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IDEALS OF HERZOG–NORTHCOTT TYPE

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Abstract This paper takes a new look at ideals generated by 2×2 minors of 2×3 matrices whose entries are powers of three elements not necessarily forming a regular sequence. A special case of this is the ideals determining monomial curves in three-dimensional space, which were studied by Herzog. In the broader context studied here, these ideals are identified as Northcott ideals in the sense of Vasconcelos, and so their liaison properties are displayed. It is shown that they are set-theoretically complete intersections, revisiting the work of Bresinsky and of Valla. Even when the three elements are taken to be variables in a polynomial ring in three variables over a field, this point of view gives a larger class of ideals than just the defining ideals of monomial curves. We then characterize when the ideals in this larger class are prime, we show that they are usually radical and, using the theory of multiplicities, we give upper bounds on the number of their minimal prime ideals, one of these primes being a uniquely determined prime ideal of definition of a monomial curve. Finally, we provide examples of characteristic-dependent minimal prime and primary structures for these ideals.

Keywords: Herzog ideal; Northcott ideal; almost complete intersection; liaison; associative law of multiplicities

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1. Introduction

Let A be a commutative Noetherian ring (with identity) and x_1, x_2, x_3 be a sequence of elements of A generating a proper ideal of height 3. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and take $a = (a_1, a_2, a_3) \in \mathbb{N}_0^3$ and $b = (b_1, b_2, b_3) \in \mathbb{N}_0^3$. Let $c = a + b$, $c = (c_1, c_2, c_3)$. Let $\mathcal{M}$ be the matrix

$$\mathcal{M} = \begin{pmatrix} x_1^{a_1} & x_2^{a_2} & x_3^{a_3} \\ x_2^{b_2} & x_3^{b_3} & x_1^{b_1} \end{pmatrix},$$

and $v_1 = x_1^{c_1} - x_2^{b_2}x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1}x_3^{b_3}$ and $D = x_3^{c_3} - x_1^{b_1}x_2^{a_2}$, the 2×2 minors of $\mathcal{M}$ up to a change of sign. Consider $I_2(\mathcal{M}) = (v_1, v_2, D)$, the determinantal ideal generated by the 2×2 minors of $\mathcal{M}$. Note that $x_2^{b_2}D = -x_3^{b_3}v_1 - x_1^{b_1}v_2$, so that, if $b_2 = 0$, then $I_2(\mathcal{M}) = (v_1, v_2)$.

Our motivation to consider these ideals comes from the following well-known result of Herzog in [**10**] (see also [**14**, pp. 138–139]). Take the irreducible affine space curve of

$\mathbb{A}_k^3 = k^3$, k a field, given by the parametrization $x_1 = t^{n_1}$, $x_2 = t^{n_2}$, $x_3 = t^{n_3}$, where $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ ($n_i > 0$), with $\gcd(n_1, n_2, n_3) = 1$. Let $\mathfrak{p}_n$ be the vanishing ideal of this curve, i.e. the height-two prime ideal of the polynomial ring in three variables $A = k[x_1, x_2, x_3]$ defined as the kernel of the natural morphism $\varphi\colon k[x_1, x_2, x_3] \to k[t]$, $\varphi(x_i) = t^{n_i}$. We will call $\mathfrak{p}_n$ the *Herzog ideal associated to* $n = (n_1, n_2, n_3)$ (see § 7 for more details concerning this definition; note that Huneke [**11**] used the term Herzog ideals for a different class of ideals). Herzog proved that $\mathfrak{p}_n$ is either a complete intersection or an almost complete intersection ideal (in the sense of [**9**]). Concretely, with a suitable numbering of the variables, $\mathfrak{p}_n$ has a set of generators of one of the following two types:

(ci): $v_1 = x_1^{c_1} - x_3^{a_3}$, $v_2 = x_2^{a_2} - x_1^{a_1} x_3^{b_3}$, $c_1, a_2, a_3 \in \mathbb{N}$, $a_1 b_3 \neq 0$ (here $b_2 = 0$);

(aci): $v_1 = x_1^{c_1} - x_2^{b_2} x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1} x_3^{b_3}$ and $D = x_3^{c_3} - x_1^{b_1} x_2^{a_2}$, with $a, b \in \mathbb{N}^3$.

In other words, a Herzog ideal can always be seen as an ideal of the type $I_2(\mathcal{M}) = (v_1, v_2, D)$, for some appropriate $a, b \in \mathbb{N}_0^3$. However, even in the case of a polynomial ring, an ideal of the form $I_2(\mathcal{M}) = (v_1, v_2, D)$ is not always a Herzog ideal because (as we will show) it might not be a prime ideal (see Theorem 7.8), whereas a Herzog ideal is prime by definition.

Herzog also proved that $\mathfrak{p}_n$ is always a set-theoretic complete intersection. Subsequently, Bresinsky and Valla gave constructive proofs that an ideal of the form $I_2(\mathcal{M})$ is a set-theoretic complete intersection (see [**2**] and [**20**, Theorem 3.1]), the former in the polynomial case $A = k[x_1, x_2, x_3]$, the latter in our setting and using a general result on determinantal ideals proved by Eagon and Northcott [**4**, Theorem 3].

We recently made use of Herzog ideals in order to produce a negative answer to a long-standing question about the uniform Artin–Rees property on the prime spectrum of an excellent ring. Concretely, for $s \in \mathbb{N}$, $s \geqslant 4$, and $n_1(s) = s^2 - 3s + 1$, $n_2(s) = s^2 - 3s + 3$ and $n_3(s) = s^2 - s + 1$, $n(s) = (n_1(s), n_2(s), n_3(s)) \in \mathbb{N}^3$, the one parameter family of (non-complete intersection) Herzog ideals $\mathfrak{p}_{n(s)}$ satisfies the relation that, for all $s \geqslant 4$, $\mathfrak{p}_{n(s)}^s \cap x_3 A \supsetneq \mathfrak{p}_{n(s)}(\mathfrak{p}_{n(s)}^{s-1} \cap x_3 A)$ [**16**].

On the other hand, in [**15**], Northcott considered the following situation: let $u = u_1, \dots, u_r$ and $v = v_1, \dots, v_r$ be two sets of r elements of a Noetherian ring A, connected by the relations

$$\begin{aligned} v_1 &= a_{1,1} u_1 + a_{1,2} u_2 + \cdots + a_{1,r} u_r, \\ v_2 &= a_{2,1} u_1 + a_{2,2} u_2 + \cdots + a_{2,r} u_r, \\ &\ \vdots \\ v_r &= a_{r,1} u_1 + a_{r,2} u_2 + \cdots + a_{r,r} u_r, \end{aligned}$$

with $a_{i,j} \in A$. Let D stand for the determinant of the $r \times r$ matrix $\Phi = (a_{i,j})$. Northcott proved that if $(v_1, \dots, v_r)$ has grade r and $(v_1, \dots, v_r, D)$ is proper, then the projective dimension of $A/(v_1, \dots, v_r, D)$ is r, and $(v_1, \dots, v_r, D)$ and all its associated prime ideals have grade r [**15**, Theorem 2]. Subsequently, Vasconcelos called such an ideal $(v_1, \dots, v_r, D)$ the *Northcott ideal associated to* Φ *and* u (see, for example, [**21**, p. 100]).

