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THE TORSION SUBMODULE OF A CYCLIC MODULE SPLITS OFF

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A prominent question in the study of modules over an integral domain has been: "When is the torsion submodule $t(A)$ of a module A a direct summand of A?" A module is said to split when its torsion module is a direct summand. Clearly, every cyclic module over an integral domain splits. Interesting splitting problems have been explored by Kaplansky [**14**; **15**], Rotman [**20**], Chase [**4**], and others.

Recently, many concepts of torsion have been proposed for modules over arbitrary associative rings with identity. Two of the most important of these concepts are Goldie's torsion theory (see [**1**; **12**; **22**]) and the simple torsion theory (see [**5**; **6**; **8**; **9**; **23**], and their references). Both Goldie's torsion theory $(\mathscr{G}, \mathscr{N})$ and the simple torsion theory $(\mathscr{S}, \mathscr{F})$ are hereditary "torsion theories" in the sense of Dickson [**5**] and have an associated topologizing and idempotent filter of left ideals in the sense of Gabriel [**10**] (also see [**16**]). The main purpose of this paper is to continue the study of the splitting properties of these two types of torsion for modules over a commutative ring.

The first two sections are devoted to terminology and preliminary results, which relate splitting properties to the existence of idempotent left ideals. It is interesting to note that the "idempotence" techniques, which are developed in § 2, work for both the Goldie torsion theory and the simple torsion theory, but not for hereditary torsion theories in general.

The major result of § 3 (Theorem 3.3) proves the equivalence of the following three properties for a commutative ring R such that $R \in \mathscr{N}$: (1) $\mathscr{G}(Rx)$ is a summand of each cyclic module Rx; (2) every closed ideal I of R has the form $I = Re \oplus S$, where $e^2 = e$ and where S is a direct sum of simple R-modules or zero; (3) $\mathscr{G}(M)$ is a summand of each module M such that $M/\mathscr{G}(M)$ is a direct sum of cyclic modules. (It was shown in [**23**, p. 272] that there is no loss of generality in assuming $R \in \mathscr{N}$.) This result generalizes the fact that every cyclic module over an intergral domain splits.

In § 4 it is proved (Lemma 4.1) that if $\mathscr{S}(Rx)$ is a summand of each cyclic module Rx and $R \in \mathscr{F}$, then $\mathscr{G} = \mathscr{S}$ for ${}_R\mathscr{M}$. This enables us to see (Theorem 4.3) that $(\mathscr{S}, \mathscr{F})$ splits cyclic modules over a commutative ring R if and only if R is a direct sum of a semi-artinian ring and finitely many integral domains D_i such that $\mathscr{G} = \mathscr{S}$ for ${}_{D_i}\mathscr{M}$. As corollaries we are able to obtain several known results on splitting for $(\mathscr{S}, \mathscr{T})$. Moreover, we are able to characterize the

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commutative rings R for which either of the following properties hold: (1) $\mathscr{S}(M)$ is a summand of each finitely generated R-module M (Theorem 4.6); (2) $(\mathscr{S},\mathscr{F})$ is stable and has the bounded splitting property (see definitions below and Corollary 4.5).

1. Definitions and preliminary results. In this paper all rings R are associative with identity. Unless stated otherwise, all modules will be unitary left R-modules, and the category of left R-modules will be denoted by ${}_R\mathscr{M}$.

A *torsion theory* $(\mathscr{T},\mathscr{F})$ is a pair of subclasses $(\mathscr{T},\mathscr{F})$ of ${}_R\mathscr{M}$ satisfying the following conditions:

(1) $\mathscr{T} \cap \mathscr{F} = 0$.
(2) $B \subseteq A$ and $A \in \mathscr{T}$ imply $A/B \in \mathscr{T}$.
(3) $B \subseteq A$ and $A \in \mathscr{F}$ imply $B \in \mathscr{F}$.
(4) For each $A \in {}_R\mathscr{M}$, there exists an exact sequence

$$0 \rightarrow T \rightarrow A \rightarrow F \rightarrow 0$$

such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

Then $\mathscr{T}$ is called the *torsion class*, and $\mathscr{F}$ is called the *torsionfree class*. $\mathscr{T}$ is closed under homomorphic images, direct sums, and the extensions of one module in $\mathscr{T}$ by another. $\mathscr{F}$ is closed under submodules, direct products, and extensions of one module in $\mathscr{F}$ by another. Each module $A \in {}_R\mathscr{M}$ has a (necessarily unique) largest submodule in $\mathscr{T}$; this submodule is denoted by $\mathscr{T}(A)$. A class is called *hereditary* if it is closed under submodules; and a torsion theory $(\mathscr{T},\mathscr{F})$ is called *hereditary* if $\mathscr{T}$ is a hereditary class. A hereditary torsion theory $(\mathscr{T},\mathscr{F})$ is uniquely associated with a topologizing and idempotent *filter* of left ideals $F(\mathscr{T}) = \{I|R/I \in \mathscr{T}\}$. We shall use $F(\mathscr{T})$ as our standard notation for the filter associated with $\mathscr{T}$. For further properties of $(\mathscr{T},\mathscr{F})$ the reader is referred to [**5**; **13**; **16**; **23**]. Properties of filters may be found in [**9**; **10**; **13**; **22**].

In this paper, we deal with two important hereditary torsion theories, namely Goldie's torsion theory $(\mathscr{G},\mathscr{N})$ and Dickson's simple theory $(\mathscr{S},\mathscr{F})$.

Goldie's torsion class $\mathscr{G}$ for ${}_R\mathscr{M}$ (see [**1**; **12**; **22**]) is the smallest torsion class containing all modules B/A, where A is an essential submodule of B. In case R is an integral domain, then $\mathscr{G}$ coincides with the usual torsion class. $F(\mathscr{G})$ is the smallest filter containing all the essential left ideals of R. The torsionfree class $\mathscr{N}$ associated with $\mathscr{G}$ is precisely the class of non-singular modules. In case R has zero left singular submodule, then $\mathscr{G}$ coincides with the $E(R)$-torsion class defined by Jans [**13**, p. 1255].

Dickson's simple torsion class $\mathscr{S}$ for ${}_R\mathscr{M}$ (see [**5**; **6**; **9**]) is the smallest torsion class containing all the simple modules. If R is a Dedekind domain, then $\mathscr{S}$ coincides with the usual torsion class for ${}_R\mathscr{M}$. For a commutative Noetherian ring, $\mathscr{S}$ also coincides with Matlis' class of modules with "maximal orders" (**17**). $\mathscr{S}$ has been useful in the study of Lowey series (see [**8**]). $F(\mathscr{S})$ is the smallest filter containing the maximal left ideals of R, and the torsionfree class $\mathscr{F}$ associated with $\mathscr{S}$ is precisely the class of modules with zero socle.

Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory. A module A is said to *split* (relative to $(\mathscr{T}, \mathscr{F})$) if $\mathscr{T}(A)$ is a summand of A. We say that $(\mathscr{T}, \mathscr{F})$ has the *cyclic splitting property* (*CSP*) if every cyclic module splits. $(\mathscr{T}, \mathscr{F})$ is said to have the *finitely generated splitting property* (*FGSP*) if every finitely generated module splits. We say that $(\mathscr{T}, \mathscr{F})$ has the *splitting property* (*SP*) if every module splits. A module A is said to have *bounded order* if $IA = 0$ for some $I \in F(\mathscr{T})$. Then $(\mathscr{T}, \mathscr{F})$ is said to have the *bounded splitting property* (*BSP*) if every module A, whose torsion submodule $\mathscr{T}(A)$ has bounded order, splits.

If $(\mathscr{G}, \mathscr{N})$ has *CSP* (respectively, *FGSP*, *SP*) for ${}_R\mathscr{M}$, then $R = \mathscr{G}(R) \oplus R'$, $R' \in \mathscr{N}$, and $(\mathscr{G}, \mathscr{N})$ for ${}_{R'}\mathscr{M}$ has *CSP* (*FGSP*, *SP*). A similar result holds for $(\mathscr{S}, \mathscr{F})$. So in characterizing the rings R for which $(\mathscr{G}, \mathscr{N})$ has *CSP* (*FGSP*, *SP*) for ${}_R\mathscr{M}$, it is sufficient to study R'. Hence there is no loss of generality in assuming $R \in \mathscr{N}$. Similarly, in studying splitting for $(\mathscr{S}, \mathscr{F})$ we may assume $R \in \mathscr{F}$ without loss of generality. Moreover, if R is a commutative ring such that $R = \oplus \sum_{i=1}^{n} R_i$, then $(\mathscr{G}, \mathscr{N})$ for ${}_R\mathscr{M}$ has *CSP* (*FGSP*, *BSP*, *SP*) if and only if, for each $i = 1, 2, \ldots, n$, $(\mathscr{G}, \mathscr{N})$ for ${}_{R_i}\mathscr{M}$ has *CSP* (*FGSP*, *BSP*, *SP*).

We now record an elementary, but basic, result.

PROPOSITION 1.1 *Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. Then the following statements are equivalent*:
(1) *$(\mathscr{T}, \mathscr{F})$ has CSP.*
(2) *$\mathscr{T}(M)$ is a summand of each module M such that $M/\mathscr{T}(M)$ is a direct sum of cyclic modules.*

Proof. (2) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2). Let $M/\mathscr{T}(M)$ be a direct sum of cyclic modules. Since

$$\operatorname{Ext}_R\left(\oplus \textstyle\sum_{\alpha \in \mathscr{A}} Rx_\alpha, T\right) \cong \textstyle\prod_{\alpha \in \mathscr{A}} \operatorname{Ext}_R(Rx_\alpha, T)$$

for every $T \in \mathscr{T}$, it is sufficient to show that $\operatorname{Ext}_R(Rx, T) = 0$ for all $Rx \in \mathscr{F}$ and $T \in \mathscr{T}$. Consider the exact sequence

$$\text{(E)} \qquad 0 \rightarrow T \xrightarrow{\chi} Y \xrightarrow{\sigma} Rx \rightarrow 0.$$

Then $\chi(T) \cong \mathscr{T}(Y)$. Choose $y \in Y$ such that $\sigma(y) = x$. From (1) we see that $Ry = \mathscr{T}(Ry) \oplus F$, and hence $Y = \chi(T) + F$. Since $\chi(T) \cap F \in \mathscr{T} \cap \mathscr{F} = 0$, the last sum is direct and (E) splits. Therefore, $\operatorname{Ext}_R(Rx, T) = 0$ as desired.

Finally, we will make frequent use of the following well-known result (see [**11**, p. 58] for a proof):

LEMMA 1.2. *Let R be a commutative ring. If I is a finitely generated idempotent ideal of R, then $I = Re$, where $e^2 = e$.*

2. Lemmas on splitting and idempotence. Let $(\mathscr{T},\mathscr{F})$ be a hereditary torsion theory with the cyclic splitting property (CSP), i.e. $\mathscr{T}(Rx)$ is a summand of each cyclic module Rx. This condition places restrictions on left ideals I such that $R/I \in \mathscr{F}$. In case $(\mathscr{T},\mathscr{F})$ is the simple torsion theory $(\mathscr{S},\mathscr{F})$, we show that such left ideals I must be idempotent. And in the case where $(\mathscr{T},\mathscr{F})$ is Goldie's torsion theory $(\mathscr{G},\mathscr{N})$, we show that CSP and $R/I \in \mathscr{N}$ imply that $I = I^2 \oplus A$, where A is semi-simple or zero. If R is a commutative ring, then $A = 0$, i.e. I is idempotent.

The results of this section will be used frequently in the proofs of the main results in §§ 3 and 4.

We begin with an examination of the simple theory $(\mathscr{S},\mathscr{F})$.

PROPOSITION 2.1. *Let $(\mathscr{S},\mathscr{F})$ have CSP for ${}_R\mathscr{M}$. If $R/I \in \mathscr{F}$, then $I^2 = I$.*

Proof. If $I^2 \neq I$, then let $0 \neq x \in I - I^2$. Choose $K \subseteq I$ maximal with respect to $I^2 \subseteq K$ and $x \notin K$. Since $R/I \in \mathscr{F}$, then $\mathscr{S}(R/K) \subseteq I/K$; and since $(Rx + K)/K$ is simple, then $\mathscr{S}(R/K) \neq 0$.

By CSP, we can write

$$(*) \qquad R/K = \mathscr{S}(R/K) \oplus F/K.$$

Let $1 + K = (e + K) + (f + K)$, where $e + K \in \mathscr{S}(R/K)$ and $f + K \in F/K$. From (*) it follows that

$$e + K = e^2 + K \subseteq I^2 + K \subseteq K.$$

But this implies $\mathscr{S}(R/K) = 0$, which is the desired contradiction.

