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ON POSITIVE SOLUTIONS OF SOME SEMILINEAR ELLIPTIC EQUATIONS

SHIN-HWA WANG and NICHOLAS D. KAZARINOFF

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Abstract

The existence of positive solutions of some semilinear elliptic equations of the form $-\Delta u = \lambda f(u)$ is studied. The major results are a nonexistence theorem which gives a $\lambda^* = \lambda^*(f, \Omega) > 0$ below which no positive solutions exist and a lower bound theorem for $u_{\max}$ for Ω a ball. As a corollary of the nonexistence theorem that describes the dependence of the number of solutions on λ, two other nonexistence theorems, and an existence theorem are also proved.

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1. Introduction

We study the existence of positive solutions u in $C^2(\Omega) \cap C(\overline{\Omega})$ of the semilinear elliptic eigenvalue problem of the form

$$(1) \qquad -\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with $\partial\Omega$ smooth, $\lambda > 0$, is a real bifurcation parameter, f is a C^1 nonlinearity, and there are numbers $0 < a_0 < a_1$ such that the following conditions are satisfied:

(f1) $f(0) \geq 0$, or (f1′) $f(s) > 0$ on $(0, a_0)$;
(f2) $f(a_0) = f(a_1) = 0$;
(f3) $\max\{F(s) : 0 \leq s \leq a_0\} < F(a_1)$, where $F(s) \equiv \int_0^s f(\sigma)\, d\sigma$.

Note that (f1) and (f1′) allow $f(0) > 0$ and (f1′) implies (f1).

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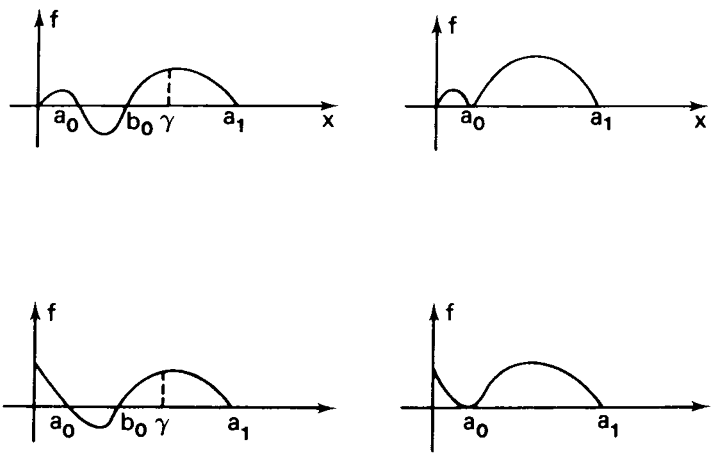

FIGURE 1. Typical f's

In part of our work we allow f to change sign on (a_0, a_1), and then we assume f also satisfies

(f4) there exists b_0 in (a_0, a_1), $f(b_0) = 0$, such that $\int_{a_0}^{b_0} f(s)\,ds < 0$ and $f(s) > 0$ on (b_0, a_1).

If (f4) is assumed, we can find a unique γ in (b_0, a_1) such that

$$\int_{a_0}^{\gamma} f(s)\,ds = 0. \tag{2}$$

It is clear that $\lambda = 0$ is not an eigenvalue of (1), and if u is a positive solution of (1) satisfying $u_{\max} \in [a_0, a_1]$, then by [1, Lemma 6.2], we know $f(u_{\max}) > 0$.

Four typical f's are as in Figure 1(a), (b), (c), (d).

Problem (1) has been recently discussed by E. N. Dancer [5], E. N. Dancer and K. Schmitt [6], C. Cosner and K. Schmitt [4], P. Clement and G. Sweers [3], P. Hess [8] and H. O. Peitgen, D. Saupe and K. Schmitt [10]. We note that a theorem of Smoller and Wasserman [13, Theorem 2.1] is an indirect motivation for our work.

