

SUPERCONGRUENCES INVOLVING MOTZKIN NUMBERS AND CENTRAL TRINOMIAL COEFFICIENTS

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Abstract Let M_n and T_n denote the n th Motzkin number and the n th central trinomial coefficient respectively. We prove that for any prime $p \geq 5$,

$$\begin{aligned}\sum_{k=0}^{p-1} M_k^2 &\equiv \left(\frac{p}{3}\right) (2-6p) \pmod{p^2}, \\ \sum_{k=0}^{p-1} kM_k^2 &\equiv \left(\frac{p}{3}\right) (9p-1) \pmod{p^2}, \\ \sum_{k=0}^{p-1} T_k M_k &\equiv \frac{4}{3} \left(\frac{p}{3}\right) + \frac{p}{6} \left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2},\end{aligned}$$

where $(-)$ is the Legendre symbol. These results confirm three supercongruences conjectured by Z.-W. Sun in 2010.

Keywords: Motzkin numbers; Catalan numbers; central trinomial coefficients; supercongruences; Legendre symbol

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1. Introduction

The Motzkin numbers $\{M_n\}_{n=0}^\infty = 1, 1, 2, 4, 9, 21, 51, 127, \dots$ first appeared in [6] in a circle chording setting, which count the number of ways of connecting a subset of n points on a circle by nonintersecting chords. The Motzkin number M_n also counts the number of lattice paths on the upper right quadrant of a grid from $(0, 0)$ to $(n, 0)$ if one is allowed to move by using only steps $(1, 1)$, $(1, 0)$ and $(1, -1)$ but forbidden from dipping below the $y = 0$ axis.

The Motzkin numbers are named after Theodore Motzkin and naturally appear in various combinatorial objects. Fourteen different manifestations of Motzkin numbers in different branches of mathematics were enumerated by Donaghey and Shapiro [2] in



their survey of Motzkin numbers. The interested readers may refer to [11] for further information on Motzkin numbers.

The famous Catalan numbers $C_n = \binom{2n}{n}/(n+1)$ are closely related to Motzkin numbers. The Motzkin numbers can be expressed in terms of Catalan numbers:

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k,$$

and inversely,

$$C_{n+1} = \sum_{k=0}^n \binom{n}{k} M_k,$$

where $\lfloor x \rfloor$ denotes the integral part of real x .

Another sequence closely related to Motzkin numbers are the central trinomial coefficients. For $n \in \mathbb{N}$, the central trinomial coefficient T_n is given by the constant term in the expansion of $(1 + x + x^{-1})^n$, which can be expressed in terms of binomial coefficients:

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}.$$

Note that $\{T_n\}_{n=0}^\infty = 1, 1, 3, 7, 19, 51, 141, 393, \dots$. We remark that the central trinomial coefficient T_n counts the number of lattice paths from $(0, 0)$ to $(n, 0)$ if one is allowed to move by using only steps $(1, 1)$, $(1, 0)$ and $(1, -1)$.

Although Catalan numbers, Motzkin numbers and central trinomial coefficients naturally arise in combinatorics, they also possess rich arithmetic properties.

Throughout the paper, let $p \geq 5$ be a prime. Sun and Tauraso [16] showed that

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2},$$

where $(-)$ denotes the Legendre symbol.

In 2014, Sun [12] proved that

$$\begin{aligned} \sum_{k=0}^{p-1} T_k^2 &\equiv \left(\frac{-1}{p}\right) \pmod{p}, \\ \sum_{k=0}^{p-1} T_k M_k &\equiv \frac{4}{3} \left(\frac{p}{3}\right) \pmod{p}. \end{aligned}$$

By the telescoping method for double summations developed by Chen, Hou and Mu [1], Sun [14] recently established the following interesting supercongruence:

$$\sum_{k=0}^{p-1} (2k + 1) M_k^2 \equiv 12p \left(\frac{p}{3}\right) \pmod{p^2}. \tag{1.1}$$

The motivation of the paper is the following three conjectural supercongruences of Sun [12, Conjecture 1.1]:

$$\sum_{k=0}^{p-1} M_k^2 \equiv \left(\frac{p}{3}\right) (2 - 6p) \pmod{p^2}, \tag{1.2}$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv \left(\frac{p}{3}\right) (9p - 1) \pmod{p^2}, \tag{1.3}$$

$$\sum_{k=0}^{p-1} T_k M_k \equiv \frac{4}{3} \left(\frac{p}{3}\right) + \frac{p}{6} \left(1 - 9 \left(\frac{p}{3}\right)\right) \pmod{p^2}. \tag{1.4}$$

‘The three supercongruences look curious and challenging’, as described by Sun [14] in his recent paper. Although the three supercongruence conjectures were officially announced by Sun [12] in 2014, they first appeared in arXiv version of Sun’s paper in 2010 (see <https://arxiv.org/abs/1008.3887>).

Note that (1.3) follows immediately from (1.1) and (1.2). In this paper, we aim to prove (1.2) and (1.4).

Theorem 1.1. *The supercongruences (1.2)–(1.4) are true.*

We remark that both Motzkin numbers and central trinomial coefficients have many different expressions (see A001006 and A002426 in [10]). The following two expressions will be used in the proof of Theorem 1.1:

$$M_n = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} C_{k+1}, \tag{1.5}$$

$$T_n = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \binom{2k}{k}. \tag{1.6}$$

The rest of the paper is organized as follows. Section 2 is devoted to proving some preliminary results. In § 3, we establish three congruences for triple sums which play an important role in the proof of Theorem 1.1. We show (1.2) in § 4, whereas (1.4) is proved in § 5.

2. Preliminaries

Let $p \geq 5$ be a prime. In the proof of Theorem 1.1, we will frequently use Wolstenholme’s theorem [19]:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}, \tag{2.1}$$

which is equivalent to

$$\binom{2p}{p} \equiv 2 \pmod{p^3}, \tag{2.2}$$

and Lehmer’s congruences [5]:

$$H_{\lfloor p/6 \rfloor} \equiv H_{\lfloor 5p/6 \rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod{p}, \tag{2.3}$$

$$H_{\lfloor p/3 \rfloor} \equiv H_{\lfloor 2p/3 \rfloor} \equiv -\frac{3}{2}q_p(3) \pmod{p}, \tag{2.4}$$

$$H_{\lfloor p/2 \rfloor} \equiv -2q_p(2) \pmod{p}, \tag{2.5}$$

where $q_p(a)$ denotes the Fermat quotient $(a^{p-1} - 1)/p$.

In addition, we require some congruences related to central binomial coefficients and Catalan numbers.