COROLLARY 2.2. *Let $(\mathscr{S},\mathscr{F})$ have CSP for ${}_R\mathscr{M}$. If $R/I \in \mathscr{F}$ and $R/K \in \mathscr{F}$, then $I \cap K = IK \cap KI$.*

Proof. Since $R/I \in \mathscr{F}$ and $R/K \in \mathscr{F}$, then $R/(I \cap K) \in \mathscr{F}$ by the closure properties of $\mathscr{F}$. So by Proposition 2.1, $(I \cap K)^2 = I \cap K$. Hence $IK \supseteq (I \cap K)^2 = I \cap K$. By symmetry, $KI \supseteq I \cap K$. Thus

$$I \cap K \supseteq KI \cap IK \supseteq I \cap K,$$

which gives the desired result.

Next we examine the behavior of certain left ideals when Goldie's torsion theory $(\mathscr{G},\mathscr{N})$ has CSP.

PROPOSITION 2.3. *Let $(\mathscr{G},\mathscr{N})$ have CSP for ${}_R\mathscr{M}$. If $R/I \in \mathscr{N}$, then $I^2 = I^3$ and $I = I^2 \oplus A$, where A is a direct sum of simple modules or zero.*

Proof. If $I^2 \neq I$, choose $A \subseteq I$ maximal with respect to $A \cap I^2 = 0$. Hence $I^2 \oplus A$ is essential in I. Since $R/I \in \mathscr{N}$, then $\mathscr{G}(R/(I^2 \oplus A)) = I/(I^2 \oplus A)$. By CSP for $(\mathscr{G},\mathscr{N})$,

$$R/(I^2 \oplus A) = I/(I^2 \oplus A) \oplus F/(I^2 \oplus A).$$

Choose $e \in I$ and $f \in F$ such that

$$1 + (I^2 \oplus A) = (e + (I^2 \oplus A)) + (f + (I^2 \oplus A)).$$

Thus $e + (I^2 \oplus A) = e^2 + (I^2 \oplus A) \subseteq I^2 + A$, from which it follows that $I/(I^2 \oplus A) = 0$, i.e. $I = I^2 \oplus A$.

Since $IA \subseteq I^2 \cap A = 0$, then we obtain

$$I^2 = I(A \oplus I^2) = IA + I^3 = I^3.$$

Choose an essential submodule B of A. Thus $\mathscr{G}(R/(I^2 \oplus B)) = I/(I^2 \oplus B)$. By *CSP* for $(\mathscr{G}, \mathscr{N})$, we obtain

$$R/(I^2 \oplus B) = I/(I^2 \oplus B) \oplus F/(I^2 \oplus B).$$

Choose $u \in I$ and $v \in F$ such that

$$1 + (I^2 \oplus B) = (u + (I^2 \oplus B)) + (v + (I^2 \oplus B)).$$

Then $u + (I^2 \oplus B) = u^2 + (I^2 \oplus B) \subseteq I^2 \oplus B$, from which it follows that $I = I^2 \oplus B$. Hence $B = A$ is the only essential submodule of A. Thus A is a direct sum of simple modules or zero.

PROPOSITION 2.4. *Let R be a commutative ring such that $R \in \mathscr{N}$, and let $(\mathscr{G}, \mathscr{N})$ have CSP for ${}_R\mathscr{M}$. If $R/I \in \mathscr{N}$, then $I^2 = I$.*

Proof. Choose the ideal C maximal with respect to $C \cap I = 0$. Then $C \oplus I \in F(\mathscr{G})$. By Proposition 2.3, $I = I^2 \oplus A$. Since R is commutative, then $(C \oplus I)A = 0$, and hence $A \subseteq \mathscr{G}(R) = 0$.

We close this section with a technical result, which will be used in the proofs of our main results in the next two sections.

LEMMA 2.5. *Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for ${}_R\mathscr{M}$. Let $R \in \mathscr{F}$, and let $(\mathscr{T}, \mathscr{F})$ have CSP. Suppose that $R/I \in \mathscr{F}$ and $\oplus \sum_{\alpha \in \mathscr{A}} Rx_\alpha \subseteq I$, where $\mathscr{A}$ is an index set. Then there exists a collection $\{I_\alpha\}_{\alpha \in \mathscr{A}}$ of left ideals of R satisfying the following conditions*:

(1) *$Rx_\alpha \subseteq I_\alpha \subseteq I$ for each $\alpha \in \mathscr{A}$.*
(2) *$R/I_\alpha \in \mathscr{F}$ for each $\alpha \in \mathscr{A}$.*
(3) *I_α is generated by (at most) two elements for each $\alpha \in \mathscr{A}$.*
(4) *$\sum_{\alpha \in \mathscr{A}} I_\alpha$ is a direct sum.*
(5) *$I_\alpha/Rx_\alpha \in \mathscr{T}$ for each $\alpha \in \mathscr{A}$.*

Proof. Define $I_\alpha/Rx_\alpha = \mathscr{T}(R/Rx_\alpha)$ for each $\alpha \in \mathscr{A}$, which immediately yields (2) and (5). Since $R \in \mathscr{F}$, then each I_α is an essential extension of Rx_α. Hence $I_\beta \cap (\sum_{\alpha \in \mathscr{A} - \{\beta\}} Rx_\alpha) = 0$ for each $\beta \in \mathscr{A}$; thus it is not hard to show that (4) holds. Since $(\mathscr{T}, \mathscr{F})$ has *CSP*, then $I_\alpha/Rx_\alpha = \mathscr{T}(R/Rx_\alpha)$ is a summand of R/Rx_α for each $\alpha \in \mathscr{A}$. Then I_α/Rx_α is cyclic for each α, and hence (3) is satisfied. Finally, since

$$I_\alpha/(I \cap I_\alpha) \cong (I_\alpha + I)/I \subseteq R/I \in \mathscr{F}$$

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and since $I_\alpha/(I \cap I_\alpha)$ is a homomorphic image of $I_\alpha/Rx_\alpha \in \mathscr{T}$, it follows that $I_\alpha/(I \cap I_\alpha) = 0$. Hence $I_\alpha \subseteq I$, and (1) holds.

3. Goldie's torsion theory. The main purpose of this section is to characterize the commutative rings for which the Goldie torsion theory $(\mathscr{G}, \mathscr{N})$ has *CSP*. Before proving this result (Theorem 3.3), we establish two lemmas.

