Padé approximation. Recall that if $n \ge 1$, the (n, n) Padé approximant to a function f analytic at 0, is a rational function

$$[n/n](z) = (p_n/q_n)(z),$$

where p_n, q_n are polynomials of degree $\le n$ with q_n not identically zero, and

$$(f q_n - p_n)(z) = O(z^{2n+1}).$$

The order relation indicates that the coefficients of $1, z, z^2, \ldots, z^{2n}$ in the Maclaurin series of the left-hand side vanish. For an introduction to the subject, see [1].

The convergence theory of Padé approximation is rich and complex. It is known that if f is meromorphic in the whole plane, then $\{[n/n]\}_{n=1}^{\infty}$ converges in measure, and in capacity—the Nuttall-Pommerenke Theorem. There are deeper analogues for functions with branchpoints [15]. On the other hand, there is no such theorem for functions with finite radius of meromorphy or analyticity [9, 13]. See [10], [12], [16] for reviews of various aspects of the convergence theory.

As a first step towards positive results for functions with finite radius of meromorphy, it was shown in [10] that a subsequence of $\{[n/n]\}_{n=1}^{\infty}$ displays a weak convergence in capacity property in a small neighbourhood of 0. Here we shall show that Theorem 1 implies a positive result for the full diagonal sequence:

Theorem 3 *Let f be meromorphic in the unit ball and analytic at* 0*. Let $0 < \delta < 1$. There exists n_0 with the following property: for $n \ge n_0$, there exists a set $\mathcal{S}_n \subset (0, 1/2)$ of measure $\ge \exp(-40/\delta)$ such that for $r \in \mathcal{S}_n$,*

$$\max_{|z|=r} \Big| \frac{f(z) - [n/n](z)}{z^n} \Big| \le (1 + \delta)^n. \tag{1.9}$$

Moreover, there exists a set $\mathcal{A}_n$ *within the ball centre* 0*, radius* $\frac{1}{2}$*, of planar measure* $\geq \pi \exp(-80/\delta)$ *such that*

$$\sup_{z \in \mathcal{A}_n} \left| \frac{f - [n/n]}{z^n} \right| (z) \leq (1 + \delta)^n. \tag{1.10}$$

Thus $[n/n]$ is uniformly close to f on circles whose radii have linear measure bounded below independent of n and consequently on a set whose planar measure is bounded below independently of n. It shows that while we cannot expect good approximation *outside* a set of small measure or capacity (as in the Nuttall-Pommerenke theorem), nevertheless, we can expect good approximation *on* a set which is a positive proportion (depending only on δ) of the ball of meromorphy, at least in the sense of planar measure.

Strictly speaking, even (1.1) leads to an estimate of the form (1.10), but with $1 + \delta$ replaced by the much worse factor $25e$.

One may replace the factor z^n in the denominator in (1.9), (1.10) by an error of rational approximation on a suitable disc; this leads one to expect that at least for an infinite subsequence of integers, one should be able to replace z^n by z^{2n}.

This paper is organised as follows: we prove Theorem 2 in Section 2. Then in Section 3, we prove Theorem 1. Finally in Section 4, we prove Theorem 3.

2 Proof of Theorem 2

We shall base our proof on Cartan's Lemma applied to the metric space in the following proposition.

Proposition 4 *Let* $X := (0, \infty)$ *and*

$$d(x, t) := \left| \frac{x - t}{x + t} \right|, \quad x, t \in X.$$

Then (X, d) *is a metric space.*

Proof Let

$$\rho(z, w) := \left| \frac{z - w}{1 - \bar{w}z} \right|, \quad |z|, |w| < 1$$

denote the pseudohyperbolic metric on the unit ball, and ψ the conformal map of the right-half plane onto the unit ball:

$$\psi(z) := \frac{z - 1}{z + 1}.$$

Then it is easy to check that

$$d(x, t) = \rho\big(\psi(x), \psi(t)\big), \quad x, t \in X.$$

The fact that ρ is a metric [6, p. 4] then implies the same for d. ■

We now turn to a general Cartan type lemma.

Lemma 5 *Let* (X, d) *be a metric space, let* $0 < r_1 < r_2 < \cdots < r_n$ *and let* $a_1, a_2, \ldots, a_n \in X$*. There exist positive integers* $p \leq n$*,* $\{\lambda_j\}_{j=1}^p$ *and closed balls* $\{B_j\}_{j=1}^p$ *in* X *such that*

(i) $\lambda_1 + \lambda_2 + \cdots \lambda_p = n$.
(ii) B_j *has radius* $2r_{\lambda_j}$, $1 \le j \le p$.
(iii)

$$\prod_{j=1}^{n} d(z, a_j) > \prod_{j=1}^{n} r_j, \quad z \in X \Big\backslash \bigcup_{j=1}^{p} B_j. \tag{2.1}$$

Proof This is exactly the same as the usual form for monic polynomials ([11, p. 201] or [3, p. 350]), one simply replaces $(z - a_j)$ by $d(z, a_j)$. Nevertheless, we provide the details for the reader's convenience. Let A denote the sequence $a_1, a_2, \ldots, a_n$. The multiplicity of a member of A is the number of times it is repeated in the sequence. We divide this into four steps:

Step 1 We show that there exists $\lambda_1 \le n$ and a circle C_1 of radius r_{λ_1} containing exactly λ_1 members of A, counting multiplicity.