Lemma 2.1. *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \tag{2.6}$$

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2}, \tag{2.7}$$

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv 0 \pmod{p^2}, \tag{2.8}$$

$$\sum_{k=0}^{p-1} k \binom{2k}{k} \equiv \frac{2}{3} \left(p - \left(\frac{p}{3}\right)\right) \pmod{p^2}, \tag{2.9}$$

$$\sum_{k=0}^{p-2} \frac{\binom{2k}{k}}{(k+1)^2} \equiv 3 \left(\frac{p}{3}\right) + 1 \pmod{p}. \tag{2.10}$$

Proof. Congruences (2.6)–(2.8) were proved by Sun and Tauraso (see [16, (1.7) and (1.9)] and [15, Theorem 1.3]). It is easily proved by induction on n that

$$\sum_{k=0}^{n-1} (3k+2) \binom{2k}{k} = n \binom{2n}{n}. \tag{2.11}$$

Letting $n = p$ in (2.11) and using (2.2) gives

$$\sum_{k=0}^{p-1} (3k+2) \binom{2k}{k} \equiv 2p \pmod{p^4}. \tag{2.12}$$

Then the proof of (2.9) follows from (2.6) and (2.12).

From the following identity:

$$\binom{2k}{k+1} = \frac{1}{2} \binom{2k+2}{k+1} - \binom{2k}{k},$$

we deduce that

$$\begin{aligned} \sum_{k=0}^{p-2} \frac{\binom{2k}{k+1}}{k+1} &= \frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} - \sum_{k=0}^{p-2} C_k \\ &\equiv \frac{1}{2} - \frac{3}{2} \left(\frac{p}{3}\right) + C_{p-1} \\ &\equiv -\frac{1}{2} - \frac{3}{2} \left(\frac{p}{3}\right) \pmod{p}, \end{aligned} \tag{2.13}$$

where we have used (2.7) and (2.8).

By (2.7), (2.13) and the identity $C_k = \binom{2k}{k} - \binom{2k}{k+1}$, we have

$$\begin{aligned} \sum_{k=0}^{p-2} \frac{\binom{2k}{k}}{(k+1)^2} &= \sum_{k=0}^{p-2} C_k - \sum_{k=0}^{p-2} \frac{\binom{2k}{k+1}}{k+1} \\ &\equiv 3 \left(\frac{p}{3}\right) + 1 \pmod{p}, \end{aligned}$$

as desired. □

We also need some congruences involving harmonic numbers $H_n = \sum_{k=1}^n 1/k$.

Lemma 2.2 *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} \binom{2k}{k} H_k \equiv -\left(\frac{p}{3}\right) q_p(3) \pmod{p}, \tag{2.14}$$

$$\sum_{k=0}^{p-1} k \binom{2k}{k} H_k \equiv \frac{1}{3} \left(\frac{p}{3}\right) \left(\left(\frac{p}{3}\right) + 2q_p(3) - 1\right) \pmod{p}, \tag{2.15}$$

$$\sum_{k=0}^{p-1} C_k H_k \equiv -\frac{3}{2} \left(\frac{p}{3}\right) q_p(3) \pmod{p}. \tag{2.16}$$

Proof. Congruence (2.14) is a known result. We begin with the following congruence due to Tauraso [17, (9)]:

$$\sum_{k=0}^{p-1} \binom{2k}{k} H_k x^k \equiv -(1-4x)^{\frac{p-1}{2}} \sum_{j=1}^{p-1} \binom{2j}{j} \frac{y^j}{j} \pmod{p}, \tag{2.17}$$

where $y = -x/(1 - 4x)$. Letting $x = 1$ in (2.17) and using the congruence [15, (1.13)], we arrive at

$$\sum_{k=0}^{p-1} \binom{2k}{k} H_k \equiv -(-3)^{\frac{p-1}{2}} \sum_{j=1}^{p-1} \binom{2j}{j} \frac{1}{j3^j} \equiv -\left(\frac{p}{3}\right) q_p(3) \pmod{p},$$

which is (2.14).

By using the software package **Sigma** developed by Schneider [9], we can automatically discover and prove the following three combinatorial identities:

$$\sum_{k=0}^n (-4)^k \binom{n}{k} H_k = (-3)^n \left(H_n - \sum_{k=1}^n \frac{1}{k(-3)^k} \right), \tag{2.18}$$

$$\sum_{k=0}^n (-4)^k k \binom{n}{k} H_k = \frac{1 - (-3)^n}{3} + \frac{4n(-3)^n}{3} \left(H_n - \sum_{k=1}^n \frac{1}{k(-3)^k} \right), \tag{2.19}$$

and

$$\begin{aligned} & \sum_{k=0}^n \frac{(-4)^k}{k+1} \binom{n}{k} H_k \\ &= \frac{1}{4(n+1)} \left((-1 + 3(-3)^n) H_n - 3(-3)^n \sum_{k=1}^n \frac{1}{k(-3)^k} + \sum_{k=1}^n \frac{(-3)^k}{k} \right). \end{aligned} \tag{2.20}$$

By (2.18), we can rewrite (2.19) and (2.20) as

$$\sum_{k=0}^n (-4)^k k \binom{n}{k} H_k = \frac{1 - (-3)^n}{3} + \frac{4n}{3} \sum_{k=0}^n (-4)^k \binom{n}{k} H_k, \tag{2.21}$$

and

$$\begin{aligned} & \sum_{k=0}^n \frac{(-4)^k}{k+1} \binom{n}{k} H_k \\ &= \frac{1}{4(n+1)} \left(\frac{3((-3)^n + 1)}{(-3)^n} \sum_{k=0}^n (-4)^k \binom{n}{k} H_k \right) \\ &+ \frac{1}{4(n+1)} \left(3 \sum_{k=1}^n \frac{1}{k(-3)^k} + \sum_{k=1}^n \frac{(-3)^k}{k} - 4H_n \right). \end{aligned} \tag{2.22}$$

Using Fermat’s little theorem, we obtain

$$\sum_{k=1}^{(p-1)/2} \frac{(-3)^k}{k} + 3 \sum_{k=1}^{(p-1)/2} \frac{1}{k(-3)^k}$$

$$\begin{aligned} &\equiv \sum_{k=1}^{(p-1)/2} \frac{(-3)^k}{k} + \sum_{k=1}^{(p-1)/2} \frac{(-3)^{p-k}}{p-k} \\ &= \sum_{k=1}^{p-1} \frac{(-3)^k}{k} \pmod{p}. \end{aligned}$$

From Granville’s congruence [4]:

$$\sum_{j=1}^{p-1} \frac{x^j}{j} \equiv \frac{1 - x^p + (x - 1)^p}{p} \pmod{p},$$

we deduce that

$$\sum_{k=1}^{p-1} \frac{(-3)^k}{k} \equiv 3q_p(3) - 8q_p(2) \pmod{p},$$

which was also mentioned in the proof of [13, Lemma 3.3]. Thus,

$$\sum_{k=1}^{(p-1)/2} \frac{(-3)^k}{k} + 3 \sum_{k=1}^{(p-1)/2} \frac{1}{k(-3)^k} \equiv 3q_p(3) - 8q_p(2) \pmod{p}. \tag{2.23}$$

Finally, letting $n = (p - 1)/2$ in (2.21)–(2.22) and using (2.5), (2.14), (2.23) and the facts that $H_{p-1} \equiv 0 \pmod{p^2}$ and

$$\begin{aligned} (-3)^{(p-1)/2} &\equiv \left(\frac{p}{3}\right) \pmod{p}, \\ \binom{(p-1)/2}{k} &\equiv \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}, \\ \binom{2k}{k} &\equiv 0 \pmod{p} \text{ for } (p+1)/2 \leq k \leq p-1, \end{aligned}$$

we arrive at the desired congruences (2.15) and (2.16). □

For an assertion A , we set

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

The following two known congruences play important roles in the proof of Theorem 1.1 (see [7, Theorem 1.2]).