In [5], Dancer showed that if f satisfies (f1$'$), (f2) and (f3) then for large λ, (1) has a positive solution $u(x)$ satisfying

$$u_{\max} = \max_{x\in\Omega} u(x) \in (a_0, a_1). \tag{3}$$

In [6], Dancer and Schmitt showed that if f satisfies (f2)–(f4), then the positive solution of (1) with $u_{\max} \in (b_0, a_1)$ satisfies

$$u_{\max} \geq \gamma \tag{4}$$

for general domains Ω in $\mathbb{R}^n$ $(n \geq 1)$.

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In this paper we prove a nonexistence theorem which gives a $\lambda^* = \lambda^*(f, \Omega) > 0$ below which problem (1) does *not* possess any positive solutions satisfying (3); that is, we provide a lower bound for the least positive eigenvalue of (1). If Ω is a ball in $\mathbb{R}^n$ $(n \geq 1)$, and u is a positive solution of (1) with $u_{\max} \in (b_0, a_1)$, we improve (4) to

$$u_{\max} > \gamma \tag{5}$$

by modifying a technique used in [6]. Finally, by degree theory we prove a corollary of the nonexistence theorem which describes the dependence of the number of positive solutions of problem (1) on λ and we prove two nonexistence theorems for $-\Delta u = M(x, u)f(u)$.

We point our that Clement and Sweers [3] have independently shown (5) by techniques different from ours and Cosner and Schmitt's [4], provided Ω satisfies a "uniform interior sphere condition." Their method, however, seems to require more regularity of f.

Since Ω is a bounded domain, we can find a ball B with least radius R such that $\Omega \subset B$. Let $c \equiv \int_0^{a_1} f^+(s)\,ds$, where $f^+ = \max(0, f)$. Since f satisfies (f2) and (f3), we define λ^* as follows:

$$\lambda^* = \begin{cases} \gamma^2/2cR^2 & \text{if } f \text{ satisfies (f4)} \\ & \text{(see Figure 1(a), (c)), where } \gamma \text{ is defined by (2),} \\ a_0^2/2cR^2 & \text{otherwise (see Figure 1(b), (d)).} \end{cases} \tag{6}$$

We first prove Theorem 2 below in Section 2 in the case where Ω is a ball in $\mathbb{R}^n$ $(n \geq 1)$ centered at the origin by employing the famous theorem of Gidas, Ni and Nirenberg [7] on radial symmetry of positive solutions of (1) and a lower bound theorem for $u_{\max}$. Then we use a modified technique of [6] to prove (1) has no positive solutions for $\lambda \in [0, \lambda^*)$ for general domains Ω. We prove our lower bound theorem for $u_{\max}$ at the end of Section 2. In Section 3, we prove a corollary establishing the dependence of the number of solutions of problem (1) on λ. Finally, in Section 4 we extend our results to equations of the form $-\Delta u = M(x, u)f(u)$.

2. Main results

THEOREM 1 (Nonexistence of positive solutions). *There exists a number λ^* defined by* (6) *such that problem* (1), *with f satisfying* (f2) *and* (f3), *does not possess any positive solutions satisfying* (3) *if $0 \leq \lambda < \lambda^*$. Moreover, if f also satisfies* (f1), *then problem* (1) *does not possess any positive solutions satisfying* (3) *if $0 \leq \lambda \leq \lambda^*$.*

REMARK. While it is possible to give a much shorter proof of the existence of λ^*, the proof here gives quite a good estimate for the best λ^*. For example, the table below shows that our λ^* is often a much better estimate than $\lambda^{**} = \lambda_1/d$, where $d = \max_{x\in(a_0, a_1)} f'(x)$ and λ_1 is the first eigenvalue of Laplacian $-\Delta$ subject to Dirichlet boundary condition. Our estimate tends to be better if n is small and the domain is nearly circular.

