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PSEUDODIFFERENTIAL RESOLVENT FOR A CERTAIN NON-LOCALLY-SOLVABLE OPERATOR

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Introduction. In this note we construct a pseudo-differential resolvent for $P = D_x^2 + x^2D_y^2 - \lambda D_y$ by the method of [**3**] and study the dependence on the parameter λ as $\lambda \to 1$. Grushin [**2**] first pointed out that P is solvable and hypoelliptic if λ is not an odd integer, whereas P is neither locally solvable at the origin nor hypoelliptic if λ is an odd integer. Gilioli and Treves [**1**] showed that this discrete nature of the condition for solvability persists to a more general class of operators. But when λ is an odd integer, adding a nonreal constant term to P recovers solvability; thus a description of how the resolvent depends on λ would be of interest. In particular, this paper comprises a proof of the *Proposition*: $(zI - P)^{-1}$ has a pseudodifferential symbol which is expressible in closed form if z is not a nonnegative real. This symbol can be used to compute the λ dependence of the symbol of the spectral resolution of P, which reveals the non-local-solvability of P as $\lambda \to 1$.

1. $P = D_x^2 + x^2D_y^2 - \lambda D_y$ is a symmetric operator on $S(R^2)$ when λ is real, and thus it extends to a self-adjoint operator on $L^2(R^2)$ [**4**]. This guarantees the existence of a resolution of the identity, E, and a method of computation:

$$E((b, c)) = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{b+\delta}^{c-\delta} R(r - i\epsilon) - R(r + i\epsilon)dr.$$

Here $R(z) = (zI - P)^{-1}$ for z a complex number, b and c are real numbers, possibly infinite, and the limits are in the strong operator topology.

Construction of resolvent for $\operatorname{Re} z < 0$. First we presume that $(z - P)^{-1}$ is a pseudodifferential operator with symbol $k(x, \xi, \eta; \lambda; z)$, and thus we try to solve

$$(1) \quad 1 = \sum_\alpha \frac{1}{\alpha!} (z - (\xi^2 + x^2\eta^2 - \lambda\eta))^{(\alpha)} D_{x,y}{}^\alpha k.$$

As in [**3**] we look for a solution of the form

$$k = - \int_0^\infty \exp(-f)dt$$

with $f(x, \xi, \eta, \lambda, z, t)$ satisfying the t-boundary conditions: $f(0) = 0$ and $\operatorname{Re} f \to \infty$ as $t \to \infty$. This leads to an equation for f

$$\frac{\partial f}{\partial t} = \xi^2 + x^2\eta^2 - \lambda\eta - z + 2i\xi \frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial f}{\partial x}\right)^2$$

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which is easily solved when $\operatorname{Re} z < 0$ and yields

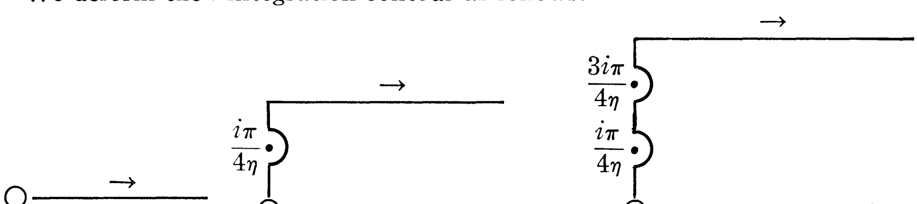
$$f = (\xi^2 + x^2\eta^2)\,\frac{\tanh 2\eta t}{2} + ix\xi(1 - \operatorname{sech} 2\eta t) + \tfrac{1}{2}\log\cosh 2\eta t - \lambda\eta t - zt.$$

Notice that f is invariant under the change $\eta \to -\eta$, $\lambda \to -\lambda$ so we will only consider $\eta > 0$ and $-1 \leqq \lambda \leqq 1$; also $\eta = 0$ gives $f = (\xi^2 - z)t$. Of course $\operatorname{Re} z < 0$ guarantees $\operatorname{Re} f \to \infty$ as $t \to \infty$. Lastly we note that allowing $\operatorname{Im} \lambda$ to be nonzero does not affect the validity of this representation for the symbol of the resolvent.