Take now $r = 2$, $u_1 = x_1^{a_1}$, $u_2 = -x_2^{b_2}$, and $v_1 = x_1^{c_1} - x_2^{b_2}x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1}x_3^{b_3}$, the aforementioned first two generators of $I_2(\mathcal{M}) = (v_1, v_2, D)$. Let Φ be the 2×2 matrix defined by

$$\Phi = \begin{pmatrix} x_1^{b_1} & x_3^{a_3} \\ -x_3^{b_3} & -x_2^{a_2} \end{pmatrix},$$

whose determinant, note, is just $D = x_3^{c_3} - x_1^{b_1}x_2^{a_2}$, the third generator of $I_2(\mathcal{M})$. We clearly have $\Phi \cdot [u]^{\mathrm{T}} = [v]^{\mathrm{T}}$. In other words, the ideal $I_2(\mathcal{M}) = (v_1, v_2, D)$ (in particular, a Herzog ideal) can be viewed as a Northcott ideal whenever the ideal (v_1, v_2) has grade 2.* This fact, though simple, was extremely useful in proving the main result in [**16**]. As a consequence, it awakened our interest in this family of ideals. In this paper, we study their general properties, though we will restrict ourselves just to the case when $a, b \in \mathbb{N}^3$, i.e. $a_i, b_j > 0$. For ease of reference, we state the following definition.

Definition 1.1. Let A be a commutative Noetherian ring and let $x = x_1, x_2, x_3$ be a sequence of elements of A generating an ideal of height 3. Let $a = (a_1, a_2, a_3) \in \mathbb{N}^3$ and $b = (b_1, b_2, b_3) \in \mathbb{N}^3$ $(a_i, b_j > 0)$ and set $c = a + b$, $c = (c_1, c_2, c_3)$. Let $\mathcal{M}$ be the matrix

$$\mathcal{M} = \begin{pmatrix} x_1^{a_1} & x_2^{a_2} & x_3^{a_3} \\ x_2^{b_2} & x_3^{b_3} & x_1^{b_1} \end{pmatrix},$$

and $v_1 = x_1^{c_1} - x_2^{b_2}x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1}x_3^{b_3}$ and $D = x_3^{c_3} - x_1^{b_1}x_2^{a_2}$, the 2×2 minors of $\mathcal{M}$ up to a change of sign. The ideal $I = I_2(\mathcal{M}) = (v_1, v_2, D)$ will be called the *Herzog–Northcott* (HN) *ideal associated to* $\mathcal{M}$.

The paper is organized as follows. In § 2 we start with a few preliminary results. In § 3, following Bresinsky's ideas in [**2**], we prove, in our general setting, that HN ideals are set-theoretically complete intersections, thus recovering the result of Valla. Concretely, we prove that the element g_{B} considered by Bresinsky verifies $\mathrm{rad}(I) = \mathrm{rad}(v_1, g_{\mathrm{B}})$ in complete generality (see Theorem 3.1). Comparing the element g_{B} given by Bresinsky with the element g_{V} given by Valla, we show that they are equal modulo (v_1) under mild hypotheses and are in general closely related. Section 4 is mainly devoted to studying the condition $\mathrm{grade}(v_1, v_2) = 2$. In §§ 5 and 6, and supposing that $x = x_1, x_2, x_3$ is a regular sequence and that (v) has grade 2, we prove that HN ideals are geometrically linked to a complete intersection and that they are almost complete intersections (in the sense of [**9**]). From § 7 onwards, we restrict ourselves to the case of the polynomial ring $A = k[x_1, x_2, x_3]$, where k is a field and $x = x_1, x_2, x_3$ are three variables over k. Then, in § 7, we characterize when an HN ideal is a prime ideal. To do this, we first consider an integer vector $m(I) = (m_1, m_2, m_3) \in \mathbb{N}^3$ associated to I and the corresponding Herzog

* After submitting this paper, we found that Waldi had earlier observed that the vanishing ideal of a monomial curve in affine 4-space in a Northcott ideal (Waldi used the term 'special' to refer to such ideals): see [**22**]. Subsequently, Kunz drew our attention to the thesis of his former student Gastinger, which contained the result that the property of being a Northcott ideal no longer held for monomial curves in affine space of dimension greater than 4: see [**7**]. We thank Professor Kunz for providing us with a copy of his thesis.

(prime) ideal $\mathfrak{p}_{m(I)}$. Then we prove that I being prime, $\gcd(m(I)) = 1$ and $I = \mathfrak{p}_{m(I)}$ are three equivalent conditions (see Theorem 7.8). Section 8 is devoted to finding a bound for the number of minimal components of an HN ideal. Concretely, we prove that the number of minimal components of an HN ideal I is bounded above in terms of the greatest common divisor of any pair (m_i, m_j), $i \neq j$, where $(m_1, m_2, m_3) = m(I)$ is the integer vector associated to I (Theorem 8.3). Finally, in § 9, and provided that k has characteristic zero or is big enough, we prove that an HN ideal is always radical (Theorem 9.1). We finish by giving some illustrative examples.

2. Preliminary results

In this section, A will be a commutative Noetherian ring and $x = x_1, x_2, x_3$ a sequence of elements of A generating a proper ideal of height 3. We keep the notation of § 1, i.e. $v_1 = x_1^{c_1} - x_2^{b_2}x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1}x_3^{b_3}$ and $D = x_3^{c_3} - x_1^{b_1}x_2^{a_2}$, and $u_1 = x_1^{a_1}$ and $u_2 = -x_2^{b_2}$, where $a, b \in \mathbb{N}^3$. In particular, set $I = I_2(\mathcal{M}) = (v_1, v_2, D)$.

Remark 2.1. Let I be an HN ideal. Then $\operatorname{rad}(v) = \operatorname{rad}(I) \cap \operatorname{rad}(u)$.

Proof. Let $\mathfrak{p}$ a prime ideal of A. Suppose that $\mathfrak{p} \supseteq (v)$. Then $x_2^{b_2}D = -x_3^{b_3}v_1 - x_1^{b_1}v_2 \in \mathfrak{p}$. If $x_2 \notin \mathfrak{p}$, then $D \in \mathfrak{p}$ and $I \subseteq \mathfrak{p}$. On the other hand, if $x_2 \in \mathfrak{p}$, then $x_1 \in \mathfrak{p}$ since $v_1 \in \mathfrak{p}$. It follows that $\mathfrak{p} \supseteq I$ or $\mathfrak{p} \supseteq (u)$ or, equivalently, $\mathfrak{p} \supseteq I \cap (u)$. Conversely, it is immediate that if $\mathfrak{p} \supseteq I \cap (u)$, then $\mathfrak{p} \supseteq I \supseteq (v)$ or $\mathfrak{p} \supseteq (u) \supseteq (v)$. □

For stronger results we need more restrictive conditions (see Proposition 2.2 (a) and Corollary 2.3). The following proposition is a direct consequence of the work of Northcott [**15**].

Proposition 2.2. *Let I be an HN ideal. Suppose that* $\operatorname{grade}(v) = 2$.

(a) *Then $(v) : D = (u)$ and $(v) : I = (u)$.*

(b) *The ideals (v), I and (u) are grade-unmixed of grade 2.*

(c) *Moreover, if $x = x_1, x_2, x_3$ is a regular sequence, then each of x_1, x_2, x_3 is regular modulo I.*

Proof. The first part of (a) follows from [**15**, Proposition 1] (which does not require the ring to be local), and the statement that $(v) : I = (u)$ follows from the equality $(v) : D = (u)$. By [**13**, Theorem 130], (v) is grade-unmixed and by [**15**, Theorem 2] I also has grade 2 and is grade-unmixed. From a minimal primary decomposition of (v) and the equality $(v) : D = (u)$, one can extract another minimal primary decomposition of (u), and hence (u) is grade-unmixed and clearly has grade 2. Moreover, if $x = x_1, x_2, x_3$ is a regular sequence and if some x_i were in an associated prime $\mathfrak{p}$ of I, then $\mathfrak{p}$ would contain x_1, x_2, x_3, a contradiction, since $\mathfrak{p}$ has grade 2 by part (b). □

Corollary 2.3. *Let I be an HN ideal. Suppose that $x = x_1, x_2, x_3$ is a regular sequence and that* $\operatorname{grade}(v) = 2$. *Then $I \cap (u) = (v)$.*

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Proof. Clearly, $(v) \subseteq I \cap (u)$ and, by Cramer's rule, $D \cdot (u) \subset (v)$. To see that $I \cap (u) \subseteq (v)$, it suffices to show that $(u) : D = (u)$, for if $z \in I \cap (u)$, then $z = w + qD \in (u)$, with $w \in (v) \subseteq (u)$. Thus, $qD \in (u)$ and $q \in (u) : D = (u)$ would follow. Therefore, one would deduce that $qD \in (v)$ and $z \in (v)$. But D is indeed regular modulo (u). Were D in an associated prime $\mathfrak{p}$ of (u), then $\mathfrak{p}$ would contain the regular sequence x_1, x_2, x_3, a contradiction, since $\mathfrak{p}$ has grade 2 by Proposition 2.2 (b). $\square$

3. HN ideals are set theoretically complete intersections

In this section, A will be a commutative Noetherian ring and $x = x_1, x_2, x_3$ a sequence of elements of A generating a proper ideal of height 3. We keep the notation of § 1, i.e. $v_1 = x_1^{c_1} - x_2^{b_2} x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1} x_3^{b_3}$ and $D = x_3^{c_3} - x_1^{b_1} x_2^{a_2}$, and $u_1 = x_1^{a_1}$ and $u_2 = -x_2^{b_2}$, with $a, b \in \mathbb{N}^3$.