LEMMA 3.1. *Let $R \in \mathscr{N}$, and let I be a left ideal of R having the form $I = Re \oplus S$, where $e^2 = e$ and where S is a direct sum of simple modules or zero. Then* $\mathrm{Ext}_R(R/I, G) = 0$ *for each $G \in \mathscr{G}$.*

Proof. It follows from the exactness of the sequence

$$\mathrm{Hom}_R(R, G) \xrightarrow{\sigma} \mathrm{Hom}_R(I, G) \to \mathrm{Ext}_R(R/I, G) \to 0$$

that it is sufficient to show that σ is an epimorphism, i.e. that any $f \in \mathrm{Hom}_R(I, G)$ can be lifted to some $g \in \mathrm{Hom}_R(R, G)$.

If $x \in I$, then there exists $y \in R$ and $s \in S$ such that $x = ye + s$. Thus

$$x - xe = x(1 - e) = s(1 - e) \in I \cap \mathrm{Soc}(R) = \mathrm{Soc}(I) \in \mathscr{N};$$

and any homomorphic image of $\mathrm{Soc}(I)$ is isomorphic to a submodule of $\mathrm{Soc}(I)$. Now if $f \in \mathrm{Hom}_R(I, G)$, then $f(x - xe) \in f(\mathrm{Soc}(I)) \subseteq G \in \mathscr{G}$. Since $\mathscr{G} \cap \mathscr{N} = 0$, it follows that

$$f(x) - f(xe) = f(x - xe) = 0.$$

So f can be lifted to $g \in \mathrm{Hom}_R(R, G)$ via $g(r) = f(re)$ for each $r \in R$.

LEMMA 3.2. *Let R be a commutative ring such that $R \in \mathscr{N}$. Let $(\mathscr{G}, \mathscr{N})$ have CSP for ${}_R\mathscr{M}$. Then any simple submodule of R contains an idempotent element.*

Proof. Let Rx_0 be a simple submodule of R. By Lemma 2.5, we can choose I_0 satisfying properties (1) – (5), where $I_0/Rx_0 = \mathscr{G}(R/Rx_0)$ and Rx_0 is an essential submodule of I_0. By Proposition 2.4, $I_0^2 = I_0$; thus Lemma 1.2 implies $I_0 = Re$ for some $e = e^2$. Now any simple module in $\mathscr{N}$ is projective, and $(1 - e)Rx_0 = 0$. Hence the summand of R, which is isomorphic to Rx_0, is contained in Re. But $Re = I_0$ is an essential extension of Rx_0; hence Rx_0 must be a summand of R, and $Rx_0 = I_0$.

Following the terminology used in the study of semi-simple classical quotient rings, we say that a left ideal of R is *closed* if it contains no proper essential extensions in R.

Now we come to the main result of this section.

THEOREM 3.3. *Let R be a commutative ring such that $R \in \mathscr{N}$ (see Remark (iv) following Corollary* 3.5). *Then the following statements are equivalent*:

(1) *$(\mathscr{G}, \mathscr{N})$ has CSP for ${}_R\mathscr{M}$.*
(2) *If I is a closed ideal of R, then $I = Re \oplus S$, where $e^2 = e$ and where S is a direct sum of simple modules or zero.*

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(3) *Each module M, such that $M/\mathscr{G}(M)$ is a direct sum of cyclic modules, splits (relative to $(\mathscr{G}, \mathscr{N})$).*

Proof. The equivalence of (1) and (3) is immediate from Proposition 1.1.

Assume (2) holds. Since $R \in \mathscr{N}$, an ideal I is closed if and only if $R/I \in \mathscr{N}$. Hence (2) and Lemma 3.1 yield $\text{Ext}_R(R/I, G) = 0$ for all $R/I \in \mathscr{N}$ and $G \in \mathscr{G}$. It follows that $(\mathscr{G}, \mathscr{N})$ has *CSP* for ${}_R\mathscr{M}$, i.e. (1) holds.

Finally, we shall show that (1) implies (2). Assume (1), and let I be a closed ideal of R (so that $R/I \in \mathscr{N}$).Let $\oplus \sum_{\alpha \in \Gamma} Rx_\alpha$ be a maximal direct sum of simple submodules of I. Let $\oplus \sum_{\alpha \in \Delta} Rx_\alpha$ be a direct sum of cyclic submodules of I chosen maximal with respect to

$$(\oplus \textstyle\sum_{\alpha \in \Delta} Rx_\alpha) \cap (\oplus \sum_{\alpha \in \Gamma} Rx_\alpha) = 0.$$

Let $\mathscr{A} = \Delta \cup \Gamma$. By (1) and Lemma 2.5, there is a direct sum $\oplus \sum_{\alpha \in \mathscr{A}} I_\alpha$ of ideals satisfying $Rx_\alpha \subseteq I_\alpha \subseteq I$, $R/I \in \mathscr{N}$, and I_α is generated by two elements (for all $\alpha \in \mathscr{A}$). By (1) and Proposition 2.4, $I_\alpha^2 = I_\alpha$ for all $\alpha \in \mathscr{A}$; thus by Lemma 1.2, there exists a set $\{e_\alpha\}_{\alpha \in \mathscr{A}}$ of orthogonal idempotent elements such that $I_\alpha = Re_\alpha$ for each $\alpha \in \mathscr{A}$. Since $\oplus \sum_{\alpha \in \mathscr{A}} I_\alpha$ is essential in I and $R/I \in \mathscr{N}$, then

$$\mathscr{G}(R/(\textstyle\sum_{\alpha \in \mathscr{A}} I_\alpha)) = I/(\sum_{\alpha \in \mathscr{A}} I_\alpha).$$

From (1) we obtain $g \in R$ such that $g + \sum_{\alpha \in \mathscr{A}} I_\alpha$ is an idempotent generator of $I/(\sum_{\alpha \in \mathscr{A}} I_\alpha)$ in $R/(\sum_{\alpha \in \mathscr{A}} I_\alpha)$. Hence $I = Rg + \sum_{\alpha \in \mathscr{A}} I_\alpha$ is an idempotent ideal of R.

If $I = Rg$, then it follows from Lemma 1.2 that I is generated by an idempotent element e of R. Hence $I = Re$ as desired.

If $I \neq Rg$, then there exists an e_α, $\alpha \in \mathscr{A}$, such that $e_\alpha \notin Rg$. So we may choose $M \subseteq I$ maximal with respect to the following properties:

(i) $Rg \subseteq M$.
(ii) $e_\alpha \notin M$ for each $\alpha \in \mathscr{A}$ such that $e_\alpha \notin Rg$.