For suppose such a circle does not exist. Then any circle C of radius r_1 containing 1 member of A contains at least 2. The concentric circle of radius r_2 contains 2, so must contain 3 (otherwise we could choose $\lambda_1 = 2$ and C_1 to be this circle). Continuing in this way, we eventually find that the circle concentric with C and radius r_n must contain $n + 1$ members of A, which is impossible.

Step 2 We rank the members of A.

Choose the largest λ_1 with the property in Step 1, and let C_1 be the corresponding circle. Call the λ_1 members of A inside C_1 members of rank λ_1. Next, applying the argument of Step 1 to the remaining $n - \lambda_2$ members of A, we obtain a largest positive integer $\lambda_2 \le \lambda_1$ and a circle C_2 containing exactly λ_2 of the members of A outside C_1. Call those members inside C_2 members of rank λ_2. Continuing in this way, we find $p \le n$ largest integers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p$ and corresponding circles C_j of radius r_{λ_j} containing exactly λ_j members of A outside $C_1 \cup C_2 \cup \cdots \cup C_{j-1}$. Moreover, as we eventually exhaust the members, $\lambda_1 + \lambda_2 + \cdots + \lambda_p = n$.

Step 3 We prove that if S is a circle of radius r_λ containing at least λ members of A, then at least one of these members has rank at least λ.

First if S contains more than λ_1 zeros, then at least one must lie in C_1 and so have rank $\lambda_1 \ge \lambda$. (If not, we would obtain a contradiction to the choice of λ_1 being as large as possible). Next suppose that $\lambda_j \ge \lambda > \lambda_{j+1}$, some j. If any of the members inside S lies in $C_1, C_2, \ldots, C_j$ then these have rank $\ge \lambda_j \ge \lambda$, as required. If all the members lie outside these former j circles, then the process of Step 1 yields a circle with $\ge \lambda > \lambda_{j+1}$ members contradicting the choice of λ_{j+1} being as large as possible.

Step 4 Complete the proof.

Let B_j be the (closed) ball concentric with C_j but of twice the radius, so that B_j has radius $2r_{\lambda_j}$, $1 \le j \le p$. Fix $z \in X \setminus \bigcup_{j=1}^{p} B_j$. We claim that a circle S, centre z, radius r_λ, can contain at most $\lambda - 1$ members of A. For if it contained at least λ, then by Step 3, at least one, u say, would have, say, rank $\lambda_j \ge \lambda$, and so lie in C_j and also in the concentric ball B_j of twice the radius. Then the fact that $z \notin B_j$ and u lies inside C_j forces

$$d(z,u) > \operatorname{dist}(X \setminus B_j, C_j) \geq r_{\lambda_j} \geq r_\lambda$$

contradicting our hypothesis that S contains u.

Finally rearrange the members of A in order of increasing distance from z as $a_1, a_2, \dots, a_n$. Now the circle centre z, radius r_j can contain at most $j-1$ members of A, and these could only be $a_1, a_2, \dots, a_{j-1}$ so

$$d(z, a_j) > r_j.$$

Thus

$$\prod_{j=1}^{n} d(z, a_j) > \prod_{j=1}^{n} r_j. \qquad \blacksquare$$

We turn to

The Proof of (1.7) of Theorem 2

Step 1 We first show that it suffices to consider $a_1, a_2, \dots, a_n \in [0, \infty)$.

For if not, let

$$\alpha_j := |a_j|, \quad 1 \leq j \leq n.$$

Then for $x \in [0, \infty)$,

$$\left|\frac{x - a_j}{x + a_j}\right| \geq \left|\frac{x - \alpha_j}{x + \alpha_j}\right|$$

so that

$$\left\{x \in [0,\infty) : \prod_{j=1}^{n} \left|\frac{x - a_j}{x + a_j}\right| \leq \varepsilon^n\right\} \subset \left\{x \in [0,\infty) : \prod_{j=1}^{n} \left|\frac{x - \alpha_j}{x + \alpha_j}\right| \leq \varepsilon^n\right\}.$$

So, it follows that it suffices to prove (1.7) for the case of $a_j \in [0, \infty)$.

Step 2 Prove (1.7) for $a_1, a_2, \dots, a_n \in [0, \infty)$.

