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On the Generalized Auslander–Reiten Conjecture under Certain Ring Extensions

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Abstract. We show that under some conditions a Gorenstein ring R satisfies the Generalized Auslander–Reiten conjecture if and only if $R[x]$ does. When R is a local ring we prove the same result for some localizations of $R[x]$.

1 Introduction

Convention In this paper, R is a commutative noetherian ring with identity, and all R-modules are unital.

The original version of a conjecture of Auslander and Reiten [2] is a generalized version of a conjecture of Nakayama. It states that over an Artin algebra Λ, vanishing of $\operatorname{Ext}^i_\Lambda(M, M \oplus \Lambda)$ for a finitely generated Λ-module M and for all $i > 0$ implies that M is projective. Auslander and Reiten proved this conjecture for several classes of rings. Later, Auslander, Ding, and Solberg [1] studied this conjecture for arbitrary commutative noetherian rings. This version is known as the *Auslander–Reiten conjecture* and is proved affirmatively in some special cases; see for instance [3, 5–7, 9]. A generalized version of the Auslander-Reiten conjecture, studied by several authors, including [4, 10, 11], follows.

Conjecture 1.1 *Let n be a positive integer, and let M be a finitely generated R-module. If $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for all $i > n$, then $\operatorname{pd}_R(M) \le n$.*

Related to this, the *finitistic extension degree* of the ring R, denoted $\operatorname{fed}(R)$, has been recently introduced by Diveris in [4]. This invariant is tightly connected to Conjecture 1.1 over Gorenstein rings.

Theorem 1.2 ([4, Corollary 3.2]) *If R is Gorenstein, then $\operatorname{fed}(R)$ is finite if and only if Conjecture 1.1 holds for R.*

By definition,

$$\operatorname{fed}(R) = \sup\{n \mid \operatorname{Ext}^i_R(M, M) = 0 \text{ for all } i > n \text{ and } \operatorname{Ext}^n_R(M, M) \neq 0\},$$

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where the supremum is taken over all finitely generated R-modules M such that $\mathrm{Ext}^i_R(M,M) = 0$ for $i \gg 0$. Diveris also studies the behavior of this dimension under certain base changes of rings. Among other results, he proves that when $(R,\mathfrak{m})$ is Gorenstein and local, finiteness of any of $\mathrm{fed}(R)$, $\mathrm{fed}(\widehat{R})$, $\mathrm{fed}(R[[x]])$, or $\mathrm{fed}(R[x]_{(\mathfrak{m},x)})$ implies finiteness of the others; see [4, Theorem 4.4]. In other words, when $(R,\mathfrak{m})$ is a Gorenstein local ring, if one of the rings R, $\widehat{R}$, $R[[x]]$, or $R[x]_{(\mathfrak{m},x)}$ satisfies Conjecture 1.1, then they all do.

The aim of this paper is to examine how finiteness of the finitistic extension degree of a Gorenstein ring, hence the condition of Conjecture 1.1, is preserved under certain faithfully flat ring extensions. Section 2 of this paper is devoted to the primary results and the notation that is used in the entire paper. In Section 3, we prove our main theorems (Theorems 3.4 and 3.6) that are related to the base ring extensions $R \to R[x]$ when R is a Gorenstein ring, and $R \to R[x]_{\mathfrak{m}R[x]}$ when R is a Gorenstein local ring with the maximal ideal $\mathfrak{m}$. This section also contains another base change result (Proposition 3.3) that may be of independent interest. In Section 4 we investigate Conjecture 1.1 under a different ring extension. As a consequence, we improve Theorem 3.4 slightly when R is local.

2 Preliminaries

We begin by establishing the notation that will be used in the paper.