Lemma 2.3. *Let $p \geq 5$ be a prime. For $1 \leq k \leq p$, we have*

$$\sum_{i=0}^{p-1} \binom{2i}{i+k} \equiv \binom{p-k}{3} \pmod{p}, \tag{2.24}$$

$$\sum_{i=1}^{p-1} \frac{\binom{2i}{i+k}}{i} \equiv \frac{\alpha(k) - 1}{k} \pmod{p}, \tag{2.25}$$

where $\alpha(k) = 2(-1)^k + 3[3 \mid p - k]$.

Based on Lemma 2.3, we establish the following result which seems to be crude but useful in the proof of Theorem 1.1.

Lemma 2.4. *Let $p \geq 5$ be a prime. For $1 \leq k \leq p$, we have*

$$(-1)^k \sum_{i=1}^{p-2} \frac{\binom{2i}{i+k}}{i+1} \equiv 2k + \frac{3}{2} \binom{p}{3} - \frac{3}{2} + \sum_{i=1}^{k-1} \beta(i) - k \sum_{i=1}^{k-1} \frac{\beta(i) + 2}{i} \pmod{p}, \tag{2.26}$$

where $\beta(i) = (-1)^i(3[3 \mid p - i] - 1)$.

Proof. By Pascal’s formula $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$, we have

$$\begin{aligned} \binom{2i+2}{i+1+k} &= \binom{2i+1}{i+1+k} + \binom{2i+1}{i+k} \\ &= \binom{2i}{i+1+k} + 2\binom{2i}{i+k} + \binom{2i}{i+k-1}. \end{aligned}$$

It follows that

$$\sum_{i=0}^{p-2} \frac{\binom{2i}{i+1+k}}{i+1} + 2 \sum_{i=0}^{p-2} \frac{\binom{2i}{i+k}}{i+1} + \sum_{i=0}^{p-2} \frac{\binom{2i}{i+k-1}}{i+1} = \sum_{i=1}^{p-1} \frac{\binom{2i}{i+k}}{i}. \tag{2.27}$$

Let

$$\begin{aligned} f(k) &= (-1)^k \sum_{i=0}^{p-2} \frac{\binom{2i}{i+k}}{i+1}, \\ g(k) &= (-1)^{k+1} \sum_{i=1}^{p-1} \frac{\binom{2i}{i+k}}{i}, \\ F(k) &= f(k+1) - f(k). \end{aligned}$$

We rewrite (2.27) as

$$F(k) - F(k-1) = g(k). \tag{2.28}$$

From (2.28), we deduce that

$$F(k) = f(k + 1) - f(k) = \sum_{j=1}^k g(j) + F(0),$$

and so

$$\begin{aligned} f(k) &= \sum_{l=1}^{k-1} \sum_{j=1}^l g(j) + kF(0) + f(0) \\ &= \sum_{j=1}^{k-1} (k - j)g(j) + kf(1) + (1 - k)f(0). \end{aligned} \tag{2.29}$$

By $\binom{2i}{i+1} = \binom{2i}{i} - \binom{2i}{i}/(i + 1)$, we have

$$\begin{aligned} &kf(1) + (1 - k)f(0) \\ &= -k \sum_{i=0}^{p-2} \frac{\binom{2i}{i+1}}{i + 1} + (1 - k) \sum_{i=0}^{p-2} \frac{\binom{2i}{i}}{i + 1} \\ &= (1 - 2k) \sum_{i=0}^{p-2} C_i + k \sum_{i=0}^{p-2} \frac{\binom{2i}{i}}{(i + 1)^2} \\ &\equiv \frac{3}{2} \left(\frac{p}{3}\right) + \frac{1}{2} \pmod{p}, \end{aligned} \tag{2.30}$$

where we have used (2.7) and (2.10) in the last step.

Combining (2.25), (2.29) and (2.30) gives

$$\begin{aligned} &(-1)^k \sum_{i=1}^{p-2} \frac{\binom{2i}{i+k}}{i + 1} \\ &\equiv -k \sum_{i=1}^{k-1} \frac{(-1)^i (3[3 \mid p - i] - 1) + 2}{i} + \sum_{i=1}^{k-1} (-1)^i (3[3 \mid p - i] - 1) + 2k + \frac{3}{2} \left(\frac{p}{3}\right) - \frac{3}{2} \\ &= 2k + \frac{3}{2} \left(\frac{p}{3}\right) - \frac{3}{2} + \sum_{i=1}^{k-1} \beta(i) - k \sum_{i=1}^{k-1} \frac{\beta(i) + 2}{i} \pmod{p}, \end{aligned}$$

as desired. □

3. Three key triple sums

The main idea in the proof of Theorem 1.1 is to translate the left-hand sides of (1.2) and (1.4) into three triple sums, which can be determined modulo p by using Lemmas 2.3 and 2.4. The three congruences for triple sums are stated as follows.

Lemma 3.1. *For any prime $p \geq 5$, we have*

$$\sum_{k=1}^{p-2} \sum_{j=1}^{p-2} \sum_{i=1}^{p-1} \frac{(-1)^k}{i(j+1)} \binom{2j}{j+k} \binom{2i}{i+k} \equiv \left(\frac{p}{3}\right) (1 - q_p(3)) - 1 \pmod{p}. \tag{3.1}$$

Proof. Substituting (2.25) and (2.26) into the left-hand side of (3.1) gives

$$\begin{aligned} & \sum_{k=1}^{p-2} (-1)^k \sum_{j=1}^{p-2} \frac{1}{j+1} \binom{2j}{j+k} \sum_{i=1}^{p-1} \frac{1}{i} \binom{2i}{i+k} \\ & \equiv \sum_{k=1}^{p-2} (1 - \alpha(k)) \sum_{i=1}^{k-1} \frac{\beta(i) + 2}{i} + \sum_{k=1}^{p-2} \frac{\alpha(k) - 1}{k} \sum_{i=1}^{k-1} \beta(i) \\ & + 2 \sum_{k=1}^{p-2} (\alpha(k) - 1) + \left(\frac{3}{2} \left(\frac{p}{3}\right) - \frac{3}{2}\right) \sum_{k=1}^{p-2} \frac{\alpha(k) - 1}{k} \pmod{p}. \end{aligned} \tag{3.2}$$

We shall only prove the case $p \equiv 1 \pmod{3}$ of (3.1). The proof of the case $p \equiv 2 \pmod{3}$ runs analogously, and we omit the details.