Let $0 < \varepsilon \leq \frac{1}{6}$. We choose in the lemma above (X, d) to be the metric space of Proposition 4 and let

$$r_j := \varepsilon j (n!)^{-1/n}, \quad 1 \leq j \leq n.$$

Then if B_j is the ball of radius $2r_{\lambda_j}$ of Lemma 5, $1 \leq j \leq p$, we have for $x \in [0, \infty) \setminus \bigcup_{j=1}^{p} B_j$,

$$\prod_{j=1}^{n} \left|\frac{x - a_j}{x + a_j}\right| = \prod_{j=1}^{n} d(x, a_j) > \prod_{j=1}^{n} r_j = \varepsilon^n.$$

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Thus if $\mathcal{E}$ is the set of small values defined by (1.6), it is contained in $\bigcup_{j=1}^{p} B_j$, so

$$\int_{\mathcal{E}} \frac{dx}{x} \le \sum_{j=1}^{p} \int_{B_j} \frac{dx}{x}.$$

Next, a ball B centre $a \in (0, \infty)$, radius $s < 1$ in the metric of Proposition 4 is easily seen to have the form

$$B = \left(a\Big(\frac{1-s}{1+s}\Big), a\Big(\frac{1+s}{1-s}\Big) \right)$$

so that

$$\int_{B} \frac{dx}{x} = 2\log\Big(\frac{1+s}{1-s}\Big).$$

Thus,

$$\int_{\mathcal{E}} \frac{dx}{x} \le 2\sum_{j=1}^{p} \log\Big(\frac{1+2r_{\lambda_j}}{1-2r_{\lambda_j}}\Big).$$

Now the inequality $n! \ge (n/e)^n$ implies that

$$2r_j \le 2e\varepsilon \le e/3, \quad 1 \le j \le n.$$

Since

$$f(s) := \frac{1}{s}\log\Big(\frac{1+s}{1-s}\Big) = 2\sum_{j=0}^{\infty} \frac{s^{2j}}{2j+1}$$

is an increasing function of $s \in (0,1)$, we deduce that for each j,

$$\log\Big(\frac{1+2r_j}{1-2r_j}\Big) \le 2r_j f\Big(\frac{e}{3}\Big)$$

and hence

$$\int_{\mathcal{E}} \frac{dx}{x} \le 4f\Big(\frac{e}{3}\Big)\sum_{j=1}^{p} r_{\lambda_j} = 4f\Big(\frac{e}{3}\Big)\varepsilon\Big(\frac{n^n}{n!}\Big)^{1/n} \le 4ef\Big(\frac{e}{3}\Big)\varepsilon \le 37\varepsilon. \qquad \blacksquare$$

We may improve the constant 37 a little: if instead of $\varepsilon \le \frac{1}{6}$, we assume that $\varepsilon \le \frac{\rho}{2e}$, some $\rho \in (0,1)$, then the above argument goes through with $e/3$ replaced by ρ. After setting $\rho = 2e\varepsilon$, we deduce that

$$\int_{\mathcal{E}} \frac{dx}{x} \le 4ef(2e\varepsilon)\varepsilon \text{ provided } \varepsilon < \frac{1}{2e}. \tag{2.2}$$

Here as $\varepsilon \to 0+$, $f(2e\varepsilon) \to 2$, so we obtain a constant close to $8e = 21.74\cdots$.

Before proving the sharpness part of Theorem 2, we present a generalisation of (1.7) of Theorem 2:

Theorem 6 *Let $0 < \varepsilon \leq \frac{1}{6}$, and let μ be a probability measure on $\mathbb{C}$ with compact support. Let*

$$R(x) := \int_{\mathbb{C}} \log\left|\frac{x-t}{x+t}\right| d\mu(t)$$

and let

$$\mathcal{E} := \{x \in [0,\infty) : R(x) \leq \log \varepsilon\}. \tag{2.3}$$

Then

$$\int_{\mathcal{E}} \frac{dx}{x} \leq 37\varepsilon. \tag{2.4}$$

Proof For the special case where μ is a unit measure with point masses of size $\frac{1}{n}$ at a_1, $a_2, \ldots, a_n$, this is a reformulation of Theorem 2. For the given μ, with compact support, we can find a sequence $\{\mu_n\}_{n=1}^{\infty}$ of such pure jump measures converging weakly to μ: for every continuous $g\colon \mathbb{C} \to \mathbb{R}$ with compact support,

$$\lim_{n\to\infty} \int g\, d\mu_n = \int g\, d\mu.$$

This follows from the standard fact that pure jump measures are dense in the weak-* topology. Next, for a given fixed $x \in (0,\infty)$, the function

$$t \to \log\left|\frac{x-t}{x+t}\right|, \quad t \in \mathbb{C}$$

is upper semi-continuous, being the limit of a decreasing sequence of continuous functions. Then (see *e.g.* [14, p. 5, equation (1.2)]),

$$\int \log\left|\frac{x-t}{x+t}\right| d\mu(t) \geq \limsup_{n\to\infty} \int \log\left|\frac{x-t}{x+t}\right| d\mu_n(t).$$

Now let $\eta > 1$ and

$$\mathcal{E} := \left\{x \in [0,\infty) : \int \log\left|\frac{x-t}{x+t}\right| d\mu(t) \leq \log\varepsilon\right\};$$

$$\mathcal{E}_n := \left\{x \in [0,\infty) : \int \log\left|\frac{x-t}{x+t}\right| d\mu_n(t) \leq \log(\eta\varepsilon)\right\}.$$