We compare λ^* and λ^{**} in the case $f(x) = -(x-1)(x-2)(x-4)$ which gives $\gamma = 2.614\ldots$, $c = 5.750\ldots$, and $d = 20.333\ldots$ in the following domains $\Omega \subset \mathbb{R}^n$ $(n = 2$ or $3)$ (here we assume $\int_\Omega dx = 1$ and we may remove the requirement that $\partial\Omega$ is smooth) (cf. [2]):

TABLE 1

	Ω ($\int_\Omega dx = 1$)	λ_1	R^2	λ^{**}	λ^*	λ^{**}
$n = 2$	Circle	18.168	0.318	0.894	1.868	2.089
	Square	19.739	0.500	0.971	1.188	1.187
	Rectangle, 3 : 2	21.384	0.542	1.052	1.096	1.042
$n = 3$	Ball	25.646	0.385	1.261	1.543	1.223

THEOREM 2 (Lower bound for $u_{\max}$). *Suppose f satisfies* (f2), (f3), *and* (f4). *Let γ be defined by* (2). *Let u be a positive solution of* (1) *with* $u_{\max} \in (b_0, a_1)$. *Then u satisfies* (5) *if Ω is a ball in $\mathbb{R}^n$ $(n \geq 1)$.*

REMARK 1. Similarly, it is easy to show that Theorem 2 holds if Ω is an annular domain in $\mathbb{R}^n$ $(n \geq 1)$ and u is a positive radial solution of (1).

REMARK 2. Cosner and Schmitt [4] proved (4) by an identity of Rellich for Ω satisfying some symmetry conditions. Their proof can be improved to obtain (5) if $f(0) \geq 0$. However, it seems to the authors that the requirement $f(0) \geq 0$ can *not* be removed.

REMARK 3. Since the parameter λ play *no* role in Theorem 2, we can replace λf by f in (1) in its proof.

PROOF OF THEOREM 1. It is easy to see that if $\lambda = 0$, there is a unique trivial solution $u \equiv 0$. We first assume, in addition to (f2) and (f3), that f satisfies (f1); that is $f(0) \geq 0$. Under this assumption, we first prove the result in the special case where Ω is a ball in $\mathbb{R}^n$ $(n \geq 1)$ centered at the origin.

We assume Ω is a ball B in $\mathbb{R}^n$ $(n \geq 1)$ with radius R, centered at the origin. Suppose problem (1) has a positive solution u satisfying (3) for some λ, $0 < \lambda \leq \lambda^*$. It follows from the symmetry result of the Gidas, Ni and Nirenberg Theorem ([7]) that u is radially symmetric and u has a unique maximum at $x = 0$. Hence, u is a positive solution of the following two-point boundary value problem:

$$u''(r)+\frac{n-1}{r}u'(r)+\lambda f(u(r))=0, \qquad 0<r<R, \tag{7}$$
$$u'(0)=u(R)=0,$$

and

$$u'(r)<0 \quad \text{for } 0<r<R. \tag{8}$$

If we multiply all the terms in (7) by u' and integrate the result, we obtain

$$\frac{1}{2}[u'(r)]^2+\int_{u(0)}^{u(r)}\lambda f(s)\,ds=-(n-1)\int_0^r\frac{[u'(s)]^2}{s}\,ds\le 0. \tag{9}$$

So

$$\frac{1}{2\lambda}[u'(r)]^2+\int_{u(0)}^{u(r)}f(s)\,ds\le 0. \tag{10}$$

Thus,

$$\begin{aligned}\frac{1}{2\lambda}[u'(r)]^2 &\le \int_{u(r)}^{u(0)}f(s)\,ds,\\ &\le \int_0^{a_1}f^+(s)\,ds \qquad (u(0)<a_1),\\ &= c. \qquad \text{(Here, a trick is used.)}\end{aligned} \tag{11}$$

Therefore,

$$-(2c\lambda)^{1/2}\le u'(r) \qquad (u'(r)<0,\ \text{for } 0<r<R). \tag{12}$$

Integrating (12), we obtain

$$-\int_0^R(2c\lambda)^{1/2}\,dr\le\int_0^R u'(r)\,dr; \tag{13}$$

consequently,

$$-(2c\lambda)^{1/2}R\le u(R)-u(0)=-u(0) \qquad (\text{since } u(R)=0). \tag{14}$$

Thus, by (3) and Theorem 2 (proved below), we have

$$(2c\lambda)R^2\ge u(0)^2>\begin{cases}\gamma^2, & \text{if } f \text{ satisfies (f4)},\\ a_0^2, & \text{otherwise.}\end{cases} \tag{15}$$

Hence $\lambda>\lambda^*$. This contradicts the assumption $0\le\lambda\le\lambda^*$. Theorem 1 is now proved in the special case.