Notation 2.1 Let R be an algebra over a field k and let M be a module over the polynomial ring $R[x]$. For an element $\alpha \in k$ we set

$$M_\alpha := M \otimes_{k[x]} \big(k[x]/(x-\alpha)k[x]\big).$$

Remark 2.2 We work in the setting of Notation 2.1. For each element $\alpha \in k$, the module M_α is simply a residue module of M by the submodule $(x-\alpha)M$. If M is a finitely generated $R[x]$-module, then it has a presentation of the form

$$R[x]^{\beta_1} \xrightarrow{m(x)} R[x]^{\beta_0} \longrightarrow M \longrightarrow 0,$$

where β_0, β_1 are non-negative integers and $m(x)$ is a $\beta_0 \times \beta_1$-matrix consisting of polynomials in $R[x]$. Therefore, by using the isomorphism

$$R = R[\alpha] \cong R[x]/(x-\alpha)R[x], \tag{2.1}$$

the module M_α has a presentation of the form

$$R^{\beta_1} \xrightarrow{m(\alpha)} R^{\beta_0} \longrightarrow M_\alpha \longrightarrow 0,$$

where $m(\alpha)$ is a $\beta_0 \times \beta_1$-matrix consisting of elements in R. In particular, if M is a finitely generated $R[x]$-module, then M_α is a finitely generated R-module.

For more information about M_α we refer the reader to [12].

The following lemma will be used several times in this paper.

Lemma 2.3 *Let R be an algebra over a field k, and let $\alpha \in k$. If $x-\alpha$ is a non-zero-divisor on an $R[x]$-module M, then for each i we have* $\operatorname{Ext}^i_{R[x]}(M, M_\alpha) \cong \operatorname{Ext}^i_R(M_\alpha, M_\alpha)$.

Proof This isomorphism follows from (2.1) and [8, Lemma 2(ii), p. 140]. ∎

The following two lemmas are important tools that we use in the proofs of our main results in the next section.

Lemma 2.4 *Let R be an algebra over a field k, and let M be an $R[x]$-module. Then for each ideal I of $R[x]$ and each $\alpha \in k$, we have $(IM)_\alpha = IM_\alpha$ and $(M/IM)_\alpha \cong M_\alpha/IM_\alpha$.*

Proof Without loss of generality we can assume that I is non-zero. For the first equality note that M_α has an $R[x]$-module structure that uses the natural ring homomorphism $R[x] \to R[x]_\alpha$. This gives the second equality in the following display:

$$\begin{aligned}(IM)_\alpha &= (IM) \otimes_{k[x]} k[x]/(x-\alpha)k[x] \\ &= I\big(M \otimes_{k[x]} k[x]/(x-\alpha)k[x]\big) = IM_\alpha.\end{aligned}$$

For the second equality in the statement of the lemma, let $I = (t_1, \dots, t_l)$ for some positive integer l. Then we have an exact sequence

$$\bigoplus_{j=1}^{l} M \xrightarrow{f} M \longrightarrow M/IM \longrightarrow 0,$$

where f is the matrix multiplication by $(t_1, \dots, t_l)$. By applying the right exact functor $- \otimes_{k[x]} k[x]/(x-\alpha)k[x]$ we get an exact sequence

$$\bigoplus_{j=1}^{l} M_\alpha \xrightarrow{f\otimes \mathrm{id}} M_\alpha \longrightarrow (M/IM)_\alpha \longrightarrow 0,$$

where id is the identity map on $k[x]/(x-\alpha)k[x]$. Now we have

$$(M/IM)_\alpha \cong M_\alpha/\operatorname{Im}(f\otimes \mathrm{id}) = M_\alpha/(\operatorname{Im}(f))_\alpha = M_\alpha/(IM)_\alpha = M_\alpha/IM_\alpha$$

as desired. ∎

Lemma 2.5 *Let R be an algebra over a field k, and let $\alpha \in k$. If $x-\alpha$ is a non-zero-divisor on an $R[x]$-module M, then for each $i \geq 0$ there exists an exact sequence*

$$\begin{aligned}0 \longrightarrow \operatorname{Ext}^i_{R[x]}(M,M)_\alpha &\longrightarrow \operatorname{Ext}^i_R(M_\alpha, M_\alpha) \\ &\longrightarrow \operatorname{Tor}_1^{k[x]}\big(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]\big) \longrightarrow 0.\end{aligned}$$