In the next result we produce the desired element g using an algorithm employed by Bresinsky in [**2**] (there, in the case where $A = k[x_1, x_2, x_3]$ is a polynomial ring over the field k). For this reason we will refer to this choice of g as g_{B}. This candidate for g will subsequently be contrasted in Example 3.2 and Remark 3.3 with the candidate for g, denoted g_{V}, advanced by Valla in [**20**] (in our general setting). To examine this contrast in detail, we pay close attention to the form of g.

Theorem 3.1 (cf. [2], [20]). *Let I be an HN ideal. Then there exists an algorithmically specified element g in the radical of I such that* $\operatorname{rad}(I) = \operatorname{rad}(v_1, g)$. *Moreover, if $x = x_1, x_2, x_3$ is a regular sequence and* $\operatorname{grade}(v) = 2$, *then this element g lies in I.*

Proof. We prove that there exists an element g in A of the form $g = (-1)^{c_1} x_3^r + h$, with $r \geqslant 1$ and $h \in (x_1, x_2)$, and such that $v_2^{c_1} - p v_1 = x_2^{a_1 b_2} g$, for some $p \in A$.

From this last equation, it follows that $v_2 \in \operatorname{rad}(v_1, g)$ and $g \in (v) : u_2^{a_1} \subseteq I : u_2^{a_1}$. Since I is the ideal generated by the 2×2 minors of a 2×3 matrix, by [**4**, Theorem 3] any minimal prime of I is of height at most 2. In particular, x_2 is not in any minimal prime of I. Thus, $u_2^{a_1}$ is regular modulo $\operatorname{rad}(I)$ and $g \in \operatorname{rad}(I)$. Moreover, if $x = x_1, x_2, x_3$ is a regular sequence and $\operatorname{grade}(v) = 2$, by Proposition 2.2, x_2 and hence $u_2^{a_1}$ is regular modulo I and $g \in I$ in this case. Then $D^r - (-1)^{c_1 c_3} g^{c_3} \in \operatorname{rad}(I) \cap (x_1, x_2) \subseteq \operatorname{rad}(I \cap (u))$ which, by Remark 2.1, is equal to $\operatorname{rad}(v)$ and so is included in $\operatorname{rad}(v_1, g)$. Thus, $D \in \operatorname{rad}(v_1, g)$ and $\operatorname{rad}(I) = \operatorname{rad}(v_1, g)$, and the result is then proved.

Now construct the desired g (call it g_{B}) by following Bresinsky's argument in [**2**]. His idea is to take the binomial expansion of $v_2^{c_1} = (x_2^{c_2} - x_1^{a_1} x_3^{b_3})^{c_1}$ and, by subtracting a multiple of $v_1 = x_1^{c_1} - x_2^{b_2} x_3^{a_3}$, eliminate the higher-order terms in $x_1^{a_1} x_3^{b_3}$. To assist the reader we include details that were omitted in Bresinsky's paper.

For $i = 0$, write $B_0 = v_2^{c_1}$, $p_0 = 0$ and $q_0 = 0$. For $i = 1, \dots, a_1$, write

$$B_i = \sum_{j=i}^{c_1} (-1)^{c_1 - j} \binom{c_1}{j} x_2^{j c_2} x_1^{a_1 (c_1 - j)} x_3^{b_3 (c_1 - j)},$$

$$p_i = \sum_{j=0}^{i-1} (-1)^{c_1 - j} \binom{c_1}{j} x_2^{j a_2 + (i-1) b_2} x_1^{(a_1 - i + j) c_1 - j a_1} x_3^{(i-1-j) a_3 + b_3 (c_1 - j)},$$

$$\begin{aligned} q_i &= p_i x_2^{b_2} x_3^{a_3} \\ &= \sum_{j=0}^{i-1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2+ib_2} x_1^{(a_1-i+j)c_1-ja_1} x_3^{(i-j)a_3+b_3(c_1-j)}. \end{aligned}$$

Note that B_i is just a part of the binomial expansion of $v_2^{c_1} = (x_2^{c_2} - x_1^{a_1} x_3^{b_3})^{c_1}$. Moreover,

$$\begin{aligned} &B_i - B_{i+1} + q_i - p_{i+1} x_1^{c_1} \\ &\quad = B_i - B_{i+1} + p_i x_2^{b_2} x_3^{a_3} - p_{i+1} x_1^{c_1} \\ &\quad = (-1)^{c_1-i} \binom{c_1}{i} x_2^{ic_2} x_1^{a_1(c_1-i)} x_3^{b_3(c_1-i)} \\ &\qquad + \left[\sum_{j=0}^{i-1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2+(i-1)b_2} x_1^{(a_1-i+j)c_1-ja_1} x_3^{(i-1-j)a_3+b_3(c_1-j)} \right] x_2^{b_2} x_3^{a_3} \\ &\qquad - \left[\sum_{j=0}^{i} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2+ib_2} x_1^{(a_1-i-1+j)c_1-ja_1} x_3^{(i-j)a_3+b_3(c_1-j)} \right] x_1^{c_1} \\ &\quad = 0. \end{aligned}$$

Now, for $i = 0, \dots, a_1$, let us see that $v_2^{c_1} - \sum_{j=0}^{i} p_j v_1 = B_i + q_i$. Indeed, for $i = 0$, since $B_0 = v_2^{c_1}$ and $p_0 = 0$ and $q_0 = 0$, $v_2^{c_1} - p_0 v_1 = B_0 + q_0$. Suppose the equality holds for i, $0 < i < a_1$. Then

$$\begin{aligned} v_2^{c_1} - \sum_{j=0}^{i+1} p_j v_1 &= B_i + q_i - p_{i+1} v_1 \\ &= B_{i+1} + (B_i - B_{i+1} + q_i - p_{i+1} x_1^{c_1}) + p_{i+1} x_2^{b_2} x_3^{a_3} \\ &= B_{i+1} + q_{i+1}. \end{aligned}$$

In particular, for $i = a_1$,

$$\begin{aligned} v_2^{c_1} - \sum_{j=0}^{a_1} p_j v_1 &= B_{a_1} + q_{a_1} \\ &= \sum_{j=a_1}^{c_1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{jc_2} x_1^{a_1(c_1-j)} x_3^{b_3(c_1-j)} \\ &\quad + \sum_{j=0}^{a_1-1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2+a_1b_2} x_1^{j(c_1-a_1)} x_3^{(a_1-j)a_3+b_3(c_1-j)} \\ &= x_2^{a_1b_2} \left[\sum_{j=a_1}^{c_1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2+(j-a_1)b_2} x_1^{a_1(c_1-j)} x_3^{b_3(c_1-j)} \right] \\ &\quad + x_2^{a_1b_2} \left[\sum_{j=0}^{a_1-1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2} x_1^{j(c_1-a_1)} x_3^{(a_1-j)a_3+b_3(c_1-j)} \right]. \end{aligned}$$