Suppose that $p \equiv 1 \pmod{3}$. Let $n = \lfloor p/6 \rfloor$. Then $n \equiv -1/6 \pmod{p}$. Since the sequences $\{\alpha(k)\}_{k \in \mathbb{N}}$ and $\{\beta(k)\}_{k \in \mathbb{N}}$ both have a period of 6, it is easy to check that

$$\Omega(k) = \sum_{i=1}^k (\alpha(i) - 1) = \begin{cases} 0 & \text{for } k \equiv 0, 1 \pmod{6}, \\ 1 & \text{for } k \equiv 2 \pmod{6}, \\ -2 & \text{for } k \equiv 3 \pmod{6}, \\ 2 & \text{for } k \equiv 4 \pmod{6}, \\ -1 & \text{for } k \equiv 5 \pmod{6}, \end{cases}$$

and

$$(\alpha(k) - 1) \sum_{i=1}^{k-1} \beta(i) = \begin{cases} 1 & \text{for } k \equiv 0 \pmod{6}, \\ 0 & \text{for } k \equiv 1, 5 \pmod{6}, \\ -2 & \text{for } k \equiv 2 \pmod{6}, \\ 9 & \text{for } k \equiv 3 \pmod{6}, \\ -8 & \text{for } k \equiv 4 \pmod{6}. \end{cases}$$

It follows that

$$\Omega(p - 2) = -1, \tag{3.3}$$

and

$$\sum_{k=1}^{p-2} \frac{\alpha(k) - 1}{k} \sum_{i=1}^{k-1} \beta(i) = \sum_{k=1}^{n-1} \frac{1}{6k} - 2 \sum_{k=1}^n \frac{1}{6k - 4} + 9 \sum_{k=1}^n \frac{1}{6k - 3} - 8 \sum_{k=1}^n \frac{1}{6k - 2}. \tag{3.4}$$

Furthermore, we have

$$\begin{aligned} & \sum_{k=1}^{p-2} (1 - \alpha(k)) \sum_{i=1}^{k-1} \frac{\beta(i) + 2}{i} \\ &= - \sum_{i=1}^{p-3} \frac{\beta(i) + 2}{i} \sum_{k=i+1}^{p-2} (\alpha(k) - 1) \\ &= - \sum_{i=1}^{p-3} \frac{(\beta(i) + 2) (\Omega(p - 2) - \Omega(i))}{i} \\ &= \sum_{i=1}^{p-3} \frac{(\beta(i) + 2) (1 + \Omega(i))}{i}, \end{aligned}$$

where we have used (3.3) in the last step. It is also easy to check that

$$(\beta(i) + 2) (1 + \Omega(i)) = \begin{cases} 1 & \text{for } i \equiv 0 \pmod{6}, \\ 0 & \text{for } i \equiv 1, 5 \pmod{6}, \\ 2 & \text{for } i \equiv 2 \pmod{6}, \\ -3 & \text{for } i \equiv 3 \pmod{6}, \\ 12 & \text{for } i \equiv 4 \pmod{6}, \end{cases}$$

and so

$$\begin{aligned} & \sum_{k=1}^{p-2} (1 - \alpha(k)) \sum_{i=1}^{k-1} \frac{\beta(i) + 2}{i} \\ &= \sum_{k=1}^{n-1} \frac{1}{6k} + 12 \sum_{k=1}^n \frac{1}{6k - 2} - 3 \sum_{k=1}^n \frac{1}{6k - 3} + 2 \sum_{k=1}^n \frac{1}{6k - 4}. \end{aligned} \tag{3.5}$$

Substituting (3.3)–(3.5) into the right-hand side of (3.2) gives

$$\begin{aligned}
 & \sum_{k=1}^{p-2} \sum_{j=1}^{p-2} \sum_{i=1}^{p-1} \frac{(-1)^k}{i(j+1)} \binom{2j}{j+k} \binom{2i}{i+k} \\
 & \equiv 2 \sum_{k=1}^{n-1} \frac{1}{6k} + 4 \sum_{k=1}^n \frac{1}{6k-2} + 6 \sum_{k=1}^n \frac{1}{6k-3} - 2 \\
 & \equiv \frac{1}{3} \sum_{k=1}^n \frac{1}{k} + \frac{2}{3} \sum_{k=1}^n \frac{1}{k+2n} + \sum_{k=1}^n \frac{1}{k+3n} \\
 & = \frac{1}{3}H_n - \frac{2}{3}H_{2n} - \frac{1}{3}H_{3n} + H_{4n} \pmod{p}. \tag{3.6}
 \end{aligned}$$

Finally, noting that for $p \equiv 1 \pmod{3}$, $2n = \lfloor p/3 \rfloor, 3n = \lfloor p/2 \rfloor, 4n = \lfloor 2p/3 \rfloor$ and applying (2.3)–(2.5) to the right-hand side of (3.6), we arrive at the desired result:

$$\sum_{k=1}^{p-2} \sum_{j=1}^{p-2} \sum_{i=1}^{p-1} \frac{(-1)^k}{i(j+1)} \binom{2j}{j+k} \binom{2i}{i+k} \equiv -q_p(3) \pmod{p},$$

which is the case $p \equiv 1 \pmod{3}$ of (3.1). □

Lemma 3.2. *For any prime $p \geq 5$, we have*

$$\sum_{k=1}^{p-1} \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{(-1)^k}{i} \binom{2i}{i+k} \binom{2j}{j+k} \equiv -\left(\frac{p}{3}\right) q_p(3) \pmod{p}. \tag{3.7}$$

Proof. By (2.24) and (2.25), we have

$$\sum_{k=1}^{p-1} (-1)^k \sum_{i=1}^{p-1} \frac{1}{i} \binom{2i}{i+k} \sum_{j=0}^{p-1} \binom{2j}{j+k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^k (\alpha(k) - 1)}{k} \left(\frac{p-k}{3}\right) \pmod{p}. \tag{3.8}$$

We shall only prove the case $p \equiv 1 \pmod{3}$, and the proof of the case $p \equiv 2 \pmod{3}$ runs similarly and the details are omitted.