Next given $x \in \mathcal{E}$, we have $x \in \mathcal{E}_n$ for n large enough. It follows that

$$\mathcal{E} \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \mathcal{E}_n.$$

Then if χ_k is the characteristic function of $\bigcap_{n=k}^{\infty} \mathcal{E}_n$, so that χ_k increases with k, and χ is the characteristic function of $\mathcal{E}$, we see that for all t,

$$\chi(t) \leq \lim_{k\to\infty} \chi_k(t)$$

and hence by the monotone convergence theorem, and by what we proved for jump measures,

$$\int_{\mathcal{E}} \frac{dt}{t} = \int_0^\infty \frac{\chi(t)}{t}\,dt \leq \lim_{k\to\infty} \int_0^\infty \frac{\chi_k(t)}{t}\,dt \leq \liminf_{k\to\infty} \int_{\mathcal{E}_k} \frac{dt}{t} \leq 37(\eta\varepsilon).$$

Now let $\eta \to 1+$. ∎

The function $t \to \log\left|\frac{x+t}{x-t}\right|$ is the Green's function for the right-half plane, with pole at $x \in \mathbb{R}$ (see, for example, [8], [14]). Thus Theorem 6 may be viewed as an estimate involving Green potentials. We also note that Theorem 6 admits an improvement as in (2.2): if $\mathcal{E}$ is the set defined in (2.3), then as at (2.2),

$$\int_{\mathcal{E}} \frac{dx}{x} \leq 4ef(2e\varepsilon)\varepsilon \text{ provided } \varepsilon < \frac{1}{2e}. \tag{2.5}$$

The sharpness part of Theorem 2 will follow from:

Theorem 7 *Let* $0 < a \leq b < \infty$, $0 < \varepsilon < 1$, *and*

$$\alpha := \alpha(\varepsilon) = \left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)^4. \tag{2.6}$$

Let

$$\mathcal{E} := \left\{x \in [0,\infty) : \left|\frac{(x-a)(x-b)}{(x+a)(x+b)}\right| \leq \varepsilon^2\right\}. \tag{2.7}$$

Then

$$\int_{\mathcal{E}} \frac{dx}{x} \leq \log\left(\frac{1+\sqrt{1-\alpha(\varepsilon)}}{1-\sqrt{1-\alpha(\varepsilon)}}\right), \tag{2.8}$$

with equality iff

$$\frac{4ab}{(a+b)^2} = \left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)^2. \tag{2.9}$$

Remark The interesting feature is that if equality occurs in (2.8), then necessarily $a < b$.

Proof Let us assume $a < b$. The case $a = b$ may be deduced by letting $b \to a+$. Let

$$P(x) := (x-a)(x-b); \quad R(x) := \frac{P(x)}{P(-x)}.$$

We recommend that the reader draws a graph of R. It is easily seen that R has a local minimum at some point in (a, b), a local maximum in $(-b, -a)$, and no other critical points. Moreover, $R(x)$ decreases from ∞ at $x = -a$ to 1 at $x = 0$, and then to its local minimum, after which it increases to 1 as $x \to \infty$. Then it follows that $\mathcal{E}$ consists of at most 2 intervals, and each such interval contains one of a, b.

Case I: $\mathcal{E}$ is a single interval Then $\mathcal{E} = [\xi_1, \xi_2]$, where $0 < \xi_1 < a < b < \xi_2$ and $R(\xi_1) = R(\xi_2) = \varepsilon^2$. Then the ξ_j are roots of the quadratic equation $P(x) - \varepsilon^2 P(-x) = 0$, and solving for the roots gives

$$\xi_j = \frac{1}{2}\Big(\frac{1+\varepsilon^2}{1-\varepsilon^2}\Big)(a+b)[1 + (-1)^j \sqrt{1-\beta}], \tag{2.10}$$

where

$$\beta := \frac{4ab}{(a+b)^2}\Big(\frac{1-\varepsilon^2}{1+\varepsilon^2}\Big)^2 \in (0,1).$$

Then

$$\int_{\mathcal{E}} \frac{dx}{x} = \log\frac{\xi_2}{\xi_1} = \log\Big[\frac{1+\sqrt{1-\beta}}{1-\sqrt{1-\beta}}\Big]. \tag{2.11}$$

The right-hand side is a decreasing function of $\beta \in (0, 1)$, so it suffices to find a lower bound for β. To do this we use the fact that $\mathcal{E}$ is a single interval. Then $R(x) \geq -\varepsilon^2$, $x \in [0, \infty)$ and hence $P(x) + \varepsilon^2 P(-x) \geq 0$, $x \in [0, \infty)$. This is equivalent to

$$x^2 - (a+b)\frac{1-\varepsilon^2}{1+\varepsilon^2}x + ab \geq 0, \quad x \in [0, \infty).$$

This quadratic has a minimum in $[0, \infty)$ of

$$ab - \frac{1}{4}(a+b)^2\Big(\frac{1-\varepsilon^2}{1+\varepsilon^2}\Big)^2.$$

This minimum is non-negative iff

$$\frac{4ab}{(a+b)^2} \geq \Big(\frac{1-\varepsilon^2}{1+\varepsilon^2}\Big)^2.$$

Then we deduce that $\beta \geq \alpha = \alpha(\varepsilon)$, so (2.11) yields (2.8). For equality in (2.8), we must have $\beta = \alpha$, that is (2.9) holds.