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Thus, taking $p = \sum_{j=0}^{a_1} p_j v_1$ and

$$g_{\mathrm{B}} = \sum_{j=a_1}^{c_1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2+(j-a_1)b_2} x_1^{a_1(c_1-j)} x_3^{b_3(c_1-j)} + \sum_{j=0}^{a_1-1} (-1)^{c_1-j} \binom{c_1}{j} x_2^{ja_2} x_1^{j(c_1-a_1)} x_3^{(a_1-j)a_3+b_3(c_1-j)},$$

one has $v_2^{c_1} - pv_1 = x_2^{a_1b_2} g_{\mathrm{B}}$. Moreover, note that the term $j = 0$ of g_{B} gives $(-1)^{c_1} x_3^r$, with $r = a_1a_3 + b_3c_1$, and the rest of the terms of g_{B} are in (x_1, x_2). □

Example 3.2. In [**20**, Theorem 3], Valla constructed an element g_{V} in the radical of I such that $\mathrm{rad}(I) = \mathrm{rad}(v_1, g_{\mathrm{V}})$. In concrete terms, changing Valla's notation to ours,

$$g_{\mathrm{V}} = \sum_{j=0}^{c_1} (-1)^{c_1-j} \binom{c_1}{j} x_1^s x_2^{(c_1-j)c_2+tb_2-a_1b_2} x_3^{jb_3+ta_3},$$

where for $0 \leqslant j \leqslant c_1$, $ja_1 = tc_1 + s$ with $0 \leqslant s \leqslant c_1 - 1$.

For instance, if we take

$$\mathcal{M} = \begin{pmatrix} x_1^2 & x_2 & x_3 \\ x_2^2 & x_3 & x_1^2 \end{pmatrix},$$

which is the example considered by Bresinsky in [**2**], then the element g_{B} considered in the proof of Theorem 3.1 is

$$g_{\mathrm{B}} = \sum_{j=2}^{4} (-1)^{4-j} \binom{4}{j} x_2^{j+(j-2)2} x_1^{2(4-j)} x_3^{4-j} + \sum_{j=0}^{1} (-1)^{4-j} \binom{4}{j} x_2^j x_1^{j2} x_3^{(2-j)+(4-j)},$$

so that

$$g_{\mathrm{B}} = x_2^8 - 4x_2^5 x_1^2 x_3 + 6x_2^2 x_1^4 x_3^2 - 4x_2 x_1^2 x_3^4 + x_3^6.$$

On the other hand,

$$g_{\mathrm{V}} = x_2^8 - 4x_2^5 x_1^2 x_3 + 6x_2^4 x_3^3 - 4x_2 x_1^2 x_3^4 + x_3^6.$$

We note that $g_{\mathrm{B}} - g_{\mathrm{V}} = 6x_2^2 x_3^2 v_1$. This pattern is completely general, as we now show.

Remark 3.3. If v_1, x_2 is a regular sequence, then $g_{\mathrm{B}} - (-1)^{c_1} g_{\mathrm{V}}$ is in (v_1). In particular, if, moreover, $x = x_1, x_2, x_3$ is a regular sequence and $\mathrm{grade}(v) = 2$, then g_{B} and g_{V} are in I.

Proof. By using the beginning of the proof of Valla [**20**, Theorem 3], changing his notation to ours, we have

$$(-1)^{c_1} v_2^{c_1} = x_2^{a_1b_2} g_{\mathrm{V}} \bmod (v_1).$$

On the other hand, from Theorem 3.1 we have $v_2^{c_1} = x_2^{a_1 b_2} g_{\mathrm{B}} + p v_1$. Hence,

$$x_2^{a_1 b_2}[g_{\mathrm{B}} - (-1)^{c_1} g_{\mathrm{V}}] = 0 \bmod (v_1). \tag{3.1}$$

If x_2 is regular modulo (v_1), then $g_{\mathrm{B}} - (-1)^{c_1} g_{\mathrm{V}} \in (v_1)$. If, moreover, $x = x_1, x_2, x_3$ is a regular sequence and $\mathrm{grade}(v) = 2$, by Theorem 3.1, g_{B} can be taken in I, and so can g_{V}. □

Remark 3.4. It is always the case that x_2 is regular modulo $\mathrm{rad}(I)$ for a general HN ideal I (see the proof of Theorem 3.1). Hence, it follows from Theorem 3.1 and (3.1) that $\mathrm{rad}(I) = \mathrm{rad}(v_1, g_{\mathrm{V}})$ for a general HN ideal I.

4. On the condition grade$(v) = 2$

In this section, A will be a commutative Noetherian ring and $x = x_1, x_2, x_3$ a sequence of elements of A (not necessarily generating a proper ideal of height 3). We keep the notation of § 1, i.e. $v_1 = x_1^{c_1} - x_2^{b_2} x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1} x_3^{b_3}$ and $D = x_3^{c_3} - x_1^{b_1} x_2^{a_2}$. The purpose of this section is to study the condition $\mathrm{grade}(v) = 2$ versus the condition that x_1, x_2 or x_1, x_2, x_3 forms a regular sequence. These results are of interest in view of Remark 3.3 and of results in subsequent sections.

In § 7 onwards, we shall be particularly interested in the case of a polynomial ring. As will be seen in Example 4.3, this case can be placed in a graded context. For this reason and for ease of reference, we introduce the following notation. With the assumptions of this section, we say that (A, x) satisfies the *homogeneous condition* $(*)$ if A can be graded by $\mathbb{N}_0$, with x_1, x_2, x_3 and v_1, v_2, D homogeneous elements of positive degree.

The following result is folklore (see [**3**, Corollary 1.6.19] for any unexplained notation and a proof in the local case). For the graded case use [**1**, § 9.7, Corollaire 2] and [**3**, Theorem 1.6.17 (b)].

Theorem 4.1. *Let R be a commutative Noetherian ring. Let $y = y_1, \dots, y_n$ be a sequence of elements of R. Let M be a finitely generated R-module. Suppose either that $y_1, \dots, y_n$ are in the Jacobson radical of R, or that $R = \bigoplus_{n \geqslant 0} R_n$ is $\mathbb{N}_0$-graded, $y_1, \dots, y_n$ are homogeneous elements of positive degree and $M = \bigoplus_{n \geqslant 0} M_n$ is graded over R. Then the following conditions are equivalent:*

1. $\mathrm{grade}((y_1, \dots, y_n); M) = n$;
2. $H_i(y; M) = 0$ *for all* $i \geqslant 1$;
3. $H_1(y; M) = 0$;
4. $y = y_1, \dots, y_n$ *is a regular sequence (in any order).*

We now state the desired result.

Proposition 4.2. *Let $x = x_1, x_2, x_3$ be a sequence of elements of A. Suppose either that x_1, x_2, x_3 are in the Jacobson radical of A, or that (A, x) satisfies the homogeneous condition $(*)$. Then*

(a) *x_1, x_2 is a regular sequence if and only if v_1, x_2 is a regular sequence,*

(b) *x_1, x_2, x_3 is a regular sequence if and only if v_1, v_2, x_3 is a regular sequence (so $\operatorname{grade}(v) = 2$ in either case),*

(c) *if, moreover, x_1, x_2, x_3 generate a proper ideal of height 3, $\operatorname{grade}(v) = 2$ and A satisfies the Serre condition (S_3), then x_1, x_2, x_3 is a regular sequence.*

Proof. We have $(v_1, x_2) = (x_1^{c_1}, x_2)$ and $(v_1, v_2, x_3) = (x_1^{c_1}, x_2^{c_2}, x_3)$. Thus,

$$\operatorname{grade}(v_1, x_2) = \operatorname{grade}(x_1^{c_1}, x_2) \quad \text{and} \quad \operatorname{grade}(v_1, v_2, x_3) = \operatorname{grade}(x_1^{c_1}, x_2^{c_2}, x_3).$$

By Theorem 4.1, v_1, x_2 is a regular sequence if and only if $\operatorname{grade}(v_1, x_2) = 2$, and so, by Theorem 4.1 again, if and only if $x_1^{c_1}, x_2$ is a regular sequence. Analogously, v_1, v_2, x_3 is a regular sequence if and only if $x_1^{c_1}, x_2^{c_2}, x_3$ is a regular sequence. Using [**13**, Exercise 3.1.12 (c)], one deduces (a) and (b).