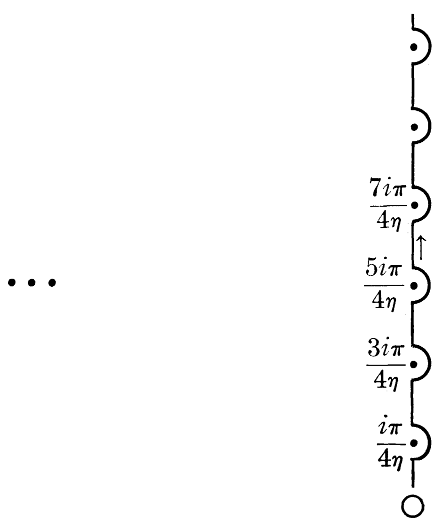
This existence of $(z - P)^{-1}$ for $\operatorname{Re} z < 0$ indicates the spectrum of P is contained in $\overline{R_+}$. Now $|k| \leqq 2^{\frac{1}{2}}|\operatorname{Re} z|^{-1}$ and clearly k is smooth since f is real analytic. In fact by differentiating and estimating as in [**3**] we find that k is in $S^{-1}_{\frac{1}{2},\frac{1}{2}}(R^2)$ when $|\lambda| < 1$, but only that k is in $S^{0}_{\frac{1}{2},\frac{1}{2}}(R^2)$ if $\lambda = \pm 1$. Incidentally this implies $z - P$ is locally solvable and hypoelliptic for $\lambda = \pm 1$ since k is also the symbol of a left inverse.

2. We analytically extend k into $\operatorname{Re} z \geqq 0$, $\operatorname{Im} z \neq 0$ by deforming the integration contour, and since everything is analytic the equation (1) will still be satisfied. Since $\operatorname{Im} z < 0$ and $\operatorname{Im} z > 0$ are handled symmetrically we will just consider the latter.

We deform the t integration contour as follows:



(1) (2) (3)



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This deforming in a series of steps was to avoid any questions about crossing branch cuts in the domain of the log. The horizontal part of the contours is estimated using $\operatorname{Re} z < 0$, $\operatorname{Im} z > 0$, and the periodicity of the trig functions. Lastly the contribution from the semicircles vanishes as the contour is pressed onto the imaginary axis, as we now show. For $t = \pi i/4\eta$ introduce polar coordinates: $2\eta t - \frac{1}{2}i\pi = R\exp(i\theta)$ and then the semicircle integration becomes

$$-\int_{-\pi/2}^{\pi/2} \exp\left(-(\xi^2 + x^2\eta^2)\frac{\coth R\exp(i\theta)}{2\eta} - ix\xi(1 + i\operatorname{csch} R\exp(i\theta))\right.$$
$$\left.-\tfrac{1}{2}\log i\sinh R\exp(i\theta) + \left(\frac{i\pi}{4} + \frac{R}{2}\exp(i\theta)\right)\left(\lambda + \frac{z}{\eta}\right)\right)\frac{iR}{2\eta}\exp(i\theta)d\theta.$$

Now for $R \to 0$ we use

$$\coth R\exp(i\theta) = \frac{1}{R}\exp(-i\theta) + \frac{R}{3}\exp(i\theta) + \ldots,$$

$$\operatorname{csch} R\exp(i\theta) = \frac{1}{R}\exp(-i\theta) - \frac{R}{6}\exp(i\theta) + \ldots,$$

$$\sinh R\exp(i\theta) = R\exp(i\theta)\left(1 + \frac{R^2}{6}\exp(2i\theta 3 + \ldots\right),$$

and find the exponent equal to

$$-\frac{\exp(-i\theta)}{R}\frac{(\xi - x\eta)^2}{2\eta} - \tfrac{1}{2}\log R + i\left(\frac{\pi}{4}(\lambda + \frac{z}{\eta} - 1) - x\xi - \tfrac{1}{2}\theta\right) + O(R).$$

Hence the integral is $O(R^{\frac{1}{2}})$.

Thus we have a principal value integral along the imaginary axis and changing variables to $2\eta t = is$ we write for $\operatorname{Re} z < 0$, $\operatorname{Im} z > 0$, $\eta > 0$

$$k(x, \xi, \eta, \lambda, z) =$$
$$\frac{-i}{2\eta} pv\int_0^\infty \exp\left(-\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s + \frac{is}{2}\left(\lambda + \frac{z}{\eta}\right)\right)ds$$

where Q denotes $\xi^2 + x^2\eta^2$. Of course we now may allow $\operatorname{Re} z \geqq 0$. Also note $\log\cos\pi l = i\pi l$ and pv is not needed since $|\cos s|^{-\frac{1}{2}}$ is locally integrable.