Let $x \in R$ such that $(Rxe_\beta + M)/M \neq 0$ for some (fixed) $\beta \in \mathscr{A}$. Since $Rxe_\beta + M \supset M$, then the definition of M requires the existence of $e_\gamma \notin Rg$ $(\gamma \in \mathscr{A})$ such that $e_\gamma \in Rxe_\beta + M$. If $\beta \neq \gamma$, then

$$e_\gamma = e_\gamma^2 \in (Rxe_\beta + M)\, e_\gamma = Rxe_\beta e_\gamma + Me_\gamma \subseteq M,$$

which contradicts (ii) in the definition of M. Therefore, $\beta = \gamma$ whence $e_\beta \in Rxe_\beta + M$. Hence

$$(Rxe_\beta + M)/M = (Re_\beta + M)/M.$$

It follows that $(Re_\beta + M)/M$ is a simple module whenever $e_\beta \notin M$. Since $\{e_\alpha\}_{\alpha \in \mathscr{A}}$ is a set of orthogonal idempotents, it is easy to see that

$$\textstyle\sum_{\alpha \in \mathscr{A}}((Re_\alpha + M)/M)$$

is a direct sum. But $I = M + \sum_{\alpha \in \mathscr{A}} Re_\alpha$; so it follows that

$$\oplus \sum_{\alpha \in \mathscr{A}} ((Re_\alpha + M)/M) \cong I/M = \mathscr{G}(R/M) \oplus \sum_{\beta \in \mathscr{B}} S_\beta,$$

where $\mathscr{B} \subseteq \mathscr{A}$ and each $S_\beta \in \mathscr{N}$ is a simple module. But each simple module in $\mathscr{N}$ is projective; so the exact sequence

$$0 \to G \to I \xrightarrow{\sigma} \oplus \sum_{\beta \in \mathscr{B}} S_\beta \to 0$$

splits, where $G/M = \mathscr{G}(R/M)$. Let τ: $\sum_{\beta \in \mathscr{B}} S_\beta \to I$ be a homomorphism such that $\sigma\tau = 1$. Clearly, im $\tau \subseteq \mathrm{Soc}(I)$. But it follows from (1) and Lemma 3.2 that there can be only one copy of each S_β in $\mathrm{Soc}(I)$, and every simple module in $\mathrm{Soc}(I)$ is some Rx_α ($\alpha \in \Gamma \subseteq \mathscr{A}$). Thus for each $\beta \in \mathscr{B}$, $\tau(S_\beta) = Rx_\alpha$ for some $\alpha \in \Gamma$. Therefore we can identify $\mathscr{B}$ with a subset of Γ. But I_α is an essential extension of the simple module Rx_α for each $\alpha \in \Gamma$, and each Rx_α contains an idempotent element by (1) and Lemma 3.2. Hence $Rx_\alpha = I_\alpha$ for each $\alpha \in \Gamma$. Therefore $I = G \oplus \sum_{\alpha \in \mathscr{B}} I_\alpha$.

It remains to show that $G = Re$ for some $e = e^2$. But G is generated by the set $\{g\} \cup \{e_\alpha\}_{\alpha \in \mathscr{A} - \mathscr{B}}$. By (1), G/M is a direct summand of R/M. Hence the set

$$\Lambda = \{\alpha | \alpha \in \mathscr{A} - \mathscr{B}, e_\alpha \in G - M\}$$

is finite. From the definition of M, it follows that G is generated by the finite set $\{g\} \cup \{e_\alpha\}_{\alpha \in \Lambda}$. So by Lemma 1.2, G is generated by an idempotent element e of R, and the proof is completed.

We now point out two obvious consequences of Theorem 3.3.

COROLLARY 3.4. *Let R be a commutative ring with no nontrivial idempotent elements. Then the following statements are equivalent*:

(1) *$(\mathscr{G}, \mathscr{N})$ has CSP for ${}_R\mathscr{M}$.*
(2) *Either R is an integral domain or else R has essential singular ideal.*
(3) *Every cyclic module in $\mathscr{N}$ is projective.*

COROLLARY 3.5. *Let R be a commutative ring such that $R \in \mathscr{N}$. If $(\mathscr{G}, \mathscr{N})$ has CSP for ${}_R\mathscr{M}$, then every closed ideal of R is projective, and hence the homological dimension of every cyclic module in $\mathscr{N}$ is $\leqq 1$.*

Remarks. (i) The hypothesis about idempotent elements in Corollary 3.4 applies to any local ring or any integral domain.

(ii) The reader may wish to compare Corollary 3.5 with a corollary of [**3**, p. 153].

(iii) Proposition 2.4 and Theorem 3.3 may provide a starting point for studying the commutative rings for which $(\mathscr{G}, \mathscr{N})$ has *FGSP*. This problem was proposed in [**3**].

(iv) As it was pointed out in § 1, there is no loss of generality in assuming $R \in \mathscr{N}$ when studying *CSP* for $(\mathscr{G}, \mathscr{N})$; but there is considerable simplification in the proofs. We now state the analogue of Theorem 3.3 without assuming $R \in \mathscr{N}$. The following statements are equivalent for a commutative ring R: (1) $(\mathscr{G},\mathscr{N})$ has *CSP* for ${}_R\mathscr{M}$; (2) $\mathscr{G}(R)$ is a direct summand of R, and every closed ideal not intersecting $\mathscr{G}(R)$ has the form $Re \oplus S$, where $e^2 = e$ and where S is a direct sum of simple modules or zero; and (3) each module M, such that $M/\mathscr{G}(M)$ is a direct sum of cyclic modules, splits.

We close this section with a non-commutative result, which is related to Corollary 3.4.

PROPOSITION 3.6. *Let R be a ring with no non-trivial idempotent left ideals. Then the following statements are equivalent*:

(1) *$(\mathscr{G},\mathscr{N})$ has CSP for ${}_R\mathscr{M}$.*
(2) *Either R is a left Ore domain or else R has essential singular ideal.*
(3) *Every cyclic module in $\mathscr{N}$ is projective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is obvious.

(1) $\Rightarrow$ (2). By (1) and the hypothesis on idempotent left ideals, either $R \in \mathscr{G}$ or $R \in \mathscr{N}$. And if $R \in \mathscr{G}$, then R has essential singular ideal.