Finally, if $\operatorname{grade}(v) = 2$, since $(v) \subseteq (x_1, x_2)$, $\operatorname{grade}(x_1, x_2) = 2$, by [**13**, Theorem 125]. By Theorem 4.1, x_1, x_2 is a regular sequence, and by [**13**, Theorem 130] the ideal (x_1, x_2) is grade-unmixed. Thus, for any associated prime $\mathfrak{p}$ of (x_1, x_2), $\operatorname{depth}(A_{\mathfrak{p}}) = \operatorname{grade}(\mathfrak{p}) = 2$ and $\operatorname{height}(\mathfrak{p}) \geqslant 2$. Since A satisfies the Serre condition (S_3),

$$2 = \operatorname{depth}(A_{\mathfrak{p}}) \geqslant \inf(3, \operatorname{height}(\mathfrak{p})).$$

Hence, $\mathfrak{p}$ has height 2, so $x_3 \notin \mathfrak{p}$ since (x_1, x_2, x_3) has height 3, by assumption. Thus, x_1, x_2, x_3 is a regular sequence. □

Example 4.3. Let $A = k[x_1, x_2, x_3]$ be the polynomial ring in three variables x_1, x_2, x_3 over a field k. Set $m_1 = c_2c_3 - a_2b_3$, $m_2 = c_1c_3 - a_3b_1$ and $m_3 = c_1c_2 - a_1b_2$. Endow A with the natural grading induced by giving x_i weight m_i. Then A is graded by $\mathbb{N}_0$ and x_1, x_2, x_3 and v_1, v_2, D are homogeneous elements of positive degree, i.e. (A, x) satisfies the homogeneous condition $(*)$. In particular, v_1, v_2 is a regular sequence in either order.

Proof. Clearly, $m_i > 0$, so x_i is a homogeneous element of positive degree. On the other hand, v_1, v_2, D are homogeneous provided that (m_1, m_2, m_3) satisfies the following system of equations:

$$\left.\begin{aligned} c_1m_1 &= b_2m_2 + a_3m_3, \\ c_2m_2 &= a_1m_1 + b_3m_3, \\ c_3m_3 &= b_1m_1 + a_2m_2. \end{aligned}\right\} \tag{4.1}$$

It is easily checked that this is indeed the case. Hence, v_1, v_2, D are homogeneous elements of positive degree and so (A, x) satisfies the homogeneous condition $(*)$. Applying Proposition 4.2, we deduce that v_1, v_2 is a regular sequence, and in either order because of Theorem 4.1. □

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Remark 4.4. In fact, note that in (4.1) each of the three equations can be obtained from the other two via addition. Moreover, since $c_1 > a_1$ and $c_2 > b_2$, the system can be reduced to the $\mathbb{Q}$-linear system of rank 2 formed by the first two equations:

$$\begin{pmatrix} c_1 & -b_2 & -a_3 \\ -a_1 & c_2 & -b_3 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By Cramer's rule, the $\mathbb{Q}$-linear subspace of solutions is generated by the non-zero vector

$$\begin{aligned} \left(\begin{vmatrix} a_3 & -b_2 \\ b_3 & c_2 \end{vmatrix}, \begin{vmatrix} c_1 & a_3 \\ -a_1 & b_3 \end{vmatrix}, \begin{vmatrix} c_1 & -b_2 \\ -a_1 & c_2 \end{vmatrix} \right) & \\ &= (a_2a_3 + a_3b_2 + b_2b_3, a_1a_3 + a_1b_3 + b_1b_3, a_1a_2 + a_2b_1 + b_1b_2) \\ &= (c_2c_3 - a_2b_3, c_1c_3 - a_3b_1, c_1c_2 - a_1b_2) \\ &= (m_1, m_2, m_3). \end{aligned}$$

Thus, $(m_1, m_2, m_3) \in \mathbb{N}^3$ is a solution of (4.1) which is unique up to a non-zero rational multiple.

One can have a Cohen–Macaulay domain A and a sequence of elements $x = x_1, x_2, x_3$ of A generating a proper ideal of height 3 (and hence grade 3), though with $\operatorname{grade}(v_1, v_2)$ equal to 1. Necessarily, by Proposition 4.2, $x = x_1, x_2, x_3$ are not in the Jacobson radical and (A, x) does not satisfy the homogeneous condition $(*)$.

Example 4.5 (Kaplansky [13, Exercise 7, p. 102]). Let $A = k[y_1, y_2, y_3]$ be the polynomial ring in three variables y_1, y_2, y_3 over a field k. Then the elements $x_3 = y_1$, $x_1 = y_2(1 - y_1)$ and $x_2 = y_3(1 - y_1)$ form a regular sequence in this order, but in the order x_1, x_2, x_3 they do not; moreover, $\operatorname{grade}(x_1, x_2) = \operatorname{grade}((x_1, x_2); A/(x_1)) + 1 = 1$. Thus, x_1, x_2, x_3 generate a proper ideal of height (and grade) 3, but $\operatorname{grade}(v_1, v_2) = 1$.

5. HN ideals are geometrically linked to complete intersections

In this section, A will be a commutative Noetherian ring and $x = x_1, x_2, x_3$ a regular sequence. Moreover, we will suppose that the ideal (v) has grade 2 (see, for example, Proposition 4.2). In spite of (v) having grade 2, it may be that v_1, v_2 is not a regular sequence. However, one can ensure that there does exist an element $w \in A$ such that $v_1 + wv_2, v_2$ is a regular sequence (see, for example, [**13**, Theorem 125] and its proof). We keep the rest of the notation as in § 1.

Lemma 5.1. *Let I be an HN ideal. Suppose that $x = x_1, x_2, x_3$ is a regular sequence and that* $\operatorname{grade}(v) = 2$. *Then $I = (v) : u_1 = (v) : u_2 = (v) : (u)$.*

Proof. By Corollary 2.3, $I \cap (u) = (v)$ and, by Proposition 2.2, x_1 is regular modulo I. Then

$$(v) : u_1 = [I \cap (u)] : u_1 = (I : u_1) \cap [(u) : u_1] = I : u_1 = I.$$

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Analogously, $I = (v) : u_2$. That $I = (v) : (u)$ follows immediately from this, since

$$(v) : (u) = ((v) : u_1) \cap ((v) : u_2).$$

□

In particular, (v) is a radical ideal if and only if I and (u) are radical ideals. Indeed, by Remark 2.1, $\mathrm{rad}(v) = \mathrm{rad}(I) \cap \mathrm{rad}(u)$ and so (v) is radical if I and (u) are radical. Conversely, if (v) is radical, then $\mathrm{rad}(I) = \mathrm{rad}((v) : u_1) \subseteq \mathrm{rad}(v) : u_1 = (v) : u_1 = I$ and $\mathrm{rad}(u) = \mathrm{rad}((v) : D) \subseteq \mathrm{rad}(v) : D = (v) : D = (u)$, by Proposition 2.2. In particular, if $a_1 > 1$ or $b_2 > 1$, then (u) is not radical, and hence (v) is not either.