Case II: $\mathcal{E}$ consists of two intervals Then $\mathcal{E} = [\xi_1, \eta_1] \cup [\eta_2, \xi_2]$, where $0 < \xi_1 < a < \eta_1 < \eta_2 < b < \xi_2$, and the ξ_j are as above, while $R(\eta_1) = R(\eta_2) = -\varepsilon^2$. Then η_j are roots of the quadratic equation $P(x) + \varepsilon^2 P(-x) = 0$, and solving gives

$$\eta_j = \frac{1}{2}\Big(\frac{1-\varepsilon^2}{1+\varepsilon^2}\Big)(a+b)[1 + (-1)^j \sqrt{1-\gamma}],$$

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where

$$\gamma := \frac{4ab}{(a+b)^2}\Big(\frac{1+\varepsilon^2}{1-\varepsilon^2}\Big)^2 = \beta/\alpha.$$

Then

$$\int_{\mathcal{E}} \frac{dx}{x} = \log\frac{\eta_1}{\xi_1} + \log\frac{\xi_2}{\eta_2} = H(\beta) - H\Big(\frac{\beta}{\alpha}\Big), \tag{2.12}$$

where

$$H(t) := \log\Big(\frac{1+\sqrt{1-t}}{1-\sqrt{1-t}}\Big).$$

Since the roots η_j are distinct, $\gamma < 1 \Rightarrow \beta < \alpha$. Now

$$\frac{d}{dt}\Big[H(t) - H\Big(\frac{t}{\alpha}\Big)\Big] = \frac{1}{t}\Big(-\frac{1}{\sqrt{1-t}} + \frac{1}{\sqrt{1-t/\alpha}}\Big) > 0, \quad t \in (0,\alpha),$$

so the right-hand side of (2.12) is an increasing function of β. Substituting the strict upper bound $\beta = \alpha$ gives (2.8) with strict inequality. ■

We turn to

The Sharpness Part of Theorem 2 A simple calculation shows that as $\varepsilon \to 0+$, the right-hand side of (2.8) behaves like $4\sqrt{2}\varepsilon\big(1+o(1)\big)$, so we may not replace 37 in (1.7) by anything smaller than $4\sqrt{2}$. ■

3 Proof of Theorem 1

We shall prove what is, essentially, a generalisation of Theorem 1:

Theorem 8 *Let μ be a probability measure on $\mathbb{C}$ with compact support. Let $\lambda > 1$. Then*

$$\operatorname{meas}\Big\{x \in [0,1] : \int_{\mathbb{C}} \log\Big|\frac{x+t}{x-t}\Big|\, d\mu(t) \le \log\lambda\Big\} \ge \frac{1}{4}\exp\Big(-\frac{12.338}{\log\lambda}\Big). \tag{3.1}$$

We preface the proof of this result with:

Lemma 9 *Let μ be a probability measure on $[0,1]$. Let $I_0 := \phi$ and for $k \ge 1$, let*

$$I_k := (2^{-k}, 2^{1-k}],$$

and

$$\sigma_k := (\log 81) \sum_{j\ge 1:|j-k|\ge 2} \mu(I_j) 2^{-|k-j|} + 4.468 \sum_{j=k-1}^{k+1} \mu(I_j). \tag{3.2}$$

Then for $k \geq 1$, there exists a subset $\mathcal{T}_k$ of I_k with

$$\operatorname{meas}(\mathcal{T}_k) \geq \frac{1}{2} \operatorname{meas}(I_k) \tag{3.3}$$

and

$$\int_0^1 \log\left|\frac{x+t}{x-t}\right| d\mu(t) \leq \sigma_k. \tag{3.4}$$

Proof We use the fact that

$$f(s) := \frac{1}{s} \log\left(\frac{1+s}{1-s}\right), \quad s \in (0,1), \tag{3.5}$$

is increasing in s, so that

$$\log\left(\frac{1+t}{1-t}\right) \leq (\log 9)t, \quad t \in \left[0, \frac{1}{2}\right].$$

Then for $x \in I_k$ and $j > k+1$, we have $t/x < 1/2$, $t \in I_j$, so

$$\begin{aligned}\int_{I_j} \log\left|\frac{x+t}{x-t}\right| d\mu(t) &= \int_{I_j} \log\left|\frac{1+t/x}{1-t/x}\right| d\mu(t) \\ &\leq \mu(I_j) \log 9 \cdot (2^{1-j}/2^{-k}) = (\log 81)\mu(I_j) 2^{-|k-j|}.\end{aligned}$$