Suppose that $p \equiv 1 \pmod{3}$. Let $n = \lfloor p/6 \rfloor$. Then $n \equiv -1/6 \pmod{p}$. It is easy to check that

$$(-1)^k (\alpha(k) - 1) \left(\frac{p-k}{3}\right) = \begin{cases} 1 & \text{for } k \equiv 0 \pmod{6}, \\ 0 & \text{for } k \equiv 1, 4 \pmod{6}, \\ -1 & \text{for } k \equiv 2 \pmod{6}, \\ 3 & \text{for } k \equiv 3 \pmod{6}, \\ -3 & \text{for } k \equiv 5 \pmod{6}. \end{cases}$$

It follows that

$$\begin{aligned}
 & \sum_{k=1}^{p-1} \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{(-1)^k}{i} \binom{2i}{i+k} \binom{2j}{j+k} \\
 & \equiv \sum_{k=1}^n \frac{1}{6k} - \sum_{k=1}^n \frac{1}{6k-4} + 3 \sum_{k=1}^n \frac{1}{6k-3} - 3 \sum_{k=1}^n \frac{1}{6k-1} \\
 & \equiv \frac{1}{6} \left(\sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^n \frac{1}{k+4n} + 3 \sum_{k=1}^n \frac{1}{k+3n} - 3 \sum_{k=1}^n \frac{1}{k+n} \right) \\
 & = \frac{1}{6} (4H_n - 3H_{2n} - 3H_{3n} + 4H_{4n} - H_{5n}) \pmod{p}. \tag{3.9}
 \end{aligned}$$

Finally, noting that for $p \equiv 1 \pmod{3}$, $2n = \lfloor p/3 \rfloor, 3n = \lfloor p/2 \rfloor, 4n = \lfloor 2p/3 \rfloor, 5n = \lfloor 5p/6 \rfloor$ and applying (2.3)–(2.5) to the right-hand side of (3.9), we obtain

$$\sum_{k=1}^{p-1} \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{(-1)^k}{i} \binom{2i}{i+k} \binom{2j}{j+k} \equiv -q_p(3) \pmod{p},$$

which is the case $p \equiv 1 \pmod{3}$ of (3.7). □

Lemma 3.3. *For any prime $p \geq 5$, we have*

$$\sum_{k=1}^{p-1} \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{(-1)^k}{i+1} \binom{2i}{i+k} \binom{2j}{j+k} \equiv -\frac{1}{3} \left(2 + \left(\frac{p}{3} \right) q_p(3) \right) \pmod{p}. \tag{3.10}$$

Proof. By (2.24) and (2.26), we have

$$\begin{aligned}
 & \sum_{k=1}^{p-1} (-1)^k \sum_{i=1}^{p-2} \frac{1}{i+1} \binom{2i}{i+k} \sum_{j=0}^{p-1} \binom{2j}{j+k} \\
 & \equiv \sum_{k=1}^{p-1} \left(-k \sum_{i=1}^{k-1} \frac{\beta(i)+2}{i} + \sum_{i=1}^{k-1} \beta(i) + 2k + \frac{3}{2} \left(\frac{p}{3} \right) - \frac{3}{2} \right) \left(\frac{p-k}{3} \right) \\
 & = - \sum_{k=1}^{p-1} k \left(\frac{p-k}{3} \right) \sum_{i=1}^{k-1} \frac{\beta(i)+2}{i} + \sum_{k=1}^{p-1} \left(\frac{p-k}{3} \right) \sum_{i=1}^{k-1} \beta(i) \\
 & + 2 \sum_{k=1}^{p-1} k \left(\frac{p-k}{3} \right) + \left(\frac{3}{2} \left(\frac{p}{3} \right) - \frac{3}{2} \right) \sum_{k=1}^{p-1} \left(\frac{p-k}{3} \right) \pmod{p}. \tag{3.11}
 \end{aligned}$$

Similarly, we only prove the case $p \equiv 1 \pmod{3}$. Suppose that $p \equiv 1 \pmod{3}$. It is easy to check that

$$k \left(\frac{p-k}{3} \right) = \begin{cases} k & \text{for } k \equiv 0 \pmod{3}, \\ 0 & \text{for } k \equiv 1 \pmod{3}, \\ -k & \text{for } k \equiv 2 \pmod{3}, \end{cases}$$

and

$$\left(\frac{p-k}{3} \right) \sum_{i=1}^{k-1} \beta(i) = \begin{cases} 1 & \text{for } k \equiv 0 \pmod{6}, \\ 0 & \text{for } k \equiv 1, 4, 5 \pmod{6}, \\ 2 & \text{for } k \equiv 2 \pmod{6}, \\ -3 & \text{for } k \equiv 3 \pmod{6}. \end{cases}$$

It follows that

$$\begin{aligned} \Psi(m) &= \sum_{k=1}^m k \left(\frac{p-k}{3} \right) \\ &= \sum_{k=1}^{\lfloor m/3 \rfloor} 3k - \sum_{k=1}^{\lfloor (m+1)/3 \rfloor} (3k-1) \\ &= \left\lfloor \frac{m+1}{3} \right\rfloor - (m+1)[3 \mid m-2], \end{aligned} \tag{3.12}$$

and

$$\sum_{k=1}^{p-1} \left(\frac{p-k}{3} \right) \sum_{i=1}^{k-1} \beta(i) = 0. \tag{3.13}$$

From (3.12), we deduce that

$$\Psi(p-1) \equiv -\frac{1}{3} \pmod{p}. \tag{3.14}$$

Furthermore, we have

$$\begin{aligned} &\sum_{k=1}^{p-1} k \left(\frac{p-k}{3} \right) \sum_{i=1}^{k-1} \frac{\beta(i)+2}{i} \\ &= \sum_{i=1}^{p-2} \frac{\beta(i)+2}{i} \sum_{k=i+1}^{p-1} k \left(\frac{p-k}{3} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^{p-2} \frac{\beta(i) + 2}{i} (\Psi(p-1) - \Psi(i)) \\
 &\equiv - \sum_{i=1}^{p-2} \frac{\beta(i) + 2}{i} \left(\frac{1}{3} + \Psi(i) \right) \pmod{p},
 \end{aligned} \tag{3.15}$$

where we have used (3.14) in the last step. It is easy to check that

$$(\beta(i) + 2) \left(\frac{1}{3} + \Psi(i) \right) = \begin{cases} \frac{i+1}{3} & \text{for } i \equiv 0 \pmod{6}, \\ 0 & \text{for } i \equiv 1 \pmod{6}, \\ -\frac{2i+1}{3} & \text{for } i \equiv 2 \pmod{6}, \\ i + 1 & \text{for } i \equiv 3 \pmod{6}, \\ \frac{4i}{3} & \text{for } i \equiv 4 \pmod{6}, \\ -(2i + 1) & \text{for } i \equiv 5 \pmod{6}. \end{cases} \tag{3.16}$$