Proposition 5.2. *Let I be an HN ideal. Suppose that $x = x_1, x_2, x_3$ is a regular sequence and that* $\mathrm{grade}(v) = 2$. *Then I is geometrically linked to (u), i.e. $(u) = (v) : I$, $I = (v) : (u)$ and $I \cap (u) = (v)$.*

Proof. By [**13**, Theorem 125], (v) can be generated by a regular sequence clearly contained in $I \cap (u)$. Moreover, $(u) = (v) : I$, $I = (v) : (u)$ and $I \cap (u) = (v)$ follow from Proposition 2.2, Lemma 5.1 and Corollary 2.3, respectively. □

In [**21**, p. 326 ff], Vasconcelos gives a proof that $I = (v) : (u)$, but it would seem that there is a hidden Gorenstein hypothesis in his Corollary 4.1.1 (see the appeal to Corollary A.9.1 in its proof; see also [**19**, Proposition 2.4], where the local Gorenstein hypothesis is used again).

In particular, one has $\mathrm{Ass}(A/I) \cup \mathrm{Ass}(A/(u)) = \mathrm{Ass}(A/(v))$ (see, for example, [**19**, Remark 2.2]).

6. HN ideals are almost complete intersections

In this section, A will again be a commutative Noetherian ring and $x = x_1, x_2, x_3$ a regular sequence. Moreover, as before, we will suppose that the ideal (v) has grade 2 (see Proposition 4.2).

Lemma 6.1. *Let I be an HN ideal. Suppose that $x = x_1, x_2, x_3$ is a regular sequence and that* $\mathrm{grade}(v) = 2$. *Then I is minimally generated by three elements.*

Proof. By Proposition 2.2, none of x_1, x_2 or x_3 is in any minimal prime of I. Localize at a minimal prime of (x_1, x_2, x_3) without changing notation. So we suppose that A is local. If I has a minimal generating set of less than three elements, then at least one element of the generating set v_1, v_2, D is redundant: D, say. In this case, $I = (v_1, v_2) \subseteq (x_1, x_2)$. By the Generalized Principal Ideal Theorem [**13**, Theorem 152], there exists a minimal prime $\mathfrak{q}$ of (x_1, x_2) of height 2. But then $\mathfrak{q}$ would be a minimal prime over I containing (x_1, x_2, x_3), a contradiction (and similarly for the possible variations on this argument).

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Alternatively, localize at a minimal prime containing (x_1, x_2, x_3) without changing notation. The resolving complex constructed in [**15**, § 2 and Theorem 2] in the case of our specific Φ, u and v yields the following free resolution of I:

$$0 \to A^2 \xrightarrow{[\varphi]} A^3 \xrightarrow{[\psi]} A \to A/(v, D) \to 0,$$

where

$$[\varphi] = \begin{pmatrix} -x_1^{a_1} & x_2^{b_2} \\ x_2^{a_2} & -x_3^{b_3} \\ x_3^{a_3} & -x_1^{b_1} \end{pmatrix} \quad \text{and} \quad [\psi] = \begin{pmatrix} -D & v_1 & v_2 \end{pmatrix}.$$

Since all the entries in the matrix maps are in the maximal ideal, this Hilbert–Burch presentation of I is minimal. □

Remark 6.2. Actually, by Valla's argument at the top of page 10 in [**20**], Lemma 6.1 holds for an arbitrary HN ideal. Note also that the second proof presented above requires only that $\operatorname{grade}(v) = 2$. From this resolution, $[\psi]\cdot[\varphi] = 0$ and so $Dx_1^{a_1} + v_1 x_2^{a_2} + v_2 x_3^{a_3} = 0$. Therefore, $Dx_1^{a_1} \in (v)$ and $IA_{x_1} = (v)A_{x_1}$. (This can also be deduced from the equality $I = (v) : (u)$.)

Proposition 6.3. *Let I be an HN ideal. Suppose that $x = x_1, x_2, x_3$ is a regular sequence and that* $\operatorname{grade}(v) = 2$. *Then I is an almost complete intersection (in the sense of [**9**]).*

Proof. On the one hand, I is minimally generated by three elements and has height 2. To see this, note that I contains (v) so has grade at least 2. On the other hand, since $\operatorname{rad}(I) = \operatorname{rad}(v_1, g)$, any minimal prime of I is a minimal prime of (v_1, g), which by the Generalized Principal Ideal Theorem will be of height at most 2. Finally, $IA_{\mathfrak{p}}$ is locally a complete intersection at primes $\mathfrak{p}$ minimal over I, because such a prime $\mathfrak{p}$ fails to contain x_1, so $IA_{\mathfrak{p}} \cong (IA_{x_1})A_{\mathfrak{p}_{x_1}}$. But $IA_{x_1} = (v)A_{x_1}$ and $IA_{\mathfrak{p}} = (v)A_{\mathfrak{p}}$. □

Remark 6.4. Let I be an HN ideal. Suppose that $x = x_1, x_2, x_3$ is a regular sequence and that $\operatorname{grade}(v) = 2$. Then I is generated by a d-sequence, I is of linear type and of strong linear type.

Proof. Indeed, since $(v) : D = (u)$ and $I \cap (u) = (v)$,

$$((v) : D) \cap I = (u) \cap I = (v).$$

Therefore, $(v) : D^2 = (v) : D$. In particular, for some $w \in A$, $v_1 + wv_2, v_2, D$ is a d-sequence which generates I (see [**12**] for a general discussion of d-sequences, and [**12**, Example 4] in particular for the result at issue here). Hence, I is an ideal of linear type, i.e. the canonical graded homomorphism $\alpha : \boldsymbol{S}(I) \to \boldsymbol{R}(I)$ between the symmetric algebra of I and the Rees algebra of I is an isomorphism (see, for example, [**9**]). In fact, a somewhat stronger property holds, namely I is of strong linear type, i.e. $H_2(A, B, \boldsymbol{G}(I)) = 0$, where $H_2(A, B, \boldsymbol{G}(I))$ stands for the second André–Quillen homology group of the A-algebra $B = A/I$ with coefficients in the B-module $\boldsymbol{G}(I)$, the associated graded ring of I.

It is well known that $\ker(\alpha_2) \cong H_2(A, B, B)$. Moreover, if $A \supset \mathbb{Q}$ and $H_2(A, B, \boldsymbol{G}(I)) = 0$, then I is of linear type. Since B as an A-module has projective dimension 2 and I is of linear type, then the converse also holds and one has $H_2(A, B, \boldsymbol{G}(I)) = 0$ [**17**, Theorem 3.4 and Proposition 3.10]. $\square$

7. When are HN ideals prime?

Henceforth, $A = k[x_1, x_2, x_3]$ is the polynomial ring in three variables $x = x_1, x_2, x_3$ over a field k. In particular, v_1, v_2 is a regular sequence in any order, by Example 4.3. Denote by $\mathfrak{m}$ the maximal ideal of A generated by x_1, x_2, x_3.

We begin with the following definition, which will play a key role. The idea behind it lies in the subsequent remark and in the proofs of Example 4.3 and Lemma 7.7.

Definition 7.1. Let J be an ideal of A, $J \subset \mathfrak{m}$, such that $x_iA + J$ is $\mathfrak{m}$-primary for all $i = 1, 2, 3$. The *integer vector associated to* J, $m(J) = (m_1(J), m_2(J), m_3(J)) \in \mathbb{N}^3$, is defined as $m_i(J) = \text{length}(A/(x_iA + J))$, for each $i = 1, 2, 3$.

Remark 7.2. Let $I = (x_1^{c_1} - x_2^{b_2}x_3^{a_3}, x_2^{c_2} - x_1^{a_1}x_3^{b_3}, x_3^{c_3} - x_1^{b_1}x_2^{a_2})$ be an HN ideal. Then, for each $i = 1, 2, 3$, $x_iA + I$ is $\mathfrak{m}$-primary and $m_i(I) = e(x_i \cdot A_{\mathfrak{m}}/I_{\mathfrak{m}})$. Moreover, $m(I)$ can be obtained directly as $m(I) = (c_2c_3 - a_2b_3, c_1c_3 - a_3b_1, c_1c_2 - a_1b_2)$. In particular, $m(I) \in \mathbb{N}^3$ generates the $\mathbb{Q}$-linear subspace of solutions of (4.1) (see Remark 4.4), and so $m_i(I)$ is the weight given to x_i, $i = 1, 2, 3$, in Example 4.3.