We now prove Theorem 1 for a general bounded domain Ω in $\mathbb{R}^n$ $(n\ge 1)$ with $\partial\Omega$ smooth. Let u be a positive solution of (1). The qualitative behavior of u does not change if we make a translation. Thus, we can

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assume B (which we consider to be the ball with least radius R that covers Ω) is centered at the origin.

Suppose problem (1), with f satisfying (f1), (f2), and (f3), has a positive solution u_0 satisfying (3) for some λ_0, with $0 < \lambda_0 \le \lambda^*$ (λ^* is defined by (6)). Consider the boundary value problem

$$-\Delta u = \lambda_0 f(u), \qquad x \in B, \qquad u = 0, \qquad x \in \partial B. \tag{16}$$

Define $\alpha(x)$ by

$$\alpha(x) = u_0(x) \quad \text{if } x \in \overline{\Omega}; \qquad \alpha(x) = 0 \quad \text{if } x \in \overline{B}\backslash\Omega. \tag{17}$$

Then, since $f(0) \ge 0$, $\alpha(x)$ is a lower solution and $\beta(x) \equiv a_1$ is an upper solution (see [11]) of (16). Hence, by the Method of Lower and Upper Solutions (see [11]), problem (16) has a positive solution v satisfying

$$a_0 < v_{\max} < a_1, \tag{18}$$

for some λ_0, with $0 < \lambda_0 \le \lambda^*$, which contradicts what we have proved above for the special case. So if f satisfies (f1), (f2) and (f3), then for general domains Ω, problem (1) has *no* positive solutions satisfying (3) if $0 \le \lambda \le \lambda^*$. Note that condition (f1) was needed to conclude that $\alpha(x)$ is a lower solution of (16).

Next we assume that f does *not* satisfy (f1); that is, $f(0) < 0$. For any $\varepsilon > 0$, we replace f by $\tilde{f}$ ($\tilde{f} = \tilde{f}(s, \varepsilon)$), where $\tilde{f} \in C^1$ satisfies

$$\begin{aligned} &\tilde{f}(s, \varepsilon) \ge f(s) \quad \text{for } 0 \le s \le a_0, \\ &\tilde{f}(s, \varepsilon) = f(s) \quad \text{for } a_0 \le s \le a_1, \\ &\tilde{f}(0, \varepsilon) \ge 0, \quad \tilde{f}(a_0, \varepsilon) = \tilde{f}(a_1, \varepsilon) = 0, \quad \text{and} \\ &c + \varepsilon = \int_0^{a_1} f^+(s)\, ds + \varepsilon \ge \int_0^{a_1} \tilde{f}^+(s, \varepsilon)\, ds \ge \int_0^{a_1} f^+(s)\, ds = c. \end{aligned} \tag{19}$$

Let $d \equiv \int_0^{a_1} \tilde{f}^+(s, \varepsilon)\, ds$. Note that $d = d(\varepsilon)$ is a function of ε. Care must be taken in choosing $\tilde{f}$ so that (19), especially the last line of (19), holds.