Assume $R \in \mathscr{N}$. If $0 \neq R/I \in \mathscr{N}$, then by (1) and Proposition 2.3, $I = I^2 \oplus A$, where $I^2 = I^3$ and where A is a direct sum of simple modules or zero. Since $I^2 = I^4$, then $I^2 = 0$. If $A \neq 0$, then R must have a projective simple module; this leads to a non-trivial idempotent left ideal, which contradicts the hypothesis on idempotent left ideals. Thus $I = 0$. Since $R \in \mathscr{N}$, it follows that R is an integral domain.

Finally, if $K \neq 0$ is a left ideal of R, choose a left ideal I maximal with respect to $I \cap K = 0$. Then $0 \neq R/I \in \mathscr{N}$; so by the preceeding paragraph, $I = 0$. Thus any non-zero left ideal K is essential in R. Hence R is an Ore domain.

4. The simple torsion theory. In this section we characterize the commutative rings, for which the simple torsion theory $(\mathscr{S},\mathscr{F})$ has *CSP*, as a direct sum of a semi-artinian ring and finitely many integral domains satisfying $\mathscr{S} = \mathscr{G}$. As an immediate corollary of this result, we can give a characterization of all commutative rings for which $(\mathscr{S},\mathscr{F})$ has *SP*. Next we characterize the commutative rings R, for which $(\mathscr{S},\mathscr{F})$ is stable and has *BSP*, as a direct sum of a semi-artinian ring and finitely many Dedekind domains. Finally, we determine the commutative rings R such that $(\mathscr{S},\mathscr{F})$ has *FGSP* for ${}_R\mathscr{M}$.

LEMMA 4.1. *Let R be a commutative ring such that $R \in \mathscr{F}$. Suppose $(\mathscr{S},\mathscr{F})$ has CSP for ${}_R\mathscr{M}$. If $R/K \in \mathscr{F}$, then K is a summand of R, and hence $\mathscr{S} = \mathscr{G}$.*

Proof. Let $R/K \in \mathscr{F}$, and let $\oplus \sum_{\alpha \in \mathscr{A}} Rx_\alpha$ be a maximal direct sum of cyclics contained in K. Choose the direct sum $\oplus \sum_{\alpha \in \mathscr{A}} I_\alpha$ satisfying (1) – (5) of Lemma 2.5. By Proposition 2.1, $I_\alpha{}^2 = I_\alpha$ for each $\alpha \in \mathscr{A}$. Hence by Lemma

1.2 there exists a set $\{e_\alpha\}_{\alpha\in\mathscr{A}}$ of orthogonal idempotents such that $I_\alpha = Re_\alpha$ for each $\alpha \in \mathscr{A}$. Let

$$I/(\oplus \sum_{\alpha\in\mathscr{A}} I_\alpha) = \mathscr{S}(R/(\oplus \sum_{\alpha\in\mathscr{A}} I_\alpha)).$$

Since $R/K \in \mathscr{F}$, then I is an essential submodule of K. By Proposition 2.1 $I^2 = I$; and by *CSP*, $I/(\oplus \sum_{\alpha\in\mathscr{A}} I_\alpha)$ has an idempotent generator $g + (\sum_{\alpha\in\mathscr{A}} I_\alpha)$ in $R/(\oplus \sum_{\alpha\in\mathscr{A}} I_\alpha)$.

If g generates I, then by Lemma 1.2, I is generated by an idempotent element of R; hence I is a summand of R. Since I is an essential submodule of K, then K must also be a summand of R.

If g does not generate I, then we choose an ideal M in the same manner that we used in the proof of Theorem 3.3 (1) $\Rightarrow$ (2). The arguments used in Theorem 3.3 show that I/M is a direct sum of simple modules. Since $R/I \in \mathscr{F}$, it follows that $\mathscr{S}(R/M) = I/M$. From *CSP* for $(\mathscr{S},\mathscr{F})$ and the fact that R/M is cyclic, it follows that I/M is a direct sum of only finitely many simple modules. Thus the set $\psi = \{\alpha | e_\alpha \notin M\}$ is finite. Hence I is generated by the set $\{g\} \cup \{e_\alpha\}_{\alpha\in\psi}$. By Lemma 1.2, I is generated by an idempotent element of R; hence I is a summand of R. Since I is an essential submodule of K, then K must also be a summand of R.

It remains to show that $\mathscr{S} = \mathscr{G}$ for ${}_R\mathscr{M}$. Since $R \in \mathscr{F}$, then every member of $F(\mathscr{S})$ is an essential ideal; hence $F(\mathscr{S}) \subseteq F(\mathscr{G})$. If B is an essential ideal and $C/B = \mathscr{S}(R/B)$, then $R/C \in \mathscr{F}$. By the first part of the lemma, C is a summand of R. Since C is an essential ideal, it follows that $C = R$, and thus $B \in F(\mathscr{S})$. Since $F(\mathscr{G})$ is the smallest filter containing all the essential ideals of R, then $F(\mathscr{G}) \subseteq F(\mathscr{S})$. Therefore $F(\mathscr{G}) = F(\mathscr{S})$. Since a hereditary torsion clas is uniquely determined by its filter, then $\mathscr{G} = \mathscr{S}$.

A ring R is called (left) *finite dimensional* if R contains no infinite direct sum of (left) ideals.

A ring R is called (left) *semi-artinian* if non-zero (left) R-modules have non-zero socles. Such rings are discussed in [**18**].

LEMMA 4.2. *Let R be a commutative ring such that $R \in \mathscr{F}$. If $(\mathscr{S},\mathscr{F})$ has CSP for ${}_R\mathscr{M}$, then R is a finite dimensional ring.*