Similarly for $\operatorname{Im} z < 0$ we deform the contour to the negative imaginary axis and then again $\operatorname{Re} z \geqq 0$ is achieved. In this case we find

$$k = \frac{i}{2\eta}\int_0^\infty \exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \frac{is}{2}\left(\lambda + \frac{z}{\eta}\right)\right)ds$$

with $\log\cos\pi l = -i\pi l$. Of course in both cases

$$|k| \leqslant \frac{1}{2\eta}\int_0^\pi |\cos s|^{-1/2}ds|\operatorname{Im} z|^{-1}$$

if $|\operatorname{Im} z| < \frac{1}{2}$.

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3. As $\operatorname{Im} z \to 0$ in $\operatorname{Re} z \geqq 0$, k should have singularities since this is where the spectrum of P lies. Now to use the formula of section 1 for the resolution of the identity we need

$$\int_{b+\delta}^{c-\delta} R(r - i\epsilon) - R(r + i\epsilon)dr$$

as a pseudo-differential operator whose symbol we can compute by using section 2. Taking the limits $\epsilon \to 0$, $\delta \to 0$ for the symbol will be immediate and actually simplify the calculation, as expected. The resulting symbol will of course not be a smooth function since $E((b, c))$ is a projection.

For example, the operator $D_x + D_y$ has a resolvent with symbol $(z - \xi - \eta)^{-1}$ which is real analytic if $\operatorname{Im} z$ is nonzero and $E((b, c))$ has the symbol $\theta(\xi + \eta - b)\theta(c - \xi - \eta)$.

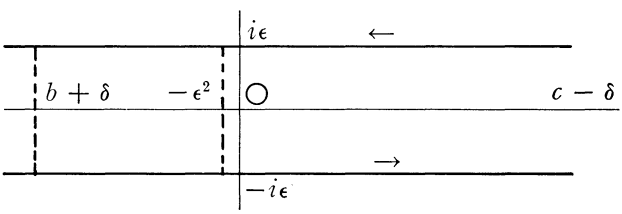
Returning to P, we first note the symbol of $R(r - i\epsilon) - R(r + i\epsilon)$ is $k(x, \xi, \eta, \lambda, r - i\epsilon) - k(x, \xi, \eta, \lambda, r + i\epsilon)$ and for $r < 0$ this is given by section 1 as $O(\epsilon)$. To take advantage of this we rewrite

$$\int_{b+\delta}^{c-\delta} k(x, \xi, \eta, \lambda, r - i\epsilon) - k(x, \xi, \eta, \lambda, r + i\epsilon)dr$$

as a contour integral

$$\int_C (x, \xi, \eta, \lambda, z)dz$$

with C the two solid lines below; b and c are tacitly taken as negative and positive, respectively.



Applying the Cauchy formula to the square with the dotted vertical sides and using the bound $O(\epsilon)$ for the left dotted side we have the symbol of

$$\int_{b+\delta}^{c-\delta} R(r - i\epsilon) - R(r + i\epsilon)dr$$

equal to

$$\int_C k(x, \xi, \eta, \lambda, z)dz + O(\epsilon)$$

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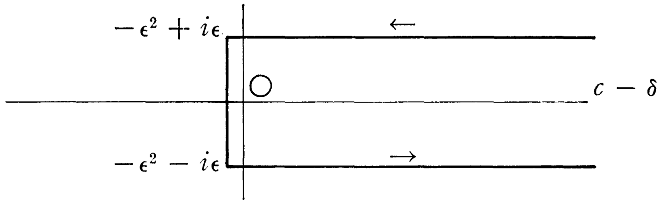
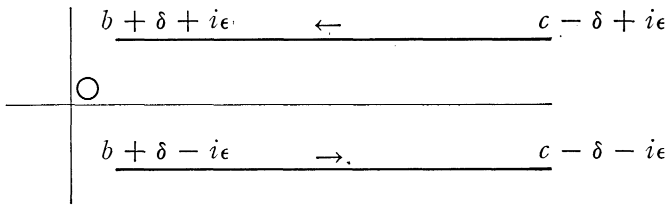
with C the solid contour below.