Assume (1) has a positive solution v satisfying (3) for some $\lambda > 0$. Clearly, $v_{\max} < a_1$. Then for $\lambda > 0$,

$$\Delta v + \lambda \tilde{f}(v, \varepsilon) \ge \Delta v + \lambda f(v) = 0. \tag{20}$$

Hence $\alpha(x) \equiv v$ is a lower solution of

$$\Delta u + \lambda \tilde{f}(u, \varepsilon) = 0, \qquad x \in \Omega, \qquad u = 0, \quad x \in \lambda\Omega. \tag{21}$$

As before, $\beta(x) \equiv a_1$ is an upper solution. Hence, (21) has a solution u which satisfies

$$v(x) < u(x) \le a_1; \tag{22}$$

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that is, u satisfies (3) for some $\lambda > 0$. But consider problem (21); since $\tilde{f}$ satisfies (f1), (f2) and (f3), by the previous result, (21) does *not* possess positive solutions satisfying (3) if

$$0 \leq \lambda \leq \begin{cases} \gamma^2/2dR^2 & \text{if } \tilde{f} \text{ satisfies (f4)}, \\ a_0^2/2dR^2 & \text{otherwise.} \end{cases}$$

Let $\varepsilon \to 0^+$. By (19) we find (1) does *not* possess any positive solutions if $0 \leq \lambda < \lambda^*$. This finishes the proof of Theorem 1.

PROOF OF THEOREM 2. For problem (1), suppose f satisfies (f2)–(f4). Let γ be defined by (2). Suppose Ω is a ball in $\mathbf{R}^n$ $(n \geq 1)$ with radius R centered at the origin, and let u be a positive solution of (1) with $u_{\max} \in (b_0, a_1)$. Then u satisfies (11). From (11), we find that

$$(23) \qquad 0 < \frac{1}{2\lambda}[u'(r)]^2 \leq \int_{u(r)}^{u(0)} f(s)\,ds \quad (\text{by } (8),\ u'(r) < 0 \text{ for } 0 < r < R).$$

Now suppose

$$(24) \qquad u_{\max} = u(0) \leq \gamma.$$

Then by (f4),

$$(25) \qquad 0 < \int_{u(r)}^{u(0)} f(s)\,ds \leq \int_{u(r)}^{\gamma} f(s)\,ds.$$

Choosing r so that $u(r) = a_0$, we obtain

$$(26) \qquad 0 < \int_{a_0}^{\gamma} f(s)\,ds$$

which contradicts (2). So if Ω is a ball centered at the origin, we obtain (5). By looking at (1) and making a translation, we can easily show (5) holds for any ball in $\mathbb{R}^n$ $(n \geq 1)$. The proof of Theorem 2 is complete.

3. A corollary

The previous results imply the following corollary.

COROLLARY 1 (Dependence of the number of positive solutions on λ). *If f satisfies $f(u) \geq 0$ for $u \in [0, a_1]$ in addition to* (f1′), (f2), *and* (f3), *then there exists a number $\bar{\lambda} > 0$ such that problem* (1) *has no positive solutions satisfying* (3) *if $\lambda < \bar{\lambda}$, at least one positive solution satisfying* (3) *if $\lambda = \bar{\lambda}$, and at least two positive solutions satisfying* (3) *if $\lambda > \bar{\lambda}$.*

(Two typical f's are given in Figure 1(b), (d).)

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Consider the map A_λ on $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) | u = 0 \text{ on } \partial\Omega\}$ defined by $A_\lambda(u) \equiv (-\Delta + \tilde{t}I)^{-1}(\lambda f(u) + \tilde{t}u)$, $\tilde{t} > 0$, is such that $\lambda f'(y) + \tilde{t} > 0$ on $[0, a_1]$. So solutions of (1) are fixed points of A_λ. The operator A_λ is compact; see [5] for details.

PROOF OF COROLLARY 1. It was shown in [5] that if f satisfies (f1′), (f2) and (f3) then (1) has at least two positive solutions satisfying (3) if λ is large. Suppose that if $\lambda = \lambda_d > 0$, there is one positive solution v satisfying (3). Then, since $f(u) \geq 0$ for $u \geq 0$, $u_- \equiv v$ is a lower solution of problem (1) and $u_+ \equiv a_1$ is an upper solution of (1) for all $\lambda > \lambda_d$. By the Method of Lower and Upper solutions [11] again, there is at least one positive solution satisfying (3) for problem (1) for all $\lambda > \lambda_d$. Now simply choose $\overline{\lambda}$ to be the infimum of all λ such that one can find one positive solution satisfying (3) for problem (1). But for $\lambda = 0$, there is a unique trivial solution $u \equiv 0$ for (1). By Theorem 1 we know that $0 < \overline{\lambda}$.