Proof Using the exact sequence

$$0 \longrightarrow M \xrightarrow{x-\alpha} M \longrightarrow M_\alpha \longrightarrow 0,$$

we get the long exact sequence

$$\begin{aligned}\operatorname{Ext}^i_{R[x]}(M,M) &\xrightarrow{x-\alpha} \operatorname{Ext}^i_{R[x]}(M,M) \\ &\longrightarrow \operatorname{Ext}^i_{R[x]}(M,M_\alpha) \longrightarrow \operatorname{Ext}^{i+1}_{R[x]}(M,M) \xrightarrow{x-\alpha} \cdots,\end{aligned}$$

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where $x-\alpha$ represents the multiplication map

$$\operatorname{Ext}^i_{R[x]}(M,M) \xrightarrow{x-\alpha} \operatorname{Ext}^i_{R[x]}(M,M).$$

By Lemma 2.3 we have an isomorphism $\operatorname{Ext}^i_{R[x]}(M,M_\alpha) \cong \operatorname{Ext}^i_R(M_\alpha,M_\alpha)$ for each integer i. Thus, for each integer i we get the following short exact sequence

$$0 \longrightarrow \operatorname{Coker}(x-\alpha) \longrightarrow \operatorname{Ext}^i_R(M_\alpha,M_\alpha) \longrightarrow \ker(x-\alpha) \to 0.$$

By definition we have $\operatorname{Coker}(x-\alpha) = \operatorname{Ext}^i_{R[x]}(M,M)_\alpha$. Now to compute $\ker(x-\alpha)$, the short exact sequence

$$0 \longrightarrow k[x] \xrightarrow{x-\alpha} k[x] \longrightarrow k[x]/(x-\alpha)k[x] \longrightarrow 0$$

gives us the following long exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{k[x]}\left(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]\right) \longrightarrow \operatorname{Ext}^{i+1}_{R[x]}(M,M) \otimes_{k[x]} k[x] \xrightarrow{x-\alpha} \operatorname{Ext}^{i+1}_{R[x]}(M,M) \otimes_{k[x]} k[x]$$

So we have $\ker(x-\alpha) \cong \operatorname{Tor}_1^{k[x]}\left(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]\right)$. This completes the proof of the lemma. ∎

We recall a straightforward fact that will be used later.

Fact 2.6 Let R be a Gorenstein ring, and let $0 \to M_1 \to F \to M \to 0$ be an exact sequence of finitely generated R-modules where F is projective. It follows from the long exact sequence

$$\operatorname{Ext}^i_R(M,F) \longrightarrow \operatorname{Ext}^i_R(M,M) \longrightarrow \operatorname{Ext}^{i+1}_R(M,M_1) \longrightarrow \operatorname{Ext}^{i+1}_R(M,F)$$

that $\operatorname{Ext}^i_R(M,M) \cong \operatorname{Ext}^{i+1}_R(M,M_1)$ for all $i > \operatorname{id}_R(F)$. Also, by using the long exact sequence

$$\operatorname{Ext}^i_R(F,M_1) \longrightarrow \operatorname{Ext}^i_R(M_1,M_1) \longrightarrow \operatorname{Ext}^{i+1}_R(M,M_1) \longrightarrow \operatorname{Ext}^{i+1}_R(F,M_1),$$

we get the isomorphism $\operatorname{Ext}^i_R(M_1,M_1) \cong \operatorname{Ext}^{i+1}_R(M,M_1)$ for all $i > 0$. Therefore, for the finitely generated module M over the Gorenstein ring R, we have $\operatorname{Ext}^i_R(M,M) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(M_1,M_1) = 0$ for all $i \gg 0$.

3 Main Results

In this section, we prove our main theorems about Conjecture 1.1; see Theorems 3.4 and 3.6. The method that we use for the proofs of both theorems uses the well-known decomposition of finitely generated modules over principal ideal domains. We begin this section with the following lemma.

Lemma 3.1 *Let k be an algebraically closed field, and let R be a Gorenstein k-algebra. Let $k(x)$ be a transcendental extension of k, and assume that both* $\operatorname{fed}(R)$ *and* $\operatorname{fed}(R \otimes_k k(x))$ *are finite. Then* $\operatorname{fed}(R[x]) < \infty$.