The case $j < k-1$ is similar, since there $x/t < 1/2$. Thus, for $x \in I_k$,

$$\int_{[0,1]\setminus(I_{k-1}\cup I_k \cup I_{k+1})} \log\left|\frac{x+t}{x-t}\right| d\mu(t) \Big/ \sum_{j\geq 1:|j-k|\geq 2} \mu(I_j) 2^{-|k-j|} \leq \log 81. \tag{3.6}$$

Next, let

$$\rho := \mu(I_{k-1} \cup I_k \cup I_{k+1}) \text{ and } \mu^* := \frac{1}{\rho} \mu_{|I_{k-1}\cup I_k \cup I_{k+1}},$$

so that μ^* is a unit measure. Let us use our Cartan Lemma (Theorem 6) in the sharper form (2.5), with

$$\varepsilon = \frac{1}{2e}\left(\frac{e^{1/8}-1}{e^{1/8}+1}\right),$$

which ensures that

$$8e\varepsilon f(2e\varepsilon) = \frac{1}{2}.$$

Here f is as in (3.5). Then (2.5) shows that there is a set $\mathcal{E} \subset I_k$ such that

$$\int \log\left|\frac{x-t}{x+t}\right| d\mu^*(t) > \log \varepsilon, \quad x \in I_k \setminus \mathcal{E},$$

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where

$$\frac{\operatorname{meas}(\mathcal{E})}{2^{1-k}} \leq \int_{\mathcal{E}} \frac{dx}{x} \leq 4e\varepsilon f(2e\varepsilon) = \frac{1}{4} \Rightarrow \operatorname{meas}(\mathcal{E}) \leq \frac{1}{4} \cdot 2^{1-k} = \frac{1}{2} \operatorname{meas}(I_k).$$

Letting

$$\mathcal{T}_k := I_k \setminus \mathcal{E},$$

we see that it fulfills (3.3) and in this set,

$$\int_{I_{k-1} \cup I_k \cup I_{k+1}} \log \left| \frac{x+t}{x-t} \right| d\mu(t) \leq (\log \varepsilon^{-1}) \mu(I_{k-1} \cup I_k \cup I_{k+1}).$$

Since

$$\log \varepsilon^{-1} = 4.46703 \cdots < 4.468,$$

this last estimate together with (3.6) gives the result. ∎

We turn to

The Proof of Theorem 8 We note first that it suffices to consider measures with support on $[0, 1]$. For if this is not the case, the measure $\mu^{\#}$ defined for measurable $S \subset [0, 1]$ by

$$\mu^{\#}(S) := \mu(\{z : |z| \in S\}) + \mu(\{z : |z| > 1\}) \int_S d\delta_1,$$

where δ_1 is a unit mass at 1, has support in $[0, 1]$, and for $x \in [0, 1]$,

$$\int \log \left| \frac{x+t}{x-t} \right| d\mu^{\#}(t) \geq \int \log \left| \frac{x+t}{x-t} \right| d\mu(t).$$

Then the measure of the set in (3.1) is no larger for $\mu^{\#}$ than for μ. So it suffices to prove (3.1) for measures μ with support in $[0, 1]$, and we assume this in the sequel. With the notation of Lemma 9, we see that

$$\begin{aligned}\sum_{k=1}^{\infty} \sigma_k &\leq (\log 81) \sum_{j=1}^{\infty} \mu(I_j) \sum_{k \geq 1 : |j-k| \geq 2} 2^{-|k-j|} + 4.468 \sum_{k=1}^{\infty} \sum_{j=k-1}^{k+1} \mu(I_j) \\ &\leq \log 81 + 4.468 \cdot 3 < 17.799.\end{aligned}$$

It follows that given $\lambda > 1$, there exists $1 \leq k \leq 17.799/\log \lambda + 1$ with

$$\sigma_k < \log \lambda.$$

By Lemma 9, we have

$$\int_0^1 \log\left|\frac{x+t}{x-t}\right| d\mu(t) \leq \sigma_k < \log\lambda, \quad x \in \mathcal{T}_k \subset I_k,$$

where

$$\begin{aligned}\operatorname{meas}(\mathcal{T}_k) &\geq \frac{1}{2}\operatorname{meas}(I_k) = 2^{-1-k}\\ &\geq 2^{-2-17.799/\log\lambda} \geq \frac{1}{4}\exp\Bigl(-\frac{12.338}{\log\lambda}\Bigr).\end{aligned}$$ ∎

For the sharpness of Theorem 1 (and hence of Theorem 8), we need a product that is often used in rational approximation to $|x|$:

