VARIETY INVARIANTS FOR MODULAR LATTICES

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A *variety* (primitive class) is a class of abstract algebras which is closed under the formation of subalgebras, homomorphic images, and products. For a given variety $\mathfrak{A}$ we shall call a function μ^*, which assigns to each algebra $A \in \mathfrak{A}$ a natural number or ∞, denoted by $\mu^*(A)$, a *variety invariant* if for every natural number n the class of all $A \in \mathfrak{A}$ with $\mu^*(A) \leqq n$ is again a variety. In this paper, a general method of finding variety invariants for the variety of all modular lattices will be developed. This method will be based on the concept of a *quotient tree* of a modular lattice. As examples of variety invariants we shall define, using the general result, the *primitive length* and the *primitive width* of modular lattices.

We start with an arbitrary lattice L. A pair $(a, b) \in L \times L$ is called a *quotient* of L and denoted by a/b if $a \geqq b$. Let $Q(L)$ be the set of all quotients of L. On $Q(L)$ we define the following binary relations (let a_1/b_1 and a_2/b_2 be quotients of L):

(i) $a_1/b_1 \lhd a_2/b_2$ if $a_1 \cap b_2 = b_1$ and $a_1 \cup b_2 = a_2$;

(ii) $a_1/b_1 \rhd a_2/b_2$ if $b_1 \cap a_2 = b_2$ and $b_1 \cup a_2 = a_1$;

(iii) $a_1/b_1 \square\, a_2/b_2$ if either $a_1/b_1 \lhd a_2/b_2$ or $a_1/b_1 \rhd a_2/b_2$; then a_1/b_1 and a_2/b_2 are said to be *transposed*;

(iv) $a_1/b_1 \approx a_2/b_2$ if there are $x_i/y_i \in Q(L)$ with $i = 1, \ldots, n$ such that $a_1/b_1 = x_1/y_1$, $a_2/b_2 = x_n/y_n$, and $x_i/y_i \square\, x_{i+1}/y_{i+1}$ for $i = 1, \ldots, n-1$; then a_1/b_1 and a_2/b_2 are said to be *projective*;

(v) $a_1/b_1 < a_2/b_2$ if $a_1 \leqq b_2$ and $a_1 < a_2$.

If we consider $Q(L)$ together with the binary relations $\square$ and $<$, we obtain $Q(L) \equiv (Q(L), \square, <)$ as a *mixed graph* (i.e., a set together with one symmetric and one anti-symmetric binary relation).

Now we are able to define a quotient tree of a modular lattice L. We shall call a pair (T, τ), where $T \equiv (T, \square, <)$ is a finite mixed graph and τ a mapping from T into $Q(L)$, a *quotient tree* of L if the following conditions are satisfied:

(i) $(T, \square)$ is a tree (i.e., a connected graph without circuits);

(ii) $s\tau \square\, t\tau$ if $s \square\, t$ for $s, t \in T$;

(iii) $s\tau < t\tau$ if and only if $s < t$ for $s, t \in T$;

(iv) If $a_t/b_t = t\tau$, then $a_t \neq b_t$.

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Every lattice homomorphism ρ from a modular lattice K into a modular lattice L induces a graph homomorphism $\hat{\rho}$ from $(Q(K), \Box)$ into $(Q(L), \Box)$ (define $(a/b)\hat{\rho} = a\rho/b\rho$). For quotient trees (S, σ) of K and (T, τ) of L we say that (S, σ) *induces* (T, τ) *under* ρ and write $(S, \sigma)\ \rho\ (T, \tau)$ if there is a graph isomorphism α from the tree $(S, \Box)$ onto the tree $(T, \Box)$ with $s\alpha\tau = s\sigma\hat{\rho}$ for all $s \in S$.

LEMMA. *Let K and L be modular lattices and ρ a lattice homomorphism from K onto L. Let (T, τ) be a quotient tree of L. Then there is a quotient tree (S, σ) of K and a graph isomorphism α from $(S, \Box, <)$ onto $(T, \Box, <)$ such that $s\alpha\tau = s\sigma\hat{\rho}$ for all $s \in S$; in particular, (S, σ) induces (T, τ) under ρ.*

Proof. We prove the lemma by induction over the cardinality of T. The case $|T| = 0$ is obvious.

$|T| = 1$. We have that $T\tau = \{a/b\}$. Take $c \in a\rho^{-1}$ and $d \in b\rho^{-1}$. Define $d' = c \cap d$. It follows that $c\rho = a$, $d'\rho = (c \cap d)\rho = c\rho \cap d\rho = a \cap b = b$, and $c > d'$ since $a \neq b$. If we choose $S = T$ and $S\sigma = \{c/d'\}$, (S, σ) is a quotient tree of K which has the desired properties.