Proof. The proof is by contradiction. Let $\oplus \sum_{\alpha\in\mathscr{A}} Rx_\alpha$ be a maximal direct sum of infinitely many non-zero principal ideals of R. Using Lemma 2.5, Proposition 2.1, and Lemma 1.2, we obtain a direct sum $\oplus \sum_{\alpha\in\mathscr{A}} Re_\alpha$, where $e_\alpha^2 = e_\alpha$ and $Re_\alpha \supseteq Rx_\alpha$. Let $L = \sum_{\alpha\in\mathscr{A}} Re_\alpha$. Since $\mathscr{S} = \mathscr{G}$ by Lemma 4.1, then $L \in F(\mathscr{S})$. Hence $A = R/L$ is a semi-artinian ring; hence $A/\text{Rad}(A)$ is von Neumann regular and $\text{Rad}(A)$ is T-nilpotent by [**18**, Theorem 3.1]. So in A, idempotents can be lifted modulo $\text{Rad}(A)$. Let S be a simple module in $\text{Soc}(A/\text{Rad}(A))$. Then S is generated by an idempotent f' of $A/\text{Rad}(A)$, which can be lifted to an idempotent element f'' of A. Let $f \in R$ such that $f'' = f + L$. Hence, for some finite subset $\{e_{\alpha_i}\}_{i=1}^n$ with $\alpha_i \in \mathscr{A}$,

$$f^2 - f = \sum_{i=1}^n r_i e_{\alpha_i},$$

where $r_i \in R$ for $i = 1, 2, \ldots, n$. Then the ideal I, generated by the set $\{f\} \cup \{e_{\alpha_i}\}_{i=1}^n$, is idempotent. By Lemma 1.2, there is an idempotent element e of R which generates I. Hence

$$\frac{Re + L}{L} = \frac{I + L}{L} = \frac{Rf + L}{L}$$

and

$$\left(\frac{Rf + L}{L}\right) \Big/ \left(\mathrm{Rad}(A) \cap \frac{Rf + L}{L}\right) \cong \frac{Af'' + \mathrm{Rad}(A)}{\mathrm{Rad}(A)} = Af' = S.$$

It follows from properties of the radical that

$$Re/(Re \cap L) \cong (Re + L)/L$$

has a unique maximal ideal.

If the set $\mathscr{B} = \{\alpha \in \mathscr{A} \mid ee_\alpha \neq 0\}$ were finite, then

$$0 \neq Re/(Re \cap L) = Re/(\textstyle\sum_{\alpha \in \mathscr{A}} Ree_\alpha) \cong R(e - \sum_{\alpha \in \mathscr{B}} ee_\alpha)$$

is contained in $\mathscr{S}(R)$. Since $R \in \mathscr{F}$, this is a contradiction to $\mathscr{S} \cap \mathscr{F} = 0$. Therefore $\mathscr{B}$ must be an infinite set.

We partition $\mathscr{B}$ into disjoint sets Δ and Γ of infinite cardinality. Choose $M \subseteq Re$ maximal with respect to

$$M \cap \textstyle\sum_{\alpha \in \Delta} Ree_\alpha = 0$$

and

$$M \supseteq \textstyle\sum_{\alpha \in \Gamma} Ree_\alpha.$$

Then $R/M \in \mathscr{F}$; hence M is a summand of Re (and of R) by Proposition 2.1. Write $Re = M \oplus N$. It follows from some routine arguments involving idempotent elements that $M \not\subseteq L$ and $N \not\subseteq L$. But this gives rise to the non-trivial direct sum decomposition

$$(Re + L)/L = ((M + L)/L) \oplus ((N \oplus L)/L).$$

Since this direct decomposition forces $(Re + L)/L$ to have at least two maximal ideals, we have the desired contradiction.

In [**19**] a ring R is called a *C-ring* if each non-zero singular module has non-zero socle. Thus R is a C-ring if and only if $\mathscr{G} \subseteq \mathscr{S}$ for ${}_R\mathscr{M}$. A commutative ring R will be called a *C-domain* if it is both a C-ring and an integral domain. If R is an integral domain (not a field), then R is a C-domain if and only if $\mathscr{G} = \mathscr{S}$ for ${}_R\mathscr{M}$. C-rings are discussed in [**19**], and C-domains are characterized in terms of their localizations in [**21**, p. 244].

Combining the two previous lemmas, we obtain the following theorem:

THEOREM 4.3. *Let R be a commutative ring. Then the following statements are equivalent*:

(1) $(\mathcal{S},\mathcal{F})$ *has CSP for* ${}_R\mathcal{M}$.
(2) *R is a direct sum of a semi-artinian ring and finitely many C-domains* D_i.
(3) *Every module M, such that* $M/\mathcal{S}(M)$ *is a direct sum of cyclic modules, splits (relative to* $(\mathcal{S},\mathcal{F})$*).*

Proof. The equivalence of (1) and (3) is immediate from Proposition 1.1; and (2) implies (1) is immediate from the properties of an integral domain.

Now assume (1). Then R is a direct sum of a semi-artinian ring and a ring R' with zero socle. Thus $(\mathcal{S},\mathcal{F})$ has *CSP* for ${}_{R'}\mathcal{M}$ and $R' \in \mathcal{F}$. By Lemma 4.1, $\mathcal{G} = \mathcal{S}$ for ${}_{R'}\mathcal{M}$. By Lemma 4.2, we can select a maximal direct sum of uniform ideals U_i $(i = 1, 2, \ldots n)$ of R'. Using Lemma 2.5, Proposition 2.1, and Lemma 1.2, we obtain $R' = \oplus \sum_{i=1}^{n} D_i$ such that each U_i is an essential R'-submodule of D_i. Since U_i is uniform and $\mathcal{G}(R') = \mathcal{S}(R') = 0$, then it is easy to verify that D_i has no zero divisors for each $i = 1, 2, \ldots, n$. Hence (1) implies (2).

Since Theorem 4.3 reduces the study of many other splitting problems to the study of the usual torsion theory over an integral domain, several theorems on splitting are immediate corollaries of Theorem 4.3 and Rotman's theorem [**20**]. These corollaries include [**5**, Theorem 1], [**9**, Theorem 3.9], [**9**, Theorem 4.6], and the following result:

COROLLARY 4.4 [**23**, Theorem 5.1]. *Let R be a commutative ring such that* $R \in \mathcal{F}$. *Then* $(\mathcal{S},\mathcal{F})$ *has SP if and only if R is a semi-artinian ring (i.e. non-zero R-modules have non-zero socles* [**18**]).

A torsion theory $(\mathcal{T},\mathcal{F})$ is called *stable* [**10**] if $\mathcal{T}$ is closed under injective envelopes. $(\mathcal{G},\mathcal{N})$ is always stable. Any hereditary torsion theory of modules over a commutative Noetherian ring is stable.

COROLLARY 4.5. *Let R be a commutative ring. Then* $(\mathcal{S},\mathcal{F})$ *is stable and has BSP for* ${}_R\mathcal{M}$ *if and only if R is a direct sum of a semi-artinian ring and finitely many Dedekind domains.*

Proof. If D is a Dedekind domain, then $\mathcal{G} = \mathcal{S}$ for ${}_D\mathcal{M}$. Hence the "if" part of the theorem follows from Kaplansky's result [**14**, p. 334].