Proof Set $f := \operatorname{fed}(R) < \infty$ and $g := \operatorname{fed}(R \otimes_k k(x)) < \infty$. Let M be a finitely generated $R[x]$-module such that $\operatorname{Ext}^i_{R[x]}(M, M) = 0$ for all $i \gg 0$. Since $R[x]$ is a Gorenstein ring, by Fact 2.6 we can replace M with its first syzygy in an arbitrary free resolution and assume that M is submodule of a free $R[x]$-module. In particular, M is assumed to be a torsion-free $R[x]$-module.

Now for each $\alpha \in k$, by using the short exact sequence

$$0 \longrightarrow M \xrightarrow{x-\alpha} M \longrightarrow M_\alpha \longrightarrow 0, \tag{3.1}$$

our vanishing assumption is equivalent to $\operatorname{Ext}^i_{R[x]}(M, M_\alpha) = 0$ for all $i \gg 0$, and by Lemma 2.3, this is equivalent to $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i \gg 0$. Since by Remark 2.2 the R-module M_α is finitely generated, our assumption implies that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i > f$. Therefore, by using the long exact sequence of Ext obtained from the short exact sequence (3.1) and the isomorphism $\operatorname{Ext}^i_{R[x]}(M, M_\alpha) \cong \operatorname{Ext}^i_R(M_\alpha, M_\alpha)$ from Lemma 2.3, we observe that $x - \alpha$ acts bijectively on $\operatorname{Ext}^i_{R[x]}(M, M)$ for all $i > f + 1$ and all $\alpha \in k$.

Now consider the set

$$S := \left\{ (x-\alpha_1)^{n_1} \dots (x-\alpha_l)^{n_l} \mid n_j \in \mathbb{N} \cup \{0\} \text{ and } \alpha_j \in k \text{ for } 1 \leq j \leq l \right\}.$$

Note that S is a multiplicatively closed subset of $k[x]$ and for each $i > f + 1$ the cohomology module $\operatorname{Ext}^i_{R[x]}(M, M)$ is an $S^{-1}k[x]$-module.

On the other hand, since k is algebraically closed, we have the equality $S^{-1}k[x] = k(x)$, and therefore,

$$S^{-1}R[x] \cong S^{-1}(R \otimes_k k[x]) = R \otimes_k (S^{-1}k[x]) = R \otimes_k k(x).$$

This isomorphism gives the last isomorphism in the next display for each $i > f + 1$:

$$\begin{aligned} \operatorname{Ext}^i_{R[x]}(M, M) &\cong S^{-1} \operatorname{Ext}^i_{R[x]}(M, M) \cong \operatorname{Ext}^i_{S^{-1}R[x]}(S^{-1}M, S^{-1}M) \\ &\cong \operatorname{Ext}^i_{R \otimes_k k(x)}(S^{-1}M, S^{-1}M). \end{aligned}$$

Hence, $\operatorname{Ext}^i_{R \otimes_k k(x)}(S^{-1}M, S^{-1}M) = 0$ for all $i \gg 0$. Our assumption implies that $\operatorname{Ext}^i_{R \otimes_k k(x)}(S^{-1}M, S^{-1}M) = 0$ for all $i > g$. Therefore, $\operatorname{Ext}^i_{R[x]}(M, N) = 0$ for all $i > h := \max\{f + 1, g\}$. Note that h is independent of the choice of the finitely generated $R[x]$-module M, and we conclude that $\operatorname{fed}(R[x]) \leq h < \infty$. ■

Remark 3.2 It is straightforward to check that if $R \to S$ is a faithfully flat ring homomorphism with $\operatorname{fed}(S) < \infty$, then $\operatorname{fed}(R) \leq \operatorname{fed}(S) < \infty$. This follows from the fact that for every finitely generated R module M and for each integer i we have $\operatorname{Ext}^i_R(M, M) = 0$ if and only if $\operatorname{Ext}^i_S(M \otimes_R S, M \otimes_R S) = 0$. Therefore, it follows from Theorem 1.2 that if R and S are Gorenstein rings such that S satisfies Conjecture 1.1, then so does R.