$|T| = n > 1$. We assume that the claim is true for every quotient tree of L with $n - 1$ elements. Since $(T, \Box)$ is a tree, there exists an element $r^* \in T$ for which there is exactly one $r \in T$ with $r \Box r^*$. Let T_r be the section graph defined by $T - \{r^*\}$ and τ_r the restriction of τ to $T - \{r^*\}$. Then (T_r, τ_r) is a quotient tree of L with $n - 1$ elements. Hence, there is a quotient tree (S_r, σ_r) of K and a graph isomorphism α_r from S_r onto T_r with $s\alpha_r\tau_r = s\sigma_r\hat{\rho}$ for all $s \in S_r$. Now we wish to extend (S_r, σ_r) and α_r to the desired (S, σ) and α. By duality, we only have to consider the case $r\tau \lhd r^*\tau$. We use the notation $a_t/b_t = t\tau$ for $t \in T$ and $c_t/d_t = t\alpha_r^{-1}\sigma_r$ for $t \in T - \{r^*\}$. Clearly, $c_t\rho = a_t$ and $d_t\rho = b_t$. Take $d \in b_{r^*}\rho^{-1}$. Define $j = \bigcup(c_t\colon t < r^*)$ and $m = \bigcap(d_t\colon r^* < t)$. Since the relation $<$ is transitive, we have that $j \leqq m$. Define $d_{r^*} = j \cup (d \cap m)$. By modularity, we also have that $d_{r^*} = (j \cup d) \cap m$ which shows that $j \leqq d_{r^*} \leqq m$. Define $d_{r^*}' = d_r \cup d_{r^*}$ and $c_{r^*} = c_r \cup d_{r^*}'$. It follows that

$$d_{r^*}'\rho = (d_r \cup j \cup (d \cap m))\rho = d_r\rho \cup j\rho \cup (d\rho \cap m\rho) =$$
$$b_r \cup \bigcup(a_t\colon t < r^*) \cup (b_{r^*} \cap \bigcap(b_t\colon r^* < t)) = b_r \cup b_{r^*} = b_{r^*},$$
$$c_{r^*}\rho = (c_r \cup d_{r^*}')\rho = c_r\rho \cup d_{r^*}'\rho = a_r \cup b_{r^*} = a_r$$

and $c_{r^*} > d_{r^*}'$ since $a_{r^*} \neq b_{r^*}$. Since $c_r \leqq m$ and $d_{r^*}' \leqq m$, we have that $c_{r^*} \leqq m$. The essential part of the proof would be completed if c_r/d_r and c_{r^*}/d_{r^*}' are transposed; but this need not be true. To obtain the desired quotient tree we have to change the d_t's. Since $(T, \Box)$ is a tree, for each $t \in T - \{r^*\}$ there exists a unique element $t^* \in T$ such that $t \Box t^*$, and there are $t_0, \ldots, t_n \in T$ with $t^* = t_n \Box \ldots \Box t_0 = r^*$ and $t \neq t_i$ for $i = 0, \ldots, n$. Now, we define by induction for $t \in T - \{r^*\}$:

$$d_t' = \begin{cases} c_t \cap d_{t^*}' & \text{if } t\tau \lhd t^*\tau, \\ d_t \cup d_{t^*}' & \text{if } t\tau \rhd t^*\tau. \end{cases}$$

Claim 1. $d_t'\rho = b_t$ *for* $t \in T$.

Proof by induction. The case $t = r^*$ is already proved. Assume that $d_{t^*}'\rho = b_t$ for some $t \in T$. It follows that
if $t\tau \lhd t^*\tau$, then $d_t'\rho = (c_t \cap d_{t^*}')\rho = c_t\rho \cap d_{t^*}'\rho = a_t \cap b_{t^*} = b_t$;
if $t\tau \rhd t^*\tau$, then $d_t'\rho = (d_t \cup d_{t^*}')\rho = d_t\rho \cup d_{t^*}'\rho = b_t \cup b_{t^*} = b_t$.

Claim 2. $d_t' \geqq d_t$ *for* $t \in T$.

Proof by induction. The case $t = r^*$ is obvious. If $t = r$, then

$$d_r' = c_r \cap d_{r^*}' = c_r \cap (d_r \cup d_{r^*}) = (c_r \cap d_{r^*}) \cup d_r \geqq d_r.$$

Assume that $d_{t^*}' \geqq d_{t^*}$ for some $t \in T$ and $t \neq r$. It follows that
if $t\tau \lhd t^*\tau$, then $d_t' = c_t \cap d_{t^*}' \geqq c_t \cap d_{t^*} = d_t$;
if $t\tau \rhd t^*\tau$, then $d_t' = d_t \cup d_{t^*}' \geqq d_t$.