Conversely, if $(\mathcal{S},\mathcal{F})$ is stable and has *BSP* for ${}_R\mathcal{M}$, then by [**23**, Lemma 3.2], $(\mathcal{S},\mathcal{F})$ also has *FGSP*. Thus by Theorem 4.3

$$R = \mathcal{S}(R) \oplus D_i \oplus D_2 \oplus \ldots \oplus D_n,$$

where each D_i is a C-domain (not a field). Then $\mathcal{S}(R)$ is a semi-artinian ring; and by [**4**, Theorem 4.2], D_i is a Dedekind domain $(i = 1, 2, \ldots, n)$.

If the torsion theory $(\mathcal{T},\mathcal{F})$ in [**23**, Theorem 4.6] is taken to be the simple theory $(\mathcal{S},\mathcal{F})$, then Corollary 4.5 shows that a cofinal subset of finitely generated ideals in $F(\mathcal{T})$ is no longer necessary for [**23**, Theorem 4.6] to be true. Hence Corollary 4.5 generalizes [**23**, Theorem 4.6] in the case of the simple theory (as a direct sum of Dedekind domains satisfies conditions (1) – (3) of [**23**, Theorem 4.6].

In Theorem 4.3, one of the conditions placed on an integral domain D is "$\mathcal{S} = \mathcal{G}$ for ${}_D\mathcal{M}$", i.e. D is a C-domain. It is necessary that all non-zero prime ideals of a C-domain be maximal. On the other hand, a Noetherian domain (not a field), such that all non-zero prime ideals are maximal, must be a C-domain. The next example shows the existence of a non-Noetherian C-domain.

Example. Let F be a field, and let K be an extension field of degree $[K:F]$ over F. Let A be the subring of $K[x]$ consisting of all polynomials whose constant term is in F. Let M be the maximal ideal of A generated by the set $\{kx|k \in K\}$. Define $R = A_M$, the localization of A by the maximal ideal M. The reader can verify the following facts:

(i) Every proper ideal of R contains a power of the (unique) maximal ideal of R.

(ii) R is a C-domain.

(iii) R is a Noetherian ring if and only if $[K:F] < \infty$.

(iv) R is Prfüer (Dedekind) domain if and only if $[K:F] = 1$.

Other examples of C-domains are given by Smith [**21**].

Another consequence of Theorem 4.3 is that we are able to classify the rings for which $(\mathcal{S},\mathcal{F})$ has *FGSP*.

THEOREM 4.6. *Let R be a commutative ring such that $R \in \mathcal{F}$. Then $(\mathcal{S},\mathcal{F})$ has FGSP for ${}_R\mathcal{M}$ if and only if R is a direct sum of finitely many Prüfer C-domains.*

Proof. First, suppose that $(\mathcal{S},\mathcal{F})$ has *FGSP* for ${}_R\mathcal{M}$. Then by Theorem 4.3, R is a direct sum of C-domains $D_i (i = 1, 2, \ldots, n)$. By Kaplansky's result [**15**], each D_i is a Prüfer domain.

Conversely, let R be a direct sum of finitely many Prüfer C-domains D_i $(i = 1, 2, \ldots, n)$. It follows from [**2**, VII, 4.1] that $(\mathcal{S},\mathcal{F})$ has *FGSP* for ${}_{D_i}\mathcal{M}$ $(i = 1, 2, \ldots, n)$. Hence $(\mathcal{S},\mathcal{F})$ has *FGSP* for ${}_R\mathcal{M}$ also.

We close this section with the following result which relates *FGSP* and *BSP* for $(\mathcal{S},\mathcal{F})$ in certain cases:

THEOREM 4.7. *Let R be a commutative ring such that $R \in \mathcal{F}$. If each maximal ideal of R is finitely generated, then the following statements are equivalent:*

(1) *$R = D_1 \oplus D_2 \oplus \ldots \oplus D_n$, where each D_i is a Dedekind domain.*
(2) *$(\mathcal{S},\mathcal{F})$ is stable and has BSP.*
(3) *$(\mathcal{S},\mathcal{F})$ has FGSP.*
(4) *R is a semi-hereditary ring such that $\mathcal{G} = \mathcal{S}$ for ${}_R\mathcal{M}$.*
(5) *R is a semi-hereditary ring, and, for every essential ideal I, R/I is an Artinian ring.*

Proof. (1) $\Rightarrow$ (2). Whenever D is a Dedekind domain, $\mathscr{S}$ coincides with usual torsion theory and hence is stable. It follows from (1) that $\mathscr{S}$ is a stable torsion class for ${}_R\mathscr{M}$. *BSP* for ${}_R\mathscr{M}$ is easily deduced from Kaplansky's result [**14**, p. 334].

(2) $\Rightarrow$ (3). This follows immediately from [**23**, Lemma 3.2].

(3) $\Rightarrow$ (4). This is a consequence of Theorem 4.6.

(4) $\Rightarrow$ (1). Since every maximal ideal is finitely generated, then $F(\mathscr{S})$ possesses a cofinal subset of finitely generated ideals. From (4), $R \in \mathscr{F}$, and [**22**, Corollary 3.7], it follows that R is a direct sum of finitely many integral domains D_i ($i = 1, 2, \ldots n$). By (4) and [**7**, Corollary 2], each D_i is integrally closed. Since $\mathscr{G} = \mathscr{S}$ by (4), then every essential prime ideal is maximal; hence every prime ideal of D_i is finitely generated. Therefore D_i is also Noetherian ($i = 1, 2, \ldots, n$). Thus each D_i is a Dedekind domain by [**24**, Theorem 13, p. 275].

(5) $\Rightarrow$ (2). By (5) and $R \in \mathscr{F}, \mathscr{S} = \mathscr{G}$; hence (2) follows from [**3**, Theorem 3.1].

(1) $\Rightarrow$ (5). This is obvious from the properties of a Dedekind domain.

Remark. In the proof of Theorem 4.7, the hypothesis, each maximal ideal of R is finitely generated, was used only in the proof of the implication (4) $\Rightarrow$ (1). Instead of using the hypothesis, each maximal ideal is finitely generated, we may use an alternate hypothesis and still retain the equivalence of *some* of the statements (1) – (5) of Theorem 4.7. Such alternate hypotheses include the following: (i) Each essential ideal is contained in only finitely many maximal ideals; or (ii) For each simple module S, there exists a finitely generated ideal I_S such that $\mathrm{Soc}(R/I_S)$ contains an isomorphic copy of S.

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