Proposition 3.3 *Let k be an uncountable field, and let R be a Gorenstein finite dimensional k-algebra. Let $k(x)$ be a transcendental extension of k, and assume that* $\operatorname{fed}(R) < \infty$. *Then* $\operatorname{fed}(R \otimes_k k(x)) < \infty$. *More precisely,* $\operatorname{fed}(R) = \operatorname{fed}(R \otimes_k k(x))$.

Proof Set $f := \operatorname{fed}(R) < \infty$. Let N be a finitely generated $R \otimes_k k(x)$-module such that $\operatorname{Ext}^i_{R \otimes_k k(x)}(N, N) = 0$ for $i \gg 0$. We know that $R \otimes_k k(x) \cong S^{-1}R[x]$ where S

is the multiplicatively closed subset $k[x]\backslash\{0\}$. Thus, there exists a finitely generated $R[x]$-submodule M of N such that

$$\begin{aligned} N \cong M \otimes_{R[x]} S^{-1}R[x] &\cong M \otimes_{R[x]} \left(R[x] \otimes_{k[x]} S^{-1}k[x]\right) \\ &\cong M \otimes_{k[x]} S^{-1}k[x] = M \otimes_{k[x]} k(x). \end{aligned}$$

Note that N is a $k(x)$-module by using the natural ring homomorphism $k(x) \to R \otimes_k k(x)$. More precisely, for each $a(x) \in k(x)$ and each $n \in N$ the scalar multiplication is defined by $a(x).n := (1 \otimes a(x))n$. Now, let $f(x) \in k[x] \subseteq k(x)$ be a non-zero polynomial such that $f(x).n = 0$ for some $n \in N$. It follows that $n = (1/f(x))f(x).n = 0$. This implies that N is a torsion-free $k[x]$-module. Hence, M is a torsion-free $k[x]$-module as a submodule of N. This implies that $x - \alpha$ is a non-zero-divisor on M for each $\alpha \in k$.

Now by the isomorphism $\operatorname{Ext}^i_{R\otimes_k k(x)}(N,N) \cong \operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x)$, we see that $\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0$ for $i \gg 0$.

On the other hand, since by assumption R is a finite dimensional k-algebra, for each $i \geq 0$ the $R[x]$-module $\operatorname{Ext}^i_{R[x]}(M,M)$ is a finitely generated $k[x]$-module; *i.e.*, $\operatorname{Ext}^i_{R[x]}(M,M)$ is finitely generated over a principal ideal domain. Therefore, $\operatorname{Ext}^i_{R[x]}(M,M)$ has a $k[x]$-module decomposition as

$$\operatorname{Ext}^i_{R[x]}(M,M) \cong \bigoplus_{j=1}^{c_i} k[x]/(w_{ij}(x))k[x] \oplus k[x]^{v_i},$$

where $0 \neq w_{ij}(x) \in k[x]$ and each v_i is a non-negative integer.

Since $\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0$ for all $i \gg 0$, we get $v_i = 0$ for $i \gg 0$. Also there are only countably many polynomials $w_{ij}(x)$. Since by assumption k is uncountable, there exists $\alpha \in k$ such that $w_{ij}(\alpha) \neq 0$ for all i, j. Therefore, $x - \alpha$ acts bijectively on $k[x]/(w_{ij}(x))k[x]$ for all i, j. Since $x - \alpha$ is a non-zero-divisor on each $k[x]^{v_i}$, we conclude that $\operatorname{Tor}_1^{k[x]}\left(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]\right) = 0$ for all i. Since $v_i = 0$ for $i \gg 0$, we obtain that $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for $i \gg 0$. Therefore, Lemma 2.5 implies that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for $i \gg 0$. Note that by Remark 2.2 the R-module M_α is finitely generated. Thus, by assumption we have $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i > f$. Since $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha$ is a submodule of $\operatorname{Ext}^i_R(M_\alpha, M_\alpha)$ by Lemma 2.5, we have $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for all $i > f$. This implies that $v_i = 0$ for $i > f$, which is equivalent to $\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0$ for $i > f$. This is also equivalent to $\operatorname{Ext}^i_{R\otimes_k k(x)}(N,N) = 0$ for $i > f$, and this shows that $\operatorname{fed}(R \otimes_k k(x)) \leq f < \infty$. Equality now follows from Remark 3.2. ∎

With the previous results in hand, we are ready to prove our first main theorem.