Claim 3. $d_t' < c_t$ *for* $t \in T$.

Proof by induction. The case $t = r^*$ is already proved. Assume that $d_{t^*}' < c_{t^*}$ for some $t \in T$. It follows that
if $t\tau \lhd t^*\tau$, then $d_t' = c_t \cap d_{t^*}' \leqq c_t$;
if $t\tau \rhd t^*\tau$, then $d_t' = d_t \cup d_{t^*}' \leqq d_t \cup c_{t^*} = c_t$.
Since $c_t\rho = a_t$, $d_t'\rho = b_t$ by Claim 1 and $a_t \neq b_t$, we have that $c_t \neq d_t'$ for $t \in T$.

Claim 4. $c_t/d_t' \,\square\, c_{t^*}/d_{t^*}'$ *for* $t \in T - \{r^*\}$.

Proof. If $t = r$, then $c_t \cap d_{t^*}' = d_t'$ and $c_t \cup d_{t^*}' = c_t$ by definition. Take $t \neq r$. It follows that
if $t\tau \lhd t^*\tau$, then

$$c_t \cap d_{t^*}' = d_t' \quad \text{and} \quad c_t \cup d_{t^*}' = c_t \cup d_{t^*} \cup d_{t^*}' = c_{t^*} \cup d_{t^*}' = c_{t^*};$$

if $t\tau \rhd t^*\tau$, then

$$d_t' \cap c_{t^*} = (d_t \cup d_{t^*}') \cap c_{t^*} = d_{t^*}' \cup (d_t \cap c_{t^*}) = d_{t^*}' \cup d_{t^*} = d_{t^*}'$$

and $d_t' \cup c_{t^*} = d_t \cup d_{t^*}' \cup c_{t^*} = d_t \cup c_{t^*} = c_t$.

Claim 5. $a_{t_1} \leqq b_{t_2}$ *if and only if* $c_{t_1} \leqq d_{t_2}'$ *for* $t_1, t_2 \in T$.

Proof. Since $a_{t_1} \leqq b_{t_2}$ implies $c_{t_1} \leqq d_{t_2}$ for $t_1, t_2 \in T$, it follows by Claim 2 that $a_{t_1} \leqq b_{t_2}$ implies $c_{t_1} \leqq d_{t_2}'$ for $t_1, t_2 \in T$. The converse is also true since $c_t\rho = a_t$ and $d_t'\rho = b_t$ by Claim 1 for all $t \in T$.

Now we are ready to state the quotient tree (S, σ) and the graph isomorphism α. Define $S = S_r \cup \{r^*\}$, $r^*\alpha = r^*$ and $s\alpha = s\alpha_r$ for $s \in S_r$. Extend $(S_r, \square, <)$ to $(S, \square, <)$ in such a way that α becomes an isomorphism from $(S, \square, <)$ onto $(T, \square, <)$. Then we define $s\sigma = c_{s\alpha}/d_{s\alpha}'$ for $s \in S$. Claims 3, 4, and 5 imply that (S, σ) is a quotient tree of K. But we also have that $s\alpha\tau = s\sigma\hat{\rho}$ for all $s \in S$ by Claim 1. Thus, the proof of the lemma is complete.

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We shall call a function μ which assigns to each quotient tree (T, τ) a natural number $\mu(T, \tau)$ a QT-*invariant* if

(i) $(S, \square, <) \cong (T, \square, <)$ implies $\mu(S, \sigma) = \mu(T, \tau)$;

(ii) $(S, \sigma)\ \rho\ (T, \tau)$ implies $\mu(S, \sigma) \leqq \mu(T, \tau)$.

We shall call μ a QT*-*invariant* if

(ii*) $(S, \sigma)\ \rho\ (T, \tau)$ implies $\mu(S, \sigma) = \mu(T, \tau)$.

THEOREM. *Let μ be a* QT- *or a* QT*-*invariant. For every modular lattice L we define $\mu^*(L) = \sup\{\mu(T, \tau)$: (T, τ) quotient tree of $L\}$. Then μ^* is a variety invariant of the variety of all modular lattices.*