Proof. We have $x_1A + I = (x_1, x_2^{c_2}, x_2^{b_2}x_3^{a_3}, x_3^{c_3})$, which is $\mathfrak{m}$-primary. Thus, $m_1(I)$ is finite and can be calculated as $\text{length}(A_{\mathfrak{m}}/(x_1A + I)_{\mathfrak{m}})$. In particular, $x_1 \cdot A_{\mathfrak{m}}/I_{\mathfrak{m}}$ is a parameter ideal of the Cohen–Macaulay local ring $A_{\mathfrak{m}}/I_{\mathfrak{m}}$ (recall that I is height-unmixed (see Proposition 2.2)). By [**18**, Proposition 11.1.10], $\text{length}(A_{\mathfrak{m}}/(x_1A + I)_{\mathfrak{m}}) = e(x_1 \cdot A_{\mathfrak{m}}/I_{\mathfrak{m}})$. On the other hand, the quotient ring $A/(x_1A + I)$ is isomorphic to $k[x_2, x_3]/(x_2^{c_2}, x_2^{b_2}x_3^{a_3}, x_3^{c_3})$, which has length $(a_2+b_2)(a_3+b_3) - a_2b_3$. There are analogous arguments for $m_2(I)$ and $m_3(I)$. $\square$

Now, let us extend the definition of Herzog ideals introduced in § 1.

Definition 7.3. Let $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ be an integer vector with greatest common divisor not necessarily equal to 1. The *Herzog ideal associated to* n is the prime ideal $\mathfrak{p}_n$ defined as the kernel of the morphism $\varphi_n \colon A \to k[t]$ sending x_i to t^{n_i} for each $i = 1, 2, 3$.

We then have the following.

Remark 7.4. Let $m = (m_1, m_2, m_3) \in \mathbb{N}^3$ and $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ such that $m = dn$ for some $d \in \mathbb{N}$. Then $\mathfrak{p}_m = \mathfrak{p}_n$.

Proof. Since $A/\mathfrak{p}_n \cong k[t^{n_1}, t^{n_2}, t^{n_3}] \subset k[t]$ is an integral extension, $\dim(A/\mathfrak{p}_n) = 1$ and $\mathfrak{p}_n$ is a prime ideal of height 2. Analogously, $\mathfrak{p}_m$ is a prime ideal of height 2. If $\gcd(n_1, n_2, n_3) = 1$, using an explicit system of generators of $\mathfrak{p}_n$ (given in [**10**]; see also [**14**, pp. 138–139]), one can easily check that $\mathfrak{p}_n \subseteq \mathfrak{p}_m$, and so they are equal. In general, factoring out the greatest common divisor $e = \gcd(n_1, n_2, n_3)$, let $r = n/e = (r_1, r_2, r_3)$, where $\gcd(r_1, r_2.r_3) = 1$. Then $n = er$ and $m = (de)r$; thus, $\mathfrak{p}_m = \mathfrak{p}_n = \mathfrak{p}_r$. $\square$

Lemma 7.5. *Let $\mathfrak{p}_n$ be the Herzog ideal associated to $n = (n_1, n_2, n_3) \in \mathbb{N}^3$. Then, for all $i = 1, 2, 3$, $x_i A + \mathfrak{p}_n$ is $\mathfrak{m}$-primary and $m_i(\mathfrak{p}_n) = e(x_i \cdot A_{\mathfrak{m}}/(\mathfrak{p}_n)_{\mathfrak{m}})$. Moreover, $m(\mathfrak{p}_n) = n/\gcd(n)$.*

Proof. By the preceding remark we clearly can suppose that $\gcd(n_1, n_2, n_3) = 1$. Since $x_2^{n_1} = x_1^{n_2} + (x_2^{n_1} - x_1^{n_2}) \in x_1 A + \mathfrak{p}_n$, and analogously for x_3, $x_1 A + \mathfrak{p}_n$ is $\mathfrak{m}$-primary. Thus, $m_1(\mathfrak{p}_n)$ is finite and can be calculated as $\operatorname{length}(A_{\mathfrak{m}}/(x_1 A + \mathfrak{p}_n)_{\mathfrak{m}})$. In particular, $x_1 \cdot A_{\mathfrak{m}}/(\mathfrak{p}_n)_{\mathfrak{m}}$ is an $\mathfrak{m}A_{\mathfrak{m}}/(\mathfrak{p}_n)_{\mathfrak{m}}$-primary ideal of the Cohen–Macaulay one-dimensional domain $A_{\mathfrak{m}}/(\mathfrak{p}_n)_{\mathfrak{m}}$. By [**18**, Proposition 11.1.10], $\operatorname{length}(A_{\mathfrak{m}}/(x_1 A+\mathfrak{p}_n)_{\mathfrak{m}}) = e(x_1 \cdot A_{\mathfrak{m}}/(\mathfrak{p}_n)_{\mathfrak{m}})$. On the other hand, the quotient ring $A/(x_1 A + \mathfrak{p}_n)$ is isomorphic to $R/t^{n_1}R$, where $R = \operatorname{Im}(\varphi) = k[t^{n_1}, t^{n_2}, t^{n_3}]$, so $\operatorname{length}(A/(x_1 A + \mathfrak{p}_n)) = \operatorname{length}(R/t^{n_1}R)$. To calculate the latter, one can localize at the maximal ideal $\mathfrak{n} = (t^{n_1}, t^{n_2}, t^{n_3})$ since $t^{n_1}R$ is $\mathfrak{n}$-primary, as $(t^{n_j})^{n_1} \in t^{n_1}R$, $j = 2, 3$. Since $t^{n_1}R_{\mathfrak{n}}$ is a parameter ideal of a Cohen–Macaulay local ring, $\operatorname{length}(R_{\mathfrak{n}}/t^{n_1}R_{\mathfrak{n}}) = e(t^{n_1}R_{\mathfrak{n}})$ (again by [**18**, Proposition 11.1.10]).

Because $1 = s_1 n_1 + s_2 n_2 + s_3 n_3$ for some $s_i \in \mathbb{Z}$, $R \subset k[t]$ is a birational integral extension. Set $S = R \backslash \mathfrak{n}$. Since $tk[t]$ is the only non-zero prime $\mathfrak{q}$ of $k[t]$ such that $\mathfrak{q} \cap S = \emptyset$, the saturation of S in $k[t]$ is $k[t] \setminus tk[t]$, so $R_{\mathfrak{n}} \subset k[t]_{(t)}$ is a birational finite extension. Then $e(t^{n_1}R_{\mathfrak{n}}) = e(t^{n_1}k[t]_{(t)})$ [**18**, Corollary 11.2.6]. By [**18**, Proposition 11.1.10] again, the latter is equal to $\operatorname{length}(k[t]_{(t)}/t^{n_1}k[t]_{(t)}) = n_1$. □

In particular, one has a kind of converse of Remark 7.4.

Corollary 7.6. *Let $m = (m_1, m_2, m_3) \in \mathbb{N}^3$ and $n = (n_1, n_2, n_3) \in \mathbb{N}^3$. Then $\mathfrak{p}_m = \mathfrak{p}_n$ if and only if m and n are linearly dependent over the field $\mathbb{Q}$.*

Proof. If $\mathfrak{p}_m = \mathfrak{p}_n$, then $m(\mathfrak{p}_m) = m(\mathfrak{p}_n)$ and, by Lemma 7.5, $\gcd(n) \cdot m = \gcd(m) \cdot n$. Conversely, if m and n are linearly dependent over the field $\mathbb{Q}$, then $rm = sn$ for some $r, s \in \mathbb{N}$. By Remark 7.4, $\mathfrak{p}_m = \mathfrak{p}_n$. □

Now, given an HN ideal I, we want to look for the 'nearest' Herzog ideal to I.