Theorem 3.4 *Let k be an uncountable algebraically closed field, and let R be a finite dimensional k-algebra that is Gorenstein. Then* $\operatorname{fed}(R) < \infty$ *if and only if* $\operatorname{fed}(R[x]) < \infty$. *Therefore, by Theorem 1.2, R satisfies Conjecture 1.1 if and only if $R[x]$ does.*

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Proof It follows from Proposition 3.3 that $\operatorname{fed}(R \otimes_k k(x)) < \infty$. Therefore, by Lemma 3.1 we conclude that $\operatorname{fed}(R[x]) < \infty$. This shows that if R satisfies Conjecture 1.1, then so does $R[x]$. The reverse implication follows directly from Remark 3.2. ∎

Remark 3.5 By [4, Corollary 3.5], for a Gorenstein ring R, finiteness of $\operatorname{fed}(R)$ implies that $\operatorname{fed}(R) = \operatorname{id}(R)$. Therefore, under the assumptions of Theorem 3.4, we have $\operatorname{fed}(R) = \operatorname{fed}(R \otimes_k k(x)) = 0$, and $\operatorname{fed}(R[x]) = 1$.

The next result is our second main theorem.

Theorem 3.6 *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring with an uncountable coefficient field k. Then $\operatorname{fed}(R) < \infty$ if and only if $\operatorname{fed}(R[x]_{\mathfrak{m}R[x]}) < \infty$. Therefore, by Theorem 1.2, R satisfies Conjecture 1.1 if and only if $R[x]_{\mathfrak{m}R[x]}$ does.*

Proof Set $f := \operatorname{fed}(R) < \infty$, and let N be a finitely generated $R[x]_{\mathfrak{m}R[x]}$-module such that $\operatorname{Ext}^i_{R[x]_{\mathfrak{m}R[x]}}(N, N) = 0$ for $i \gg 0$. Since $R[x]_{\mathfrak{m}R[x]}$ is a Gorenstein ring, by Fact 2.6 we can replace N by its first syzygy in an arbitrary free resolution and assume that N is torsion-free.

Notice that since we are working with localization, we can find a finitely generated $R[x]$-submodule M of N such that $\operatorname{Ext}^i_{R[x]_{\mathfrak{m}R[x]}}(N, N) \cong \operatorname{Ext}^i_{R[x]}(M, M)_{\mathfrak{m}R[x]}$ for all i. Therefore, by our assumption we get

$$\operatorname{Ext}^i_{R[x]}(M, M)_{\mathfrak{m}R[x]} = 0 \quad \text{for all } i \gg 0.$$

On the other hand, for each i, Nakayama's lemma implies the first equivalence in the following display:

$$\begin{aligned}\operatorname{Ext}^i_{R[x]}(M, M)_{\mathfrak{m}R[x]} = 0 &\Longleftrightarrow \big(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)\big)_{\mathfrak{m}R[x]} = 0\\ &\Longleftrightarrow \big(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)\big) \otimes_{k[x]} k(x) = 0.\end{aligned}$$

The second equivalence follows from the fact that $\mathfrak{m}R[x]$ is the zero ideal of $k[x]$. Hence, $(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)) \otimes_{k[x]} k(x) = 0$ for all $i \gg 0$.

Since for each i the $R[x]$-module $\operatorname{Ext}^i_{R[x]}(M, M)$ is finitely generated, the $k[x]$-module $\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)$ is finitely generated as well. Hence, as we explained in the proof of Proposition 3.3, each $\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)$ has a $k[x]$-modules decomposition as

$$\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M) \cong \bigoplus_{j=1}^{c_i} k[x]/(w_{ij}(x))k[x] \oplus k[x]^{\nu_i},$$

where $0 \neq w_{ij}(x) \in k[x]$ and each ν_i is a non-negative integer.