Now if c were nonpositive the symbol would be $O(\epsilon)$ and if b were nonnegative the contour would be the two solid lines below.



We now proceed with b nonnegative. For η positive we have, since $r \geqq b + \delta > 0$,

$$k(x, \xi, \eta, \lambda, r - i\epsilon) - k(x, \xi, \eta, \lambda, r + i\epsilon) = \frac{i}{2\eta}$$

$$\times \int_R \exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \frac{is}{2}\left(\lambda + \frac{r}{\eta}\right) - \frac{\epsilon}{2\eta}|s|\right) ds$$

where $\log\cos \pi l = -i\pi l$. Then using the periodicity of the trig functions we find this difference equal to

$$\frac{i}{2\eta}\sum_{l\in Z}\exp\left(i\pi l\left(1 - \lambda - \frac{r}{\eta}\right)\right)\exp\left(-\left(\epsilon|l|\frac{\pi}{\eta}\right)\right)$$

$$\times\int_0^{2\pi}\exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \frac{is}{2}\left(\lambda + \frac{r}{\eta}\right)\right)$$

$$\times \exp\left(-\left(\frac{\epsilon s}{2\eta}\right)\right) ds$$

$$(2)\qquad + \frac{i}{2\eta}\sum_{l\geqslant 1}\exp\left(-\left(i\pi l\left(1 - \lambda - \frac{r}{\eta}\right)\right)\right)\exp\left(-\epsilon l\frac{\pi}{\eta}\right)\cdot\int_0^{2\pi}\exp\left(\frac{iQ}{2\eta}\tan s - \right.$$

$$\left. ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \frac{is}{2}\left(\lambda + \frac{r}{\eta}\right)\right) 2i\sin\frac{\epsilon s}{2\eta}\, ds$$

Clearly the integrals depend analytically on ϵ and $\lambda + r/\eta$, and the sums are trivial. The first sum equals $(1 - |A|^2)/(1 + |A|^2 - 2\,\mathrm{Re}\,A)$ and the second is $A/(1 - A)$ where $A = \exp(-i\pi(1 - \lambda - r/\eta))\exp(-\epsilon\pi/\eta)$. Then taking

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the limit $\epsilon \to 0$ for

$$\int_{b+\delta}^{c-\delta} k(x, \xi, \eta, \lambda, r - i\epsilon) - k(x\cdot \xi, \eta, \lambda, r + i\epsilon)dr$$

is routine since the first sum is essentially the Poisson kernel and the second has a factor of ϵ from the second integral in (2). The result is that

$$\lim_{\epsilon \downarrow 0} \int_{b+\delta}^{c-\delta} k(x, \xi, \eta, \lambda, r - i\epsilon) - k(x, \xi, \eta, \lambda, r + i\epsilon)dr$$

$$(3) \qquad = i \sum_{m \in M} \int_0^{2\pi} \exp \left(\frac{iQ}{2\eta} \tan s - ix\xi(1 - \sec s) - \tfrac{1}{2} \log \cos s - \frac{is}{2}(2m+1)\right) ds + \tfrac{1}{2} \sum_{m \in N} \int_0^{2\pi} \exp \left(\frac{iQ}{2\eta} \tan s - ix\xi(1 - \sec s) - \tfrac{1}{2} \log \cos s - \frac{is}{2}(2m+1)\right) ds,$$

where M is the set of integers, m, such $b + \delta < (2m + 1 - \lambda)\eta < c - \delta$ and N is those m such that $(2m + 1 - \lambda)\eta$ equals either $b + \delta$ or $c - \delta$. Now b has been presumed nonnegative and λ has magnitude at most one so M and N do not contain any negative integers. Also note that the limit as $\delta \to 0$ is immediate, i.e., N disappears. In the next section we evaluate the integrals in (3).