Combining (3.15) and (3.16) yields

$$\begin{aligned}
 &\sum_{k=1}^{p-1} k \left(\frac{p-k}{3} \right) \sum_{i=1}^{k-1} \frac{\beta(i) + 2}{i} \\
 &\equiv -\frac{1}{3} \sum_{i=1}^{n-1} \frac{6i+1}{6i} + \frac{1}{3} \sum_{i=1}^n \frac{12i-7}{6i-4} - \sum_{i=1}^n \frac{6i-2}{6i-3} - \frac{1}{3} \sum_{i=1}^n 4 + \sum_{i=1}^n \frac{12i-1}{6i-1} \\
 &= -\frac{1}{3} \sum_{i=1}^n \frac{1}{6i} + \frac{1}{3} \sum_{i=1}^n \frac{1}{6i-4} - \sum_{i=1}^n \frac{1}{6i-3} + \sum_{i=1}^n \frac{1}{6i-1} + \frac{1}{3} + \frac{1}{18n} \\
 &\equiv -\frac{1}{18} \sum_{i=1}^n \frac{1}{i} + \frac{1}{18} \sum_{i=1}^n \frac{1}{i+4n} - \frac{1}{6} \sum_{i=1}^n \frac{1}{i+3n} + \frac{1}{6} \sum_{i=1}^n \frac{1}{i+n} \\
 &= \frac{1}{6} \left(-\frac{4}{3}H_n + H_{2n} + H_{3n} - \frac{4}{3}H_{4n} + \frac{1}{3}H_{5n} \right) \\
 &\equiv \frac{1}{3}q_p(3) \pmod{p},
 \end{aligned} \tag{3.17}$$

where we have used the fact that $2n = \lfloor p/3 \rfloor, 3n = \lfloor p/2 \rfloor, 4n = \lfloor 2p/3 \rfloor, 5n = \lfloor 5p/6 \rfloor$ for $p \equiv 1 \pmod{3}$ and (2.3)–(2.5) in the last step.

Finally, substituting (3.13), (3.14) and (3.17) into the right-hand side of (3.11) gives

$$\sum_{k=1}^{p-1} \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{(-1)^k}{i+1} \binom{2i}{i+k} \binom{2j}{j+k} \equiv -\frac{1}{3} (q_p(3) + 2) \pmod{p},$$

which is the case $p \equiv 1 \pmod{3}$ of (3.10). □

4. Proof of (1.2)

By (1.5), we have

$$\begin{aligned} \sum_{k=0}^{p-1} M_k^2 &= \sum_{k=0}^{p-1} \sum_{i=0}^k \sum_{j=0}^k (-1)^{i+j} \binom{k}{i} \binom{k}{j} C_{i+1} C_{j+1} \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} C_{i+1} C_{j+1} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j}. \end{aligned} \tag{4.1}$$

From the identity [8, (9), page 15]:

$$\binom{k}{i} \binom{k}{j} = \sum_{l=0}^j \binom{l+i}{j} \binom{j}{l} \binom{k}{l+i},$$

we deduce that

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} &= \sum_{l=0}^j \binom{l+i}{j} \binom{j}{l} \sum_{k=0}^{p-1} \binom{k}{l+i} \\ &= \sum_{l=0}^j \binom{i+l}{j} \binom{j}{l} \binom{p}{i+l+1}, \end{aligned} \tag{4.2}$$

where we have utilized the hockey-stick identity in the last step. Letting $l \rightarrow k - i$ on the right-hand side of (4.2), we rewrite (4.2) as

$$\sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} = p \sum_{k=0}^{i+j} \frac{1}{k+1} \binom{k}{j} \binom{j}{k-i} \binom{p-1}{k}.$$

Note that $\binom{k}{j} \binom{j}{k-i} \equiv 0 \pmod{p}$ for $0 \leq i \leq p-1, 0 \leq j \leq p-1$ and $p \leq k \leq 2p-2$, and

$$\frac{p}{k+1} \binom{p-1}{k} \equiv \frac{p(-1)^k}{k+1} \pmod{p^2}$$

for $0 \leq k \leq p-1$. It follows that

$$\sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} \equiv p \sum_{k=0}^{i+j} \frac{(-1)^k}{k+1} \binom{k}{j} \binom{j}{k-i} \pmod{p^2}. \tag{4.3}$$

Recall the following partial fraction decomposition:

$$\sum_{k=0}^{i+j} \frac{(-1)^{i+j+k}}{x+k} \binom{k}{j} \binom{j}{k-i} = \frac{(x)_i (x)_j}{(x)_{i+j+1}}, \tag{4.4}$$

where $(x)_0 = 1$ and $(x)_k = x(x + 1) \cdots (x + k - 1)$ for $k \geq 1$. Letting $x = 1$ in (4.4) gives

$$\sum_{k=0}^{i+j} \frac{(-1)^k}{k+1} \binom{k}{j} \binom{j}{k-i} = \frac{(-1)^{i+j}}{(i+j+1) \binom{i+j}{j}}. \tag{4.5}$$

It follows from (4.3) and (4.5) that

$$\sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j} \equiv \frac{p(-1)^{i+j}}{(i+j+1) \binom{i+j}{j}} \pmod{p^2}. \tag{4.6}$$

Combining (4.1) and (4.6) gives

$$\sum_{k=0}^{p-1} M_k^2 \equiv p \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{C_{i+1} C_{j+1}}{(i+j+1) \binom{i+j}{j}} \pmod{p^2}. \tag{4.7}$$

Let

$$S(i, j) = \frac{\binom{2i}{i} \binom{2j}{j}}{\binom{i+j}{i}}.$$

From the identity due to Von Szily [18] (see also [3, (3.38)]):

$$S(i, j) = \sum_k (-1)^k \binom{2i}{i+k} \binom{2j}{j+k}, \tag{4.8}$$

we deduce that $S(i, j)$ is an integer. It is easy to verify that the numbers $S(i, j)$ satisfy the recurrence:

$$4S(i, j) = S(i + 1, j) + S(i, j + 1). \tag{4.9}$$

Note that

$$\frac{C_{i+1} C_{j+1}}{(i+j+1) \binom{i+j}{j}} = \frac{(i+j+2)S(i+1, j+1)}{(i+1)(j+1)(i+2)(j+2)}. \tag{4.10}$$

It follows from (4.7) and (4.10) that

$$\begin{aligned} \sum_{k=0}^{p-1} M_k^2 &\equiv p \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{(i+j+2)S(i+1, j+1)}{(i+1)(j+1)(i+2)(j+2)} \\ &= p \sum_{i=1}^p \sum_{j=1}^p \frac{(i+j)S(i, j)}{ij(i+1)(j+1)} \end{aligned}$$

$$\begin{aligned}
 &= 2p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(i+1)(j+1)} \\
 &= 2p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} - 2p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{(i+1)(j+1)} \pmod{p^2}, \tag{4.11}
 \end{aligned}$$

where we have used the symmetry with respect to i and j in the third step.