Since $(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)) \otimes_{k[x]} k(x) = 0$ for all $i \gg 0$, we get $\nu_i = 0$ for $i \gg 0$. As in the proof of Proposition 3.3, there exists $\alpha \in k$ such that $w_{ij}(\alpha) \neq 0$ for all i, j. Hence, $x - \alpha$ acts bijectively on $k[x]/(w_{ij}(x))k[x]$ for all i, j. This implies that

$$\big(\operatorname{Ext}^i_{R[x]}(M, M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M, M)\big)_{\alpha} \cong k^{\nu_i} \tag{3.2}$$

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for all i. Since $v_i = 0$ for all $i \gg 0$, it follows that for all $i \gg 0$,

$$\left(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)\right)_\alpha = 0.$$

On the other hand, for all i we have the following isomorphism by Lemma 2.4:

$$\left(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)\right)_\alpha \cong \operatorname{Ext}^i_{R[x]}(M,M)_\alpha/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M)_\alpha. \tag{3.3}$$

Therefore, Nakayama's lemma implies that $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for all $i \gg 0$. Equivalently,

$$\operatorname{Ext}^i_{R[x]}(M,M) \xrightarrow{x-\alpha} \operatorname{Ext}^i_{R[x]}(M,M)$$

is a bijective map for all $i \gg 0$. Thus,

$$\operatorname{Tor}^{k[x]}_1\left(\operatorname{Ext}^{i+1}_{R[x]}(M,M), k[x]/(x-\alpha)k[x]\right) = 0 \quad \text{for all } i \gg 0.$$

It follows from Lemma 2.5 that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i \gg 0$. Since M_α is a finitely generated R-module by Remark 2.2, our assumption implies that $\operatorname{Ext}^i_R(M_\alpha, M_\alpha) = 0$ for all $i > f$. Again from Lemma 2.5, we get $\operatorname{Ext}^i_{R[x]}(M,M)_\alpha = 0$ for all $i > f$. In particular, $(\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M))_\alpha = 0$ for all $i > f$ by (3.3). It follows from equation (3.2) that $v_i = 0$ for $i > f$. We conclude now that

$$\operatorname{Ext}^i_{R[x]}(M,M)/\mathfrak{m}\operatorname{Ext}^i_{R[x]}(M,M) \otimes_{k[x]} k(x) = 0 \quad \text{for all } i > f.$$

As we explained above, this is equivalent to saying that $\operatorname{Ext}^i_{R[x]}(M,M)_{\mathfrak{m}R[x]} = 0$ for all $i > f$. Hence, $\operatorname{Ext}^i_{R[x]_{\mathfrak{m}R[x]}}(N,N) = 0$ for all $i > f$. Thus, $\operatorname{fed}(R[x]_{\mathfrak{m}R[x]}) \le f < \infty$. This shows that if R satisfies Conjecture 1.1, then so does $R[x]_{\mathfrak{m}R[x]}$. The reverse implication follows from Remark 3.2. ∎

4 Conjecture 1.1 Under Another Type of Ring Extension

Let R be a local ring. In this section, we investigate the condition of Conjecture 1.1 under the base ring extension $R \to R[x]_{\mathfrak{M}}$, where $\mathfrak{M}$ is a maximal ideal of the polynomial ring that contracts to the maximal ideal of R. The next proposition is our main result in this section.