4. To evaluate

$$\int_0^{2\pi} \exp \left(\frac{iQ}{2\eta} \tan s - ix\xi(1 - \sec s) - \tfrac{1}{2} \log \cos s - i\frac{s}{2}(2m+1)\right) ds$$

we split it into two parts:

$$C = \int_0^{\pi/2} + \int_{3\pi/2}^{2\pi} = \int_{-\pi/2}^{\pi/2} \quad \text{and} \quad (-)^m B = \int_{\pi/2}^{3\pi/2} = (-)^m \int_{-\pi/2}^{\pi/2}.$$

Now change variables: $w = \tan s$. This yields

$$C = \int_R \exp \left(\frac{iQ}{2\eta} w - ix\xi(1 - \sqrt{1 + w^2})\right) (1 + w^2)^{-3/4} \times \exp \left(-\left(\frac{i}{2}(2m+1) \arctan w\right)\right) dw$$

$$B = \int_R \exp \left(\frac{iQ}{2\eta} w - ix\xi(1 + \sqrt{1 + w^2})\right) (1 + w^2)^{-3/4} \times \exp \left(-\left(\frac{i}{2}(2m+1) \arctan w\right)\right) dw.$$

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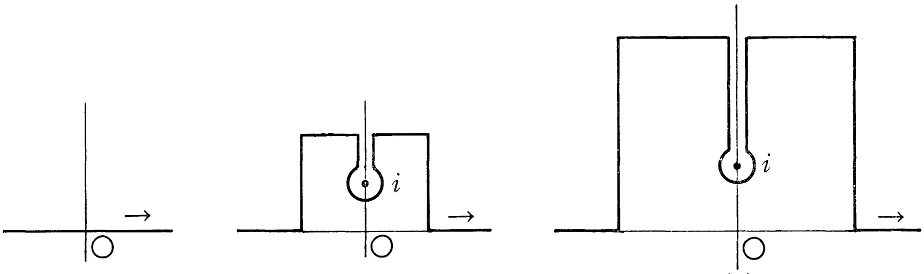
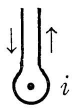
Next deform the contour as shown:



(1) (2) (3)

To verify the legitimacy of this deformation just observe that $(Q/2\eta) \pm x\xi \geqq 0$ by Cauchy's inequality (again $\eta > 0$). Thus we have B and C defined as integrals over the contour



which we split into the usual two pieces: the circle around i and the doubly used imaginary axis. For the circle we will employ a Taylor series on the integrand and then the radius will be shrunk to zero. But first observe that the imaginary axis integrals cancel in $C + (-)^m B$. To show this we let R denote the radius of the circle at i and then change variables $w = i + it$. Thus

$$\begin{aligned} C + (-)^m B = {} & \int_{|w-i|=R} \dots dw - i \exp\left(\frac{Q}{2\eta} - ix\xi\right) \int_R^\infty \exp\left(-\frac{Qt}{2\eta}\right) \\ & \times \begin{Bmatrix} \cosh x\xi(t^2+2t)^{1/2} \\ \sinh x\xi(t^2+2t)^{1/2} \end{Bmatrix} 2^{1/2} (i-1)(t^2+2t)^{-3/4} \\ & \times \exp\left(\frac{i\pi}{4}(2m+1)\right)\left(\frac{2+t}{t}\right)^{1/2m+1/4} dt + i \exp\left(-\frac{Q}{2\eta} - ix\xi\right) \\ & \times \exp \int_R^\infty \left(-\frac{Qt}{2_n}\right) \begin{Bmatrix} \cosh x\xi(t^2+2t)^{1/2} \\ \sinh x\xi(t^2+2t)^{1/2} \end{Bmatrix} 2^{1/2}(-i-1) \cdot (t^2+2t)^{-3/4} \\ & \qquad \times \exp\left(-\frac{i\pi}{4}(2m+1)\right)\left(\frac{2+t}{t}\right)^{1/2m+1/4} dt, \end{aligned}$$

where the cosh is used with m even and the sinh with m odd. Of course $\arctan w = \pm\pi/2 + i/2 \log((2+t)/t)$ was employed. Clearly the t integrals cancel and introducing polar coordinates at i, $w = i - iR\exp(i\theta)$, yields

$$\begin{aligned} C + (-)^m B = {} & \int_{-\pi}^{\pi} R \exp(i\theta) \exp\left(-\left(\frac{Q}{2\eta} + ix\xi\right)\right) \\ & \times \exp\left(\frac{QR}{2\eta}\exp(i\theta)\right) 2 \begin{Bmatrix} \cos x\xi(1+w^2)^{1/2} \\ i \sin x\xi(1+w^2)^{1/2} \end{Bmatrix} (1+w^2)^{-3/4} \\ & \qquad \times \exp\left(-\left(\frac{i}{2}(2m+1)\right)\arctan w\right) d\theta. \end{aligned}$$