By (4.9), we have

$$\begin{aligned}
 &2p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{(i+1)(j+1)} \\
 &= \frac{p}{2} \sum_{i=1}^p \sum_{j=1}^p \frac{S(i+1, j) + S(i, j+1)}{(i+1)(j+1)} \\
 &= p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i+1, j)}{(i+1)(j+1)} \\
 &= p \sum_{i=2}^{p+1} \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} \\
 &= p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} + p \sum_{j=1}^p \frac{S(p+1, j)}{(p+1)(j+1)} - p \sum_{j=1}^p \frac{S(1, j)}{j+1}. \tag{4.12}
 \end{aligned}$$

Furthermore, by (2.2), we have

$$\begin{aligned}
 &p \sum_{j=1}^p \frac{S(p+1, j)}{(p+1)(j+1)} \\
 &= \frac{2p(2p+1)}{(p+1)^2} \binom{2p}{p} \sum_{j=1}^p \frac{\binom{2j}{j}}{(j+1)\binom{p+1+j}{j}} \\
 &\equiv \frac{4p(2p+1)}{(p+1)^2} \sum_{j=1}^p \frac{\binom{2j}{j}}{(j+1)\binom{p+1+j}{j}} \pmod{p^2}.
 \end{aligned}$$

For $1 \leq j \leq p-2$, we have $\binom{p+1+j}{j} \equiv j+1 \pmod{p}$. It follows that

$$\begin{aligned}
 &p \sum_{j=1}^p \frac{S(p+1, j)}{(p+1)(j+1)} \\
 &\equiv 4p \sum_{j=1}^{p-2} \frac{\binom{2j}{j}}{(j+1)^2} + \frac{2(2p+1)}{(p+1)(2p-1)} + \frac{4p}{(p+1)^2}
 \end{aligned}$$

$$\equiv 12p \binom{p}{3} - 2 - 2p \pmod{p^2}, \tag{4.13}$$

where we have used (2.10) in the last step.

On the other hand, we have

$$\begin{aligned} p \sum_{j=1}^p \frac{S(1, j)}{j+1} &= 2p \sum_{j=1}^p \frac{\binom{2j}{j}}{(j+1)^2} \\ &= 2p \sum_{j=1}^{p-2} \frac{\binom{2j}{j}}{(j+1)^2} + \frac{2}{2p-1} \binom{2p-1}{p-1} + \frac{2p}{(p+1)^2} \binom{2p}{p} \\ &\equiv 6p \binom{p}{3} - 2 \pmod{p^2}, \end{aligned} \tag{4.14}$$

where we have used (2.1), (2.2) and (2.10) in the last step.

It follows from (4.12), (4.13) and (4.14) that

$$2p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{(i+1)(j+1)} - p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} \equiv 2p \left(3 \binom{p}{3} - 1 \right) \pmod{p^2}. \tag{4.15}$$

Combining (4.11) and (4.15) gives

$$\sum_{k=0}^{p-1} M_k^2 \equiv p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} - 2p \left(3 \binom{p}{3} - 1 \right) \pmod{p^2}. \tag{4.16}$$

Note that

$$\begin{aligned} &p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} \\ &= p \sum_{i=1}^{p-1} \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} + \sum_{j=1}^p \frac{S(p, j)}{j+1} \\ &= p \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{S(i, j)}{i(j+1)} + \sum_{j=1}^p \frac{S(p, j)}{j+1} + \sum_{i=1}^{p-1} \frac{S(i, p-1)}{i} + p \sum_{i=1}^{p-1} \frac{S(i, p)}{i(p+1)}. \end{aligned} \tag{4.17}$$

Furthermore, we have

$$p \sum_{i=1}^{p-1} \frac{S(i, p)}{i(p+1)} \equiv 2p \sum_{i=1}^{p-1} \frac{\binom{2i}{i}}{i} \equiv 0 \pmod{p^2}, \tag{4.18}$$

where we have used (2.2), (2.8) and the fact that $\binom{p+i}{i} \equiv 1 \pmod{p}$ for $1 \leq i \leq p-1$.

For $1 \leq i \leq p - 1$, we have $p / \binom{i+p-1}{i} \equiv 1 - pH_{i-1} \pmod{p^2}$, and so

$$\begin{aligned} \sum_{i=1}^{p-1} \frac{S(i, p-1)}{i} &= \frac{p}{2p-1} \binom{2p-1}{p-1} \sum_{i=1}^{p-1} \frac{\binom{2i}{i}}{i \binom{i+p-1}{i}} \\ &\equiv -(2p+1) \sum_{i=1}^{p-1} \binom{2i}{i} + p \sum_{i=1}^{p-1} \binom{2i}{i} H_{i-1} \\ &= -(2p+1) \sum_{i=1}^{p-1} \binom{2i}{i} + p \sum_{i=1}^{p-1} \binom{2i}{i} H_i - p \sum_{i=1}^{p-1} \frac{\binom{2i}{i}}{i} \\ &\equiv 2p+1 - \left(\frac{p}{3}\right) (2p+3^{p-1}) \pmod{p^2}, \end{aligned} \tag{4.19}$$

where we have used (2.1), (2.6), (2.8) and (2.14).

For $1 \leq j \leq p - 1$, we have $\binom{p+j}{j} \equiv 1 + pH_j \pmod{p^2}$, and so

$$\begin{aligned} \sum_{j=1}^p \frac{S(p, j)}{j+1} &= \frac{\binom{2p}{p}}{p+1} + \binom{2p}{p} \sum_{j=1}^{p-1} \frac{\binom{2j}{j}}{(j+1)\binom{p+j}{j}} \\ &\equiv 2 - 2p + 2 \sum_{j=1}^{p-1} C_j (1 - pH_j) \\ &\equiv 3^p \left(\frac{p}{3}\right) - 2p - 1 \pmod{p^2}, \end{aligned} \tag{4.20}$$

where we have used (2.2), (2.7) and (2.16).

It follows from (4.17)–(4.20) that

$$p \sum_{i=1}^p \sum_{j=1}^p \frac{S(i, j)}{i(j+1)} \equiv p \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{S(i, j)}{i(j+1)} + 2 \left(\frac{p}{3}\right) (3^{p-1} - p) \pmod{p^2}. \tag{4.21}$$

Combining (4.16) and (4.21) gives

$$\sum_{k=0}^{p-1} M_k^2 \equiv p \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{S(i, j)}{i(j+1)} + 2 \left(\frac{p}{3}\right) (3^{p-1} - 4p) + 2p \pmod{p^2}. \tag{4.22}$$

By (4.8), we have

$$S(i, j) = 2 \sum_{k=1}^j (-1)^k \binom{2i}{i+k} \binom{2j}{j+k} + \binom{2i}{i} \binom{2j}{j}. \tag{4.23}$$

It follows that

$$\begin{aligned}
 & p \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{S(i, j)}{i(j+1)} \\
 &= 2p \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \sum_{k=1}^j \frac{(-1)^k}{i(j+1)} \binom{2i}{i+k} \binom{2j}{j+k} + p \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{\binom{2i}{i} \binom{2j}{j}}{i(j+1)} \\
 &\equiv 2p \sum_{k=1}^{p-2} \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{(-1)^k}{i(j+1)} \binom{2i}{i+k} \binom{2j}{j+k} \pmod{p^2}, \tag{4.24}
 \end{aligned}$$

where we have used (2.8) in the last step.

Finally, combining (4.22) and (4.24) yields

$$\begin{aligned}
 \sum_{k=0}^{p-1} M_k^2 &\equiv 2p \sum_{k=1}^{p-2} \sum_{i=1}^{p-1} \sum_{j=1}^{p-2} \frac{(-1)^k}{i(j+1)} \binom{2i}{i+k} \binom{2j}{j+k} \\
 &\quad + 2 \left(\frac{p}{3}\right) (3^{p-1} - 4p) + 2p \pmod{p^2}. \tag{4.25}
 \end{aligned}$$

Then the proof of (1.2) follows from (3.1) and (4.25).