If $\lambda = \overline{\lambda}$, we choose a sequence $\{\lambda_n\}$, that $\lambda_n > \overline{\lambda}$, $\lambda_n \to \overline{\lambda}$. The sequence $\{u_n\}$ of solutions u_n of (1) evaluated at $\lambda = \lambda_n$ is relatively compact in $C_0(\overline{\Omega})$ (since $0 \leq u_n < a_1$ in Ω). Hence, we may assume (for a subsequence) that $u_n \to u$ strongly in $C_0(\overline{\Omega})$. Taking the limit for $A_{\lambda_u}(u_n) = u_n$, we find $A_\lambda(u) = u$. So, if $\lambda = \overline{\lambda}$, problem (1) has at least one positive solution satisfying (3).

If $\lambda > \overline{\lambda}$, we can first assume that there are finitely many positive solutions of $(1)_\lambda$ satisfying (3). Let $\overline{u} \equiv a_1$ and $\underline{u} \equiv u_{\overline{\lambda}}$ (we choose $u_{\overline{\lambda}}$ to be the maximal positive solution of $(1)_{\overline{\lambda}}$ satisfying (3); $u_{\overline{\lambda}}$ exists as proved above), so $\underline{u} < \overline{u}$, $\underline{u}$ is a lower solution which is not a solution of $(1)_\lambda$, and $\overline{u}$ is an upper solution, which is not a solution of $(1)_\lambda$. The strong maximum principle ensures that $\underline{u} < A_\lambda(\underline{u})$ and $A_\lambda(\overline{u}) < \overline{u}$ [12, page 97]. Thus, by [5, Theorem 2], A_λ has at least one positive solution in $(\underline{u}, \overline{u})$ isolated in $C_0(\overline{\Omega})$ with Leray-Schauder degree $+1$, which is stable. By using Theorem 1 and the homotopy invariance property of degree theory [12, page 131] and decreasing λ, we conclude that the sum of the degrees of the solutions of problem $(1)_\lambda$ in

$$D = \{u \in C_0(\overline{\Omega}) | u > 0 \text{ in } \Omega \text{ and } a_0 < u_{\max} < a_1\}$$

is 0. Therefore, by the excision property of degree theory [12, page 132], there is at least one solution of $(1)_\lambda$ in D with negative Leray-Schauder degree (which can be shown to be unstable) which hence is positive in Ω and satisfies (3). So, if $\lambda > \overline{\lambda}$, Problem $(1)_\lambda$ has at least two positive solutions satisfying (3); one is stable and one is unstable. This completes the proof of Corollary 1.

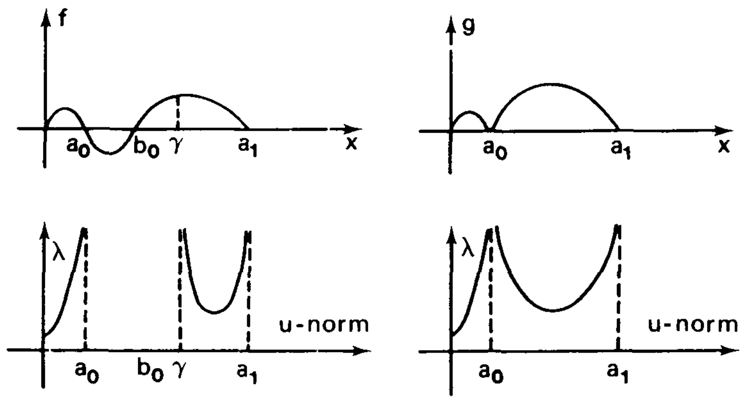

FIGURE 2. Bifurcation diagrams for the functions f and g shown as computed by Peitgen *et al.*

REMARK 1. Our results agree with the numerical results obtained by Peitgen *et al.* in [10], in which they chose two related nonlinearities f and g as in Figure 2, and used finite difference approximations to obtain numerically the set of positive solutions in some positive intervals of the corresponding boundary value problem $u'' + \lambda f(u) = 0$, $u(0) = 0 = u(\pi)$.