Lemma 10 *Let $0 < \beta < 2$ and*

$$q := \exp\Bigl(-\frac{\beta}{n\log\lambda}\Bigr)$$

and

$$R_n(x) := \prod_{j=1}^{n}\Bigl(\frac{x+q^j}{x-q^j}\Bigr).$$

(a) *Then there exists $\lambda_0(\beta) > 1$ such that for $1 < \lambda \leq \lambda_0(\beta)$, there exists $n_0(\lambda,\beta)$ such that for $n \geq n_0(\lambda,\beta)$,*

$$|R_n(x)| > \lambda^n, \quad x \in [q^n, 1]$$

and hence

$$\operatorname{meas}\{x \in [0,1] : |R_n(x)| \leq \lambda^n\} \leq \operatorname{meas}[0, q^n] = \exp\Bigl(-\frac{\beta}{\log\lambda}\Bigr). \tag{3.7}$$

(b) *Moreover, if*

$$P(z) := \prod_{j=1}^{n}(z+q^j),$$

then

$$\operatorname{meas}\Bigl\{r \in [0,1] : \frac{\max_{|z|=r}|P(z)|}{\min_{|z|=r}|P(z)|} \leq \lambda^n\Bigr\} \leq \exp\Bigl(-\frac{\beta}{\log\lambda}\Bigr).$$

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Proof (a) Let $1 \leq k \leq n$, and

$$q^k \leq x < q^{k-1}.$$

Then for $j \leq k-2$,

$$\Big|\frac{x+q^j}{x-q^j}\Big| > \frac{q^j+q^{k-1}}{q^j-q^{k-1}} = \frac{1+q^{k-1-j}}{1-q^{k-1-j}}.$$

Similarly, for $j \geq k+1$,

$$\Big|\frac{x+q^j}{x-q^j}\Big| \geq \frac{q^k+q^j}{q^k-q^j} = \frac{1+q^{j-k}}{1-q^{j-k}}.$$

Also, for $j = k-1, k$,

$$\Big|\frac{x+q^j}{x-q^j}\Big| \geq \frac{q^k+q^{k-1}}{q^k-q^{k-1}} = \frac{1+q}{1-q},$$

with strict inequality for at least one j. Then

$$\begin{aligned}|R_n(x)| &> \Big(\prod_{j=1}^{k-2}\Big(\frac{1+q^{k-1-j}}{1-q^{k-1-j}}\Big)\Big)\Big(\frac{1+q}{1-q}\Big)^2\Big(\prod_{j=k+1}^{n}\Big(\frac{1+q^{j-k}}{1-q^{j-k}}\Big)\Big)\\ &\geq \prod_{j=1}^{n}\Big(\frac{1+q^j}{1-q^j}\Big).\end{aligned}$$

Here we used the fact that $\frac{1+q^j}{1-q^j}$ decreases as j increases. Next, using the inequality

$$\log\Big(\frac{1+t}{1-t}\Big) > 2t, \quad t \in (0,1),$$

we see that we can continue this as

$$|R_n(x)| > \exp\Big(2\sum_{j=1}^{n} q^j\Big) = \exp\Big(2q\frac{1-q^n}{1-q}\Big), \quad x \in [q^n, 1].$$

(Strictly speaking, we omitted $x = 1$, but we may use continuity.) Now as $n \log\lambda \to \infty$, $q = q(n,\lambda) \to 1$, so

$$\begin{aligned}2q\frac{1-q^n}{1-q}/(n\log\lambda) &= \frac{2\big(1+o(1)\big)}{|\log q| n \log\lambda}\Big(1-\exp\Big(-\frac{\beta}{\log\lambda}\Big)\Big)\\ &= \big(1+o(1)\big)\frac{2}{\beta}\Big(1-\exp\Big(-\frac{\beta}{\log\lambda}\Big)\Big).\end{aligned}$$

Since $2/\beta > 1$, it follows that there exists $\lambda_0(\beta)$ such that for $1 < \lambda \le \lambda_0(\beta)$, there exists $n_0(\lambda, \beta)$ such that for $n \ge n_0(\lambda, \beta)$,

$$2q\frac{1-q^n}{1-q} > n\log\lambda \Rightarrow |R_n(x)| > \lambda^n, \quad x \in [q^n, 1].$$

Then (3.7) follows.

(b) This follows as

$$\frac{\max_{|z|=r}|P(z)|}{\min_{|z|=r}|P(z)|} = |R_n(r)|. \qquad \blacksquare$$

We turn to the

Proof of Theorem 1 We begin with the simple observation that it suffices to consider rational functions with real poles and zeros: let R be a rational function of type (m, n) with zeros $z_1, z_2, \dots, z_m$ and poles $w_1, w_2, \dots, w_n$ and let

$$a_j := |z_j|, \quad 1 \le j \le m; \; b_j := |w_j|, \quad 1 \le j \le n$$

and

$$S(x) := \prod_{j=1}^{m}\left(\frac{x+a_j}{x-a_j}\right) \cdot \prod_{j=1}^{n}\left(\frac{x+b_j}{x-b_j}\right).$$