Proposition 4.1 *Let $(R,\mathfrak{m})$ be a Gorenstein local ring with algebraically closed residue field, and let $\mathfrak{M}$ be a maximal ideal of $R[x]$ such that $\mathfrak{m} = \mathfrak{M} \cap R$. Then $\operatorname{fed}(R) < \infty$ if and only if $\operatorname{fed}(R[x]_{\mathfrak{M}}) < \infty$. Therefore, R satisfies Conjecture 1.1 if and only if $R[x]_{\mathfrak{M}}$ does.*

Proof Suppose that $\operatorname{fed}(R) < \infty$. By Hilbert's nullstellensatz, there exists an element $r \in R$ such that $\mathfrak{M} = (\mathfrak{m}, x-r)R[x]$. Since $R \cong R[x]/(x-r)R[x]$ and $x-r$ is a non-zero-divisor on $R[x]$, we conclude that $R[x]/(x-r)R[x]$ is a local Gorenstein ring with maximal ideal $\mathfrak{M}/(x-r)R[x]$ such that $\operatorname{fed}(R[x]/(x-r)R[x]) < \infty$; see [4, Proposition 4.3]. On the other hand, we have a ring isomorphism

$$R[x]/(x-r)R[x] \cong R[x]_{\mathfrak{M}}/(x-r)R[x]_{\mathfrak{M}},$$

which implies that $\operatorname{fed}(R[x]_{\mathfrak{M}}/(x-r)R[x]_{\mathfrak{M}}) < \infty$. Since $x-r$ is a non-zero-divisor on $R[x]_{\mathfrak{M}}$, again [4, Proposition 4.3] implies that $\operatorname{fed}(R[x]_{\mathfrak{M}}) < \infty$. This shows that if R satisfies Conjecture 1.1, then so does $R[x]_{\mathfrak{M}}$. The reverse implication follows from Remark 3.2. ∎

We conclude the paper by proving a corollary of Proposition 4.1 that improves Theorem 3.4 when R is a local ring. Before stating the corollary, we need to prove the following lemma.

Lemma 4.2 *Let R be a Gorenstein ring of finite Krull dimension. If $\operatorname{fed}(R_{\mathfrak{m}}) < \infty$ for each maximal ideal $\mathfrak{m}$ of R, then $\operatorname{fed}(R) < \infty$.*

Proof By [4, Theorem 3.1], for each maximal ideal $\mathfrak{m}$ we have $\operatorname{fed}(R_{\mathfrak{m}}) \le \operatorname{id}(R_{\mathfrak{m}}) = \dim(R_{\mathfrak{m}}) \le \dim(R)$. Therefore,

$$c := \sup\{\operatorname{fed}(R_{\mathfrak{m}}) \mid \mathfrak{m} \text{ is a maximal ideal of } R\} \le \dim(R) < \infty. \tag{4.1}$$

Let M be a finitely generated R-module, and assume that $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \gg 0$. Thus, for every maximal ideal $\mathfrak{m}$ of R we have

$$\operatorname{Ext}^i_R(M_{\mathfrak{m}}, M_{\mathfrak{m}}) \cong \operatorname{Ext}^i_R(M, M)_{\mathfrak{m}} = 0 \quad \text{for all } i \gg 0.$$

Therefore, by (4.1), for every maximal ideal $\mathfrak{m}$ of R we get $\operatorname{Ext}^i_R(M, M)_{\mathfrak{m}} = 0$ for all $i > c$. This implies that $\operatorname{Ext}^i_R(M, M) = 0$ for all $i > c$. Thus, $\operatorname{fed}(R) \le c < \infty$. ∎

Corollary 4.3 *Let $(R, \mathfrak{m})$ be an artinian Gorenstein local ring with algebraically closed residue field. Then $\operatorname{fed}(R) < \infty$ if and only if $\operatorname{fed}(R[x]) < \infty$. Therefore, R satisfies Conjecture 1.1 if and only if $R[x]$ does.*

Proof Assume that $\operatorname{fed}(R) < \infty$. Since R is artinian, for each maximal ideal $\mathfrak{M}$ of $R[x]$ we have $\mathfrak{M} \cap R = \mathfrak{m}$. It follows from Proposition 4.1 that for each maximal ideal $\mathfrak{M}$ of $R[x]$ we have $\operatorname{fed}(R[x]_{\mathfrak{M}}) < \infty$. Therefore, Lemma 4.2 implies that $\operatorname{fed}(R[x]) < \infty$. This shows that if R satisfies Conjecture 1.1, then so does $R[x]$. The reverse implication follows from Remark 3.2. ∎

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