REMARK 2. For star shaped domains Ω, P. L. Lions [9, Theorem 3.2] has related results without requiring f to keep one sign on $[0, a_1]$. For general domains Ω, if f changes sign, the result of Corollary 1 is not known. The bifurcation diagram of problem (1) could be fairly complicated. Even though f is a cubic polynomial having three distinct positive real roots and x is one-dimensional $(n = 1)$, only some partial results are known; see [13] and [15] for references.

4. Some extensions

In this section we study the nonexistence of positive solutions of nonlinear elliptic eigenvalue problems of the form

$$(27) \qquad -\Delta u = Mf(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $M = M(x, u)$ or $M(|x|, u)$, $M > 0$, and M is C^1 in u and C^α in x, $0 < \alpha < 1$. The functions f and Ω are the same as in Section 1.

We note that, by modifying the proof of [1, Lemma 6.2], if u is a positive solution of (27) satisfying $u_{\max} \in [a_0, a_1]$, we can show that $f(u_{\max}) > 0$.

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Analogously to the definition of λ^* in (6), we define λ_0^* as follows:

$$(28)\qquad \lambda_0^* = \begin{cases} b_0^2/2cR^2 & \text{if } f \text{ satisfies (f4) (see Figure 1(a), (c))}, \\ a_0^2/2cR^2 & \text{otherwise (see Figure 1(b), (d))}. \end{cases}$$

We have obtained two nonexistence theorems and one existence theorem for some classes of functions M and f and for some domains Ω. First, by modifying the proof of Theorem 1, we are able to show our Theorem 3, in which $M = M(|x|, u)$, f satisfies (f2) and (f3), and Ω is a bounded domain which is symmetric with respect to the origin. Finally, we also prove a nonexistence theorem and an existence theorem for (27) for general domains Ω in which, in addition to (f1), (f2), and (f3), f also satisfies $f(u) \geq 0$ for $u \geq 0$.

We now consider

$$(29)\qquad -\Delta u = Mf(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where we assume Ω is symmetric with respect to the origin; that is, $x \in \Omega$ implies $-x \in \Omega$. As before we can find a ball centered at the origin with smallest radius R to cover Ω. We also assume that $M = M(|x|, u)\colon (0, R)\times(0, a_1) \to \mathbb{R}$, $M > 0$ and satisfies

(M1) $M \in C^1$ in u and C^α in x, $0 < \alpha < 1$, and
(M2) M is decreasing in $r = |x|$, $0 < r < R$.

Note that (M1) and (M2) are needed in applying the Gidas, Ni and Nirenberg Theorem [7]. Let $\max M(0, y) = \lambda_0$, for $0 \leq y \leq a_1$.

Modifying the proof of Theorem 1, we obtain

Theorem 3 (Nonexistence of positive solutions). *There exists a number λ_1^* defined by* (28) *such that problem* (29), *with f satisfying* (f2), (f3) *and $M(|x|, u)$ satisfying* (M1), (M2), *does not possess any positive solution satisfying* (3) *if $0 < \lambda_0 < \lambda_0^*$. Moreover, if f also satisfies* (f1), *then problem* (1) *does not possess any positive solution satisfying* (3) *if $0 < \lambda_0 \leq \lambda_0^*$.*

Proof of Theorem 3. The proof of Theorem 3 is similar to that of Theorem 1. Thus, we only point out the differences; these lie in obtaining results analogous to (9), (10) and (14). First, assume f satisfies (f1), (f2), and (f3), and Ω is a ball centered at the origin. Suppose problem (29) has a positive solution satisfying (3). Then the Gidas, Ni and Nirenberg Theorem [7] applies. Thus, we obtain

$$(9')\qquad \frac{1}{2}[u'(r)]^2 + \int_0^r M(t, u(t))f(u(t))u'(t)\,dt = -(n-1)\int_0^r \frac{[u'(s)]^2}{s}\,ds \leq 0.$$