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Invoke Taylor series:

$$(1+w^2)^{1/2} = 2^{1/2}R^{1/2}\exp\left(\tfrac{1}{2}\theta\right)\left(1 - \sum_{n\geqslant 1}\frac{1\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{n!2^{2n}}R^n\right.$$
$$\left.\times \exp(in\theta)\right),\ (1+w^2)^{-3/4} = 2^{-3/4}R^{-3/4}\exp\left(-\frac{3i}{4}\theta\right)$$
$$\times\left(1 + \sum_{n\geqslant 1}\frac{3\cdot 7\cdot 11\cdots(4n-1)}{n!2^{3n}}R^n\exp(in\theta)\right),$$
$$\arctan w = \tfrac{1}{2}\theta + \tfrac{1}{2}\sum_{n\geqslant 1}\frac{R^n\sin n\theta}{n2^n} + \frac{i}{2}\log\frac{2}{R} + \frac{i}{4}\log(1 - R\cos\theta + R^2).$$

This yields $C + (-)^m B$ by just expanding

$$\exp\left(\frac{QR}{2\eta}\exp(i\theta)\right),\ \cos x\xi(1+w^2)^{1/2},\ (1 - R\cos\theta + R^2)^{(2m+1)/8},$$

and $\exp(-\frac{1}{2}(2m+1)\sum)$ up to $m/2$ powers of R and doing the θ integrals which almost all vanish. Of course negative powers of R have zero θ integrals, as is obvious by inspection. We compute for $m = 0$, 1, and 2 and guess the result for $m \geqq 3$.

$$\begin{aligned}(C+B)(m=0) &= 2\pi\sqrt{2}\exp(-Q/2\eta - ix\xi)\\ &= 2\pi\psi_0(x)\overline{\hat{\psi}_0(\xi)}\exp(-ix\xi)\\ (C-B)(m=1) &= 2\pi 2\sqrt{2}ix\xi\exp(-Q/2\eta - ix\xi)\\ &= 2\pi\psi_1(x)\overline{\hat{\psi}_1(\xi)}\exp(-ix\xi)\\ (C+B)(m=2) &= 2\pi 2\sqrt{2}(Q/2\eta - x^2\xi^2 - \tfrac{1}{4})\exp(-Q/2\eta - ix\xi)\\ &= 2\pi\psi_2(x)\overline{\hat{\psi}_2(\xi)}\exp(-ix\xi)\end{aligned}$$

where $\psi_j(x) = A_j H_j(x\sqrt{\eta})\exp(-\frac{1}{2}x^2\eta)$ and A_j is chosen so

$$\int_R |\psi_j|^2 dx = 1.$$

Hence the Hermite functions appear, as expected.

So we have found the integral at the beginning of the section to be $2\pi\psi_m(x)\overline{\hat{\psi}_m(\xi)}\exp(-ix\xi)$ and now we have $E((b,c))$ in hand for $0 \leqq b < c < \infty$. Explicitly for $\eta = 0$ it is easy to see that the procedure of sections 3 and 4 gives a symbol of $\theta(c - \xi^2)\theta(\xi^2 - b)$, and for $\eta < 0$ we find the symbol by using $-\eta$ and $-\lambda$ in place of η and λ.

Thus for $\varphi \in L^2(R^2)$ we have

$$E((b,c))\hat{\phi}(\xi,\eta) = \sum_M \hat{\psi}_m(\xi)\int_R \overline{\hat{\psi}_m(\zeta)}\hat{\phi}(\zeta,\eta)\frac{d\zeta}{2\pi} \qquad \text{if } \eta > 0,$$

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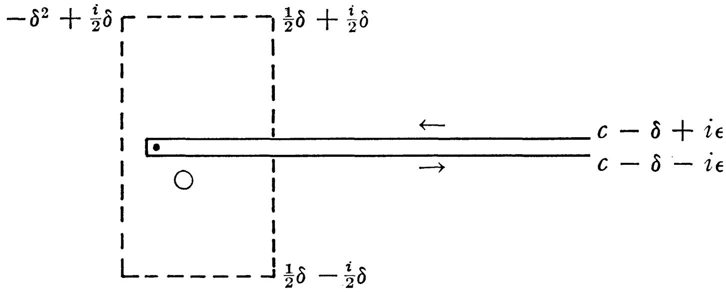
and the analogue if $\eta < 0$. Clearly this is a projection; also recall M and ψ_m depend on η. Of course this result could have been found by directly solving for the eigenfunctions of P.