Then it is easily seen that

$$\frac{\max_{|z|=r}|R(z)|}{\min_{|z|=r}|R(z)|} \le |S(r)|.$$

Then

$$\left\{r \in [0,1] : \frac{\max_{|z|=r}|R(z)|}{\min_{|z|=r}|R(z)|} \le \lambda^{m+n}\right\} \supseteq \{r \in [0,1] : |S(r)| \le \lambda^{m+n}\}$$

$$\Rightarrow \operatorname{meas}\left(\left\{r \in [0,1] : \frac{\max_{|z|=r}|R(z)|}{\min_{|z|=r}|R(z)|} \le \lambda^{m+n}\right\}\right) \ge \operatorname{meas}(\{r \in [0,1] : |S(r)| \le \lambda^{m+n}\}).$$

If we let μ be the unit measure with mass $\frac{1}{m+n}$ at $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n$, then we see that

$$\{r \in [0,1] : |S(r)| \le \lambda^{m+n}\} = \left\{x \in [0,1] : \int_0^\infty \log\left|\frac{x+t}{x-t}\right| d\mu(t) \le \log\lambda\right\}$$

and now Theorem 8 gives the estimate (1.3). For the sharpness of the estimate in Theorem 1, we simply apply Lemma 10(b) with $\beta = 2 - \varepsilon$. $\blacksquare$

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4 Proof of Theorem 3

We turn directly to:

The Proof of Theorem 3 Let $0 < \delta < 1$ and $\tau := (1+\delta)^{-1/4}$. Let S be a monic polynomial, of degree ℓ say, such that fS is analytic in $|z| \le \tau$. We assume that f itself is analytic on $|z| = \tau$. (If not, alter δ a little). We use the well known error formula for Padé approximation,

$$(f - [n/n])(z) = \frac{1}{2\pi i} \int_{|t|=\tau} \frac{(fSq_n)(t)}{(Sq_n)(z)} \left(\frac{z}{t}\right)^{2n+1} \frac{dt}{t-z}, \quad |z| < \tau.$$

This is a simple consequence of Cauchy' integral formula, see *e.g.* [1, 10]. We deduce that for $r \le \frac{1}{2}$,

$$\max_{|z|=r} |f - [n/n]|(z) \le C\left(\frac{r}{\tau}\right)^{2n+1} \frac{\max_{|t|=\tau} |Sq_n|(t)}{\min_{|z|=r} |Sq_n|(z)},$$

where

$$C := \frac{\tau}{\tau - \frac{1}{2}} \max_{|t|=\tau} |f(t)|$$

depends only on f, δ. Using first Bernstein's inequality to bound Sq_n on the circle $|t| = \tau$ in terms of its maximum on $|t| = r$, and then the remark (1.5) after Theorem 1, we obtain

$$\frac{\max_{|t|=\tau} |Sq_n|(t)}{\min_{|z|=r} |Sq_n|(z)} \le \left(\frac{\tau}{r}\right)^{n+\ell} \frac{\max_{|t|=r} |Sq_n|(t)}{\min_{|z|=r} |Sq_n|(z)} \le \left(\frac{\tau}{r}\right)^{n+\ell} (1+\delta)^{\frac{n+\ell}{2}},$$

for $r \in \mathcal{S}_n$, where $\mathcal{S}_n \subset [0, \frac{1}{2}]$ and

$$\text{(4.1)} \qquad \operatorname{meas}(\mathcal{S}_n) \ge \frac{1}{8} \exp\left(-\frac{13}{\log(1+\delta)^{1/2}}\right) \ge \frac{1}{8} \exp\left(-\frac{26}{(\log 2)\delta}\right).$$

We have used here the inequality $\log(1 + x) \ge (\log 2)x$, $x \in [0, 1]$. Then for $r \in \mathcal{S}_n$, we deduce that

$$\max_{|z|=r} |f - [n/n]|(z) \le C_1 \left(r(1+\delta)^{3/4}\right)^{n-\ell},$$

where

$$C_1 := C(1+\delta)^{\ell+1/4}$$

depends only on f and δ, recall that $\tau = (1 + \delta)^{-1/4}$. Then

$$\max_{|z|=r} |f - [n/n]|(z) \le \left(r(1+\delta)\right)^n,$$

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for

$$r \in [C_1^{1/\ell}(1+\delta)^{-\frac{3}{4}-\frac{n}{4\ell}}, \infty) \cap \mathcal{S}_n.$$

For $n \geq n_0(\delta)$, this has by (4.1), measure at least

$$\frac{1}{8}\exp\Big(-\frac{37.52}{\delta}\Big) \geq \exp\Big(-\frac{40}{\delta}\Big).$$

(Recall that $\delta < 1$). So we have (1.9). Finally, if

$$\mathcal{A}_n := \{z : |z| \in \mathcal{S}_n\},$$

then $\mathcal{A}_n$ has planar measure

$$\geq 2\pi \int_{\mathcal{S}_n} r\,dr \geq \pi \exp\Big(-\frac{80}{\delta}\Big)$$

and (1.10) follows.

Note added in proof The sharp form of Theorem 1, for each λ, involves condenser capacity. ■

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