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From $(9')$, we know

$$
\begin{aligned}
0 &> \int_0^r M(t, u(t)) f(u(t)) u'(t)\,dt \\
&\geq \int_0^r M(t, u(t)) f^+(u(t)) u'(t)\,dt \quad (M > 0 \text{ and } u' < 0) \\
&= M(d, u(d)) \int_0^r f^+(u(t)) u'(t)\,dt \text{ (for some } d,\ 0 < d < r; \text{ by} \\
&\qquad\qquad\qquad\qquad\qquad \text{the Mean Value Theorem for Integrals)} \\
&\geq M(0, u(d)) \int_0^r f^+(u(t)) u'(t)\,dt \quad \left(\text{by (M2) and } \int_0^r f^+(u(t)) u'(t)\,dt < 0\right) \\
&\geq \lambda_0 \int_{u(0)}^{u(r)} f^+(s)\,ds.
\end{aligned}
$$

So, by the above, we obtain

$$
(10') \qquad \frac{1}{2\lambda_0}[u'(r)]^2 + \int_{u(0)}^{u(r)} f^+(s)\,ds \leq 0
$$

(note the difference between (10) and $(10')$). Therefore,

$$
(11') \qquad
\begin{aligned}
\frac{1}{2\lambda_0}[u'(r)]^2 &\leq \int_{u(r)}^{u(0)} f^+(s)\,ds \\
&\leq \int_0^{a_1} f^+(s)\,ds \quad (u(0) < a_1) \\
&= c.
\end{aligned}
$$

Following the argument between (11) and (14) of the proof of Theorem 1, we only obtain

$$
(15') \qquad (2c\lambda_0)R^2 \geq u(0)^2 > \begin{cases} b_0^2, & \text{if } f \text{ satisfies (f4)}, \\ a_0^2, & \text{otherwise}, \end{cases}
$$

so $\lambda_0 > \lambda_0^*$, which is slightly different from (15). This contradicts the assumption $0 < \lambda_0 \leq \lambda_0^*$. The case in which Ω is a ball is finished.

The rest of the proof for the cases in which Ω is not a ball and f does *not* satisfy (f1) is quite similar to that in Theorem 1. The proof of Theorem 3 is complete.

Applying the Method of Lower and Upper Solutions [11] again with Theorem 1, we can easily obtain the following for general smooth bounded domains Ω.

THEOREM 4 (Nonexistence of positive solutions). *In addition to* $(\mathrm{f1}')$, (f2), *and* (f3), *if* f *satisfies* $f(u) \geq 0$ *for* $u \in [0, a_1]$, *then problem* (27)

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does not possess any positive solutions satisfying (3) *if* $\sup M(x, y) \le \lambda^*$ (λ^* *is defined by* (6)) *for* $(x, y) \in \Omega \times (0, a_1)$.

Similarly, applying the Method of Lower and Upper Solutions [10] again with Dancer's [5, Theorem 3], we can easily obtain the following for general smooth bounded domains.

THEOREM 5 (Existence of positive solutions). *In addition to* (f1′), (f2), *and* (f3), *if* f *satisfies* $f(u) \ge 0$ *for* $u \in [0, a_1]$, *then problem* (27) *possesses at least one positive solution satisfying* (3) *if there exists a positive number* $\tilde{\lambda}$ (*cf.* [8, *page* 952]), *large enough, such that* $\tilde{\lambda} < \inf M(x, y)$ *for* $(x, y) \in \Omega \times (0, a_1)$.

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Department of Mathematics
National Tsing Hua University
Hsinchu
Taiwan 300, R.O.C.

Department of Mathematics
State University of New York
Buffalo, New York, 14214-3093
U.S.A.