Proof. Let L be a sublattice of the modular lattice K. Since every quotient tree of L can be considered as a quotient tree of K, we have that $\mu^*(L) \leqq \mu^*(K)$. Let L be a homomorphic image of the modular lattice K and (T, τ) a quotient tree of L. By the lemma there is a quotient tree (S, σ) of K such that $(S, \square, <) \cong (T, \square, <)$ and $(S, \sigma)\ \rho\ (T, \tau)$. Hence, $\mu(S, \sigma) = \mu(T, \tau)$. Therefore, we have that $\mu^*(L) \leqq \mu^*(K)$. Let L be the product of the modular lattices K_ω with $\omega \in \Omega$ and (T, τ) a quotient tree of L. Then there is a projection π_{ω_0}: $L \rightarrow K_{\omega_0}$ such that $t\tau\hat{\pi}_{\omega_0} = a_t/b_t$ implies that $a_t \neq b_t$ for $t \in T$. Extend $(T, \square, <)$ to $(T_0, \square_0, <_0)$, leaving $(T, \square) = (T_0, \square_0)$, such that $(T_0, \tau\hat{\pi}_{\omega_0})$ becomes a quotient tree of K_{ω_0}. Obviously, $(T, \tau)\pi_{\omega_0}(T_0, \tau\hat{\pi}_{\omega_0})$. Hence, $\mu(T, \tau) \leqq \mu(T_0, \tau\hat{\pi}_{\omega_0})$. Thus, for every quotient tree (T, τ) of L there is an $\omega \in \Omega$ and a quotient tree (T_ω, τ_ω) of K_ω with $\mu(T, \tau) \leqq \mu(T_\omega, \tau_\omega)$. All together, this shows that the class of all modular lattices L with $\mu^*(L) \leqq n$ for a natural number n is closed under the formation of sublattices, homomorphic images and products. Therefore, μ^* is a variety invariant of the variety of all modular lattices.

Examples. (i) For a quotient tree (T, τ), the *length* $\lambda(T, \tau)$ is defined as the maximal cardinality of an index set I such that there are $t_i \in T$ $(i \in I)$ with $t_i < t_j$ or $t_j < t_i$ for $i, j \in I$ and $i \neq j$. Clearly, $(S, \square, <) \cong (T, \square, <)$ implies that $\lambda(S, \sigma) = \lambda(T, \tau)$. But we also have that $(S, \sigma)\ \rho\ (T, \tau)$ implies that $\lambda(S, \sigma) \leqq \lambda(T, \tau)$ since $\sigma\hat{\rho}$ preserves the relation $<$. Therefore, λ is a QT-invariant. For every modular lattice L let us define the *primitive length* $\lambda^*(L) = \sup\{\lambda(T, \tau)$: (T, τ) *a quotient tree of* $L\}$. Then, by the theorem, the primitive length of modular lattices is a variety invariant. Obviously, the primitive length of a modular lattice L is the supremum of all numbers n such that there are $a_i/b_i \in Q(L)$ $(1 \leqq i \leqq n)$ with

$$a_i/b_i < a_{i+1}/b_{i+1} \quad \text{and} \quad a_i/b_i \approx a_{i+1}/b_{i+1}$$

for $i = 1, \ldots, n - 1$. For a simple modular lattice of finite length (in this case, every two prime quotients are projective), the length and the primitive length coincide.

(ii) For a quotient tree (T, τ), the *width* $\omega(T, \tau)$ is defined as the maximal cardinality of an index set I such that there are $s_{ij}, t_{ij} \in T$ $(i, j \in I$ and $i \neq j)$ with $s_{ij} \square t_{ji}$ and $s_{ij} < t_{ik}$ for $i, j, k \in I$. It easily follows that ω is a QT-

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invariant. For every modular lattice L let us define the *primitive width* $\omega^*(L) = \sup\{\omega(T, \tau): (T, \tau)$ *a quotient tree* of $L\}$. Then, by the theorem, the primitive width of modular lattices is a variety invariant. For the lattice L_n ($3 \leqq n < \infty$) of subspaces of a projective line P_n with n points $p_1, \ldots, p_n$ we obtain $\omega^*(L_n) = n$; the following quotient tree (T, τ) of L_n has width n:

$$T = \{s_{ij}: 1 \leqq i, j \leqq n \quad \text{and} \quad i \neq j\} \cup \{t_{ij}: 1 \leqq i, j \leqq n \quad \text{and} \quad i \neq j\},$$

$s_{ij} \square t_{ji}$, $s_{12} \square t_{ij}$ for $2 \leqq i \leqq n$, $s_{23} \square t_{1j}$, $s_{ij} < t_{ik}$, $s_{ij}\tau = p_i/\emptyset$ and $t_{ij}\tau = P_n/p_i$. Conversely, let (T, τ) be a quotient tree of L_n with the desired $s_{ij}, t_{ij} \in T$ ($i, j \in I$ and $i \neq j$); then $s_{ij}\tau = s_{kl}\tau$ ($t_{ij}\tau = t_{kl}\tau$, respectively) if and only if $i = k$; hence, $|I| \leqq n$, and therefore $\omega(T, \tau) \leqq n$.

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