5. Proof of (1.4)

By (1.5) and (1.6), we have

$$\begin{aligned}
 \sum_{k=0}^{p-1} T_k M_k &= \sum_{k=0}^{p-1} \sum_{i=0}^k \sum_{j=0}^k (-1)^{i+j} \binom{k}{i} \binom{k}{j} \binom{2j}{j} C_{i+1} \\
 &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} \binom{2j}{j} C_{i+1} \sum_{k=0}^{p-1} \binom{k}{i} \binom{k}{j}. \tag{5.1}
 \end{aligned}$$

Applying (4.6) to the right-hand side of (5.1), we obtain

$$\sum_{k=0}^{p-1} T_k M_k \equiv p \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{\binom{2j}{j} C_{i+1}}{(i+j+1) \binom{i+j}{j}} \pmod{p^2}.$$

Noting that

$$\frac{\binom{2j}{j} C_{i+1}}{(i+j+1) \binom{i+j}{j}} = \frac{S(i+1, j)}{(i+1)(i+2)},$$

we have

$$\begin{aligned} \sum_{k=0}^{p-1} T_k M_k &\equiv p \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{S(i+1, j)}{(i+1)(i+2)} \\ &= p \sum_{i=1}^p \sum_{j=0}^{p-1} \frac{S(i, j)}{i(i+1)} \\ &= p \sum_{i=1}^p \sum_{j=0}^{p-1} \frac{S(i, j)}{i} - p \sum_{i=1}^p \sum_{j=0}^{p-1} \frac{S(i, j)}{i+1} \pmod{p^2}. \end{aligned} \tag{5.2}$$

Note that

$$p \sum_{i=1}^p \sum_{j=0}^{p-1} \frac{S(i, j)}{i} = p \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{S(i, j)}{i} + \sum_{j=0}^{p-1} S(p, j), \tag{5.3}$$

and

$$p \sum_{i=1}^p \sum_{j=0}^{p-1} \frac{S(i, j)}{i+1} = p \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{S(i, j)}{i+1} + \sum_{j=0}^{p-1} S(p-1, j) + p \sum_{j=0}^{p-1} \frac{S(p, j)}{p+1}. \tag{5.4}$$

For $0 \leq j \leq p-1$, we have $\binom{p+j}{j} \equiv 1 + pH_j \pmod{p^2}$, and so

$$\begin{aligned} \sum_{j=0}^{p-1} S(p, j) &= \binom{2p}{p} \sum_{j=0}^{p-1} \frac{\binom{2j}{j}}{\binom{p+j}{j}} \\ &\equiv 2 \sum_{j=0}^{p-1} \binom{2j}{j} (1 - pH_j) \\ &\equiv 2 \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}, \end{aligned} \tag{5.5}$$

where we have used (2.2), (2.6) and (2.14).

By (5.5), we have

$$p \sum_{j=0}^{p-1} \frac{S(p, j)}{p+1} \equiv p \sum_{j=0}^{p-1} S(p, j) \equiv 2p \left(\frac{p}{3}\right) \pmod{p^2}. \tag{5.6}$$

By (2.1) and $p/\binom{p+j-1}{j} \equiv j(1 - pH_{j-1}) \pmod{p^2}$ for $1 \leq j \leq p-1$, we have

$$\sum_{j=0}^{p-1} S(p-1, j) = \frac{p}{2p-1} \binom{2p-1}{p-1} \sum_{j=0}^{p-1} \frac{\binom{2j}{j}}{\binom{j+p-1}{j}}$$

$$\begin{aligned}
 &\equiv -(2p + 1) \sum_{j=0}^{p-1} j \binom{2j}{j} + p \sum_{j=0}^{p-1} j \binom{2j}{j} H_{j-1} \\
 &= -(2p + 1) \sum_{j=0}^{p-1} j \binom{2j}{j} + p \sum_{j=0}^{p-1} j \binom{2j}{j} H_j - p \sum_{j=0}^{p-1} \binom{2j}{j} \\
 &\equiv -\frac{p}{3} + \frac{2}{3} \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}, \tag{5.7}
 \end{aligned}$$

where we have used (2.6), (2.9) and (2.15) in the last step.

It follows from (5.2)–(5.7) that

$$\begin{aligned}
 \sum_{k=0}^{p-1} T_k M_k &\equiv p \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{S(i, j)}{i} - p \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{S(i, j)}{i+1} \\
 &\quad + \frac{p}{3} - 2p \left(\frac{p}{3}\right) + \frac{4}{3} \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}. \tag{5.8}
 \end{aligned}$$

By (4.23), we have

$$\begin{aligned}
 &p \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{S(i, j)}{i} \\
 &= 2p \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \sum_{k=1}^j \frac{(-1)^k}{i} \binom{2i}{i+k} \binom{2j}{j+k} + p \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{\binom{2i}{i} \binom{2j}{j}}{i} \\
 &\equiv 2p \sum_{k=1}^{p-1} \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{(-1)^k}{i} \binom{2i}{i+k} \binom{2j}{j+k} \pmod{p^2}, \tag{5.9}
 \end{aligned}$$

where we have used (2.8) in the last step.

On the other hand, we have

$$\begin{aligned}
 &p \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{S(i, j)}{i+1} \\
 &= 2p \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \sum_{k=1}^j \frac{(-1)^k}{i+1} \binom{2i}{i+k} \binom{2j}{j+k} + p \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{\binom{2i}{i} \binom{2j}{j}}{i+1} \\
 &\equiv 2p \sum_{k=1}^{p-1} \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{(-1)^k}{i+1} \binom{2i}{i+k} \binom{2j}{j+k} + \frac{p}{2} \left(3 - \left(\frac{p}{3}\right)\right) \pmod{p^2}, \tag{5.10}
 \end{aligned}$$

where we have used (2.6), (2.7) and $C_{p-1} \equiv -1 \pmod{p}$ in the last step.

It follows from (5.8)–(5.10) that

$$\begin{aligned}
 & \sum_{k=0}^{p-1} T_k M_k \\
 & \equiv 2p \sum_{k=1}^{p-1} \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \frac{(-1)^k}{i} \binom{2i}{i+k} \binom{2j}{j+k} \\
 & \quad - 2p \sum_{k=1}^{p-1} \sum_{i=1}^{p-2} \sum_{j=0}^{p-1} \frac{(-1)^k}{i+1} \binom{2i}{i+k} \binom{2j}{j+k} \\
 & \quad - \frac{7p}{6} - \frac{3p}{2} \binom{p}{3} + \frac{4}{3} \binom{p}{3} 3^{p-1} \pmod{p^2}.
 \end{aligned} \tag{5.11}$$

Then the proof of (1.4) follows from (3.7), (3.10) and (5.11).

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