Now for $\lambda = 1$ the $m = 0$ term is missing from the $\eta > 0$ sum and if $\lambda = -1$ the $m = 0$ term does not appear in the $\eta < 0$ sum. This is explained in the next section where we finally allow b to be negative.

5. In this section b is presumed negative; in fact we also presume δ is so small that $b + \delta < -\frac{1}{2}$ and $c - \delta > \frac{1}{2}\delta$. Then the contour of section 3, shown as a solid line, is deformed to the dotted line for each ϵ.



Thus the limit $\epsilon \to 0$ yields a principal value integral around the rectangle plus the same sums (3) with M and N defined using $\frac{1}{2}\delta$ in place of $b + \delta$. Lastly the limit $\delta \to 0$ will give the symbol of $E((0, c))$ plus a residue at the origin which we now compute. The principal value integral is

$$\int_{-1/2\delta}^{1/2\delta} k(x, \xi, \eta, \lambda, -\delta^2 + is)ids$$

$$(4) \quad + \int_{-\delta^2}^{1/2\delta} k(x, \xi, \eta, \lambda, s - i\tfrac{1}{2}\delta) - k(x, \xi, \eta, \lambda, s + i\tfrac{1}{2}\delta)ds$$

$$+ \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{1/2\delta} k(x, \xi, \eta, \lambda, \tfrac{1}{2}\delta - is) - k(x, \xi, \eta, \lambda, \tfrac{1}{2}\delta + is)ids.$$

The first integral of (4) is evaluated by applying Fubini after using section 1 for k, and we find it equal to

$$\int_0^\infty \exp\left(-\frac{Q}{2\eta}\tanh 2\eta t - ix\xi(1 - \operatorname{sech} 2\eta t) - \tfrac{1}{2}\log\cosh 2\eta t + \lambda\eta t - \delta^2 t\right) \times 2\,\frac{\sin\frac{1}{2}\delta t}{t}\,dt.$$

The limit $\delta \to 0$ is trivial by changing variables, $w = \frac{1}{2}\delta t$, and including

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$1/2\pi i$ equals

$$\begin{cases} 0, & \lambda < 1 \\ \frac{1}{2}\sqrt{2}\exp\left(-\dfrac{Q}{2\eta} - ix\xi\right), & \lambda = 1. \end{cases}$$

The second integral of (4) is split into two pieces:

$$\int_0^{1/2\delta} + \int_{-\delta^2}^0 .$$

The latter piece is seen to be $O(\delta)$ by using the estimate $|k| \leqq C|\operatorname{Im} z|^{-1}$ which appears at the end of section 2. The former piece is evaluated using (2) and integrating after summing and approximating:

$$\int_0^{1/2\delta} k(x, \xi, \eta, \lambda, r - i\tfrac{1}{2}\delta) - k(x, \xi, \eta, \lambda, r + i\tfrac{1}{2}\delta)dr$$

$$= \int_0^{1/2\delta} \frac{i}{2\eta} \sum_{l \in Z} \exp\left(i\pi l\left(1 - \lambda - \frac{r}{\eta}\right)\right) \exp\left(-\left(\frac{\delta\pi|l|}{2\eta}\right)\right) \cdot$$

$$\int_0^{2\pi} \exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \tfrac{1}{2}is\left(\lambda + \frac{r}{\eta}\right)\right)$$

$$\times \exp\left(-\frac{\delta s}{4\eta}\right) ds + \frac{i}{2}\sum_{l \geqslant 1} \exp\left(-\left(i\pi l\left(1 - \lambda - \frac{r}{\eta}\right)\right)\right) \exp\left(-\left(\frac{\delta\pi|l|}{2\eta}\right)\right)$$

$$\times \int_0^{2\pi} \exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \tfrac{1}{2}is\left(\lambda + \frac{r}{\eta}\right)\right)$$

$$\times 2i\sin\frac{\delta s}{4\eta} dsdr$$

$$= \int_0^{1/2\delta} \frac{i}{2\eta}\left(1 + \frac{A}{1 - A} + \frac{\bar{A}}{1 - \bar{A}}\right)$$

$$\times \int_0^{2\pi} \exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \tfrac{1}{2}is\left(\lambda + \frac{r}{\eta}\right)\right)$$

$$\times \exp\left(-\frac{\delta s}{4\eta}\right) ds + \frac{i}{2}\frac{A}{1 - A}$$

$$\times \int_0^{2\pi} \exp\left(\frac{iQ}{2\eta}\tan s - ix\xi(1 - \sec s) - \tfrac{1}{2}\log\cos s - \tfrac{1}{2}is\left(\lambda + \frac{r}{\eta}\right)\right).$$

$$\times 2i\sin\frac{\delta s}{4\eta} dsdr,$$

where $A = \exp(-(i\pi(1 - \lambda - r/\eta) - \delta\pi/2\eta))$. Now the $0 - 2\pi$ integrals depend on r analytically so we write them as the value at $r = 0$ plus $O(\delta)$. This $O(\delta)$ is negligible and integrating yields

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$$\frac{i}{2\eta}\left(\tfrac{1}{2}\delta + \frac{\eta}{i\pi}\log\left(\frac{1-\exp\left(i\pi\left(1-\lambda-\frac{\delta}{2\eta}\right)-\frac{\delta\pi}{2\eta}\right)}{1-\exp\left(i\pi(1-\lambda)-\frac{\delta\pi}{2\eta}\right)}\right)-\frac{\eta}{i\pi}\right.$$

$$\times\log\left(\frac{1-\exp\left(-i\pi\left(1-\lambda-\frac{\delta}{2\eta}\right)-\frac{\delta\pi}{2\eta}\right)}{1-\exp\left(-i\pi(1-\lambda)-\frac{\delta\pi}{2\eta}\right)}\right)\cdot\int_0^{2\pi}\cdots ds-\frac{i}{2\eta}\frac{\eta}{i\pi}$$

$$\times\ \log\left(\frac{1-\exp\left(-i\pi\right)1-\lambda-\frac{\delta}{2\eta}\left(-\frac{\delta\pi}{2\eta}\right)}{1-\exp\left(-i\pi(1-\lambda)-\frac{\delta\pi}{2\eta}\right)}\right)$$

$$\left.\times\int_0^{2\pi}\exp\left(\frac{iQ}{2\eta}\tan s-ix\xi(1-\sec s)-\tfrac{1}{2}\log\cos s-\tfrac{1}{2}\,is\lambda\right)2i\sin\frac{\delta s}{4\eta}\,ds\right).$$

Now the limit $\delta \to 0$ is easy and with the $1/(2\pi i)$ included we have

$$\begin{cases} 0, & \lambda < 1 \\ \tfrac{1}{4}\sqrt{2}\exp\left(-\dfrac{Q}{2\eta}-ix\xi\right), & \lambda = 1. \end{cases}$$

The $\lambda = -1$ case here corresponds to $m = -1$ in section 4 and is found to vanish. Recall $\eta > 0$ is still presumed so only $\lambda = 1$ is expected to be trouble.

The third integral of (4) is essentially the same as the second and gives the same result.

Thus for $|\lambda| < 1$ the spectral resolution is continuous, but for $\lambda = 1$ there is a mass at the origin that projects onto $\psi_0(x)$ for $\eta > 0$. Recalling how ψ_0 depends on η we see that $E(\{0\})$ projects onto the subspace spanned by $\exp(-(s(\frac{1}{2}x^2 - iy)))$ for $s > 0$. Of course these functions are homogeneous solutions for P and are used in showing non-local-solvability for ${}^tP$.

Lastly for $E((b, \infty))$ with b small positive the symbol includes ψ_0 for $b < (1-\lambda)\eta < \infty$, i.e., for $\eta > b/(1-\lambda)$. Thus as $\lambda \to 1$ the $\eta > a > 0$ part of the ψ_0 projection is contained in $E((0, a(1-\lambda)))$ which becomes $E(\{0\})$ and causes the solvability trouble.

For $\lambda = -1$ the analogues with $\eta < 0$ hold.

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