Lemma 7.7. *Let I be an HN ideal and $m(I) \in \mathbb{N}^3$ its associated integer vector. Then $\mathfrak{p}_{m(I)}$ is the unique Herzog ideal containing I. In particular, $\mathfrak{p}_{m(I)}$ is a minimal prime of I.*

Proof. Given any $m = (m_1, m_2, m_3) \in \mathbb{N}^3$,

$$I = (x_1^{c_1} - x_2^{b_2} x_3^{a_3}, x_2^{c_2} - x_1^{a_1} x_3^{b_3}, x_3^{c_3} - x_1^{b_1} x_2^{a_2}) \subseteq \mathfrak{p}_m$$

if and only if m satisfies (4.1), whose $\mathbb{Q}$-linear subspace of solutions is generated by $m(I) \in \mathbb{N}^3$ (see Example 4.3 and Remark 7.2). Thus, $I \subseteq \mathfrak{p}_{m(I)}$. Since both ideals have height 2, $\mathfrak{p}_{m(I)}$ is a minimal prime of I.

Suppose now that $I \subseteq \mathfrak{p}_m$ and $I \subseteq \mathfrak{p}_n$, where $m, n \in \mathbb{N}^3$. By Remark 7.4 we can suppose that $\gcd(m) = 1$ and $\gcd(n) = 1$. In particular, since $I \subseteq \mathfrak{p}_m, \mathfrak{p}_n$, then m and n are solutions of (4.1). So, there exist $p, q \in \mathbb{Q}$, $p, q > 0$, such that $m = pm(I)$ and $n = qm(I)$, i.e. $rm = sn$ for some $r, s \in \mathbb{N}$ (see Remark 4.4). Taking the greatest common divisor, $r = s$, $m = n$ and $\mathfrak{p}_m = \mathfrak{p}_n$. □

The next result characterizes when an HN ideal is a prime ideal.

Theorem 7.8. *Let I be an HN ideal and let $m(I) \in \mathbb{N}^3$ be its associated integer vector. Then the following conditions are equivalent:*

(i) *I is prime;*

(ii) *$I = \mathfrak{p}_{m(I)}$;*

(iii) $\gcd(m(I)) = 1$.

Proof. By Lemma 7.7, $\mathfrak{p}_{m(I)}$ is a minimal prime of I. Thus, I is prime if and only if $I = \mathfrak{p}_{m(I)}$. Consider the exact sequence

$$0 \to L \to A/I \to A/\mathfrak{p}_{m(I)} \to 0,$$

where $L = \mathfrak{p}_{m(I)}/I$. Tensoring it with A/x_1A and using the fact that $x_1 \notin \mathfrak{p}_{m(I)}$, one obtains

$$0 \to L/x_1L \to A/(x_1A + I) \to A/(x_1A + \mathfrak{p}_{m(I)}) \to 0.$$

Endow A with the natural grading induced by giving x_i weight $m_i(I)$, $i = 1, 2, 3$ (see Remark 7.2). Then I and $\mathfrak{p}_{m(I)}$ are homogeneous ideals in this grading. By the graded variant of Nakayama's Lemma, $L = 0$ if and only if $L = x_1L$. Therefore, $I = \mathfrak{p}_{m(I)}$ if and only if the Artinian rings $A/(x_1A + I)$ and $A/(x_1A + \mathfrak{p}_{m(I)})$ have the same length. But, by Lemma 7.5, the length of $A/(x_1A+\mathfrak{p}_{m(I)})$ is equal to $m_1(\mathfrak{p}_{m(I)}) = m_1(I)/\gcd(m(I))$, where we recall that $m_1(I)$ is by definition the length of $A/(x_1A + I)$. So the result follows. □

Example 7.9. Let I_r be the HN ideal associated to

$$\mathcal{M}_r = \begin{pmatrix} x_1^{ra_1} & x_2^{ra_2} & x_3^{ra_3} \\ x_2^{rb_2} & x_3^{rb_3} & x_1^{rb_1} \end{pmatrix},$$

where $a, b \in \mathbb{N}^3$ and $r \in \mathbb{N}$. Then $m(I_r) = r^2 m(I_1)$. In particular, I_r is not prime for all $r > 1$. In fact, $I_r \subseteq I_1$, since

$$x_1^{rc_1} - x_2^{rb_2}x_3^{ra_3} = (x_1^{c_1} - x_2^{b_2}x_3^{a_3})\left(\sum_{i=1}^{r} x_1^{(r-i)c_1}x_2^{(i-1)b_2}x_3^{(i-1)a_3}\right).$$

Note that the I_r are all distinct and all have the same associated Herzog ideal.

Remark 7.10. One has the maps

$$m\colon \{\text{HN ideals}\} \to \mathbb{N}^3 \quad \text{and} \quad \mathfrak{p}_\bullet\colon \mathbb{N}^3 \to \{\text{Herzog ideals}\}$$

defined by $I \mapsto m(I)$ and $m \mapsto \mathfrak{p}_m$.

The map m is not injective. For example, as regards the HN ideal given by the triples $(a_1, a_2, a_3) = (1, 1, 3)$, $(b_1, b_2, b_3) = (3, 2, 3)$, we get $m = (15, 15, 10)$, which we also get whenever $(a_1, a_2, a_3) = (2, 3, 3)$, $(b_1, b_2, b_3) = (1, 1, 3)$. The second ideal contains a binomial with a pure term in x_1^3, whereas the first ideal does not. And indeed m is not surjective even if we restrict the range to triples of positive integers, each of which is at least 3; for example, $(3, 4, 4)$ is not in the image of m, since if it were, $a_2 = a_3 = b_2 = b_3 = 1$, and a contradiction would follow easily. The map $\mathfrak{p}_\bullet$ is not injective by Remark 7.4 and is surjective by definition.

The composition $\mathfrak{p}_\bullet \circ m$, which assigns to each HN ideal its associated Herzog ideal, is clearly not injective (because m is not injective (see also Example 7.9)).

Is $\mathfrak{p}_\bullet \circ m$ surjective? What is the image of m?

Remark 7.11. As regards a description of $\mathrm{Im}(\mathfrak{p}_\bullet \circ m)$ and $\mathrm{Im}(m)$, at the moment we have only partial results giving necessary conditions for triples of positive integers to belong to these image sets. These results have elementary but somewhat lengthy and technical proofs. So for the moment we confine ourselves to the following observations.

(1) Whenever $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ has $\gcd(n) = 1$, $n \in \mathrm{Im}(m)$ if and only if the subsemigroup H of $(\mathbb{N}, +)$ generated by n_1, n_2, n_3 is not symmetric. Indeed, suppose that there exists an HN ideal I with $m(I) = n$. By Theorem 7.8, I is prime and equal to $\mathfrak{p}_n$. Thus, $\mathfrak{p}_n$ is an HN ideal and so is not a complete intersection. By [**10**] (see also [**14**, p. 139]), H is not symmetric. Conversely, if H is not symmetric, $\mathfrak{p}_n$ is not a complete intersection. Thus, $\mathfrak{p}_n$ is an HN ideal and, by Remark 7.5, $m(\mathfrak{p}_n) = n$, so $n \in \mathrm{Im}(m)$.

(2) If (m_1, m_2, m_3) is in $\mathrm{Im}(m)$ and m_1 and m_2 are bounded above by r, say, then it is easy to see that m_3 is bounded above by $3r^2 - 2$. On the other hand, one can show that if $n = (n_1, n_2, n_3) \in \mathbb{N}^3$ is such that $\gcd(n_2, n_3) = 1$, so that in particular $\gcd(n_1, n_2, n_3) = 1$, and that n_1 is contained in $\mathbb{N}n_2 + \mathbb{N}n_3$, then $\mathfrak{p}_n$ lies in $\mathrm{Im}(\mathfrak{p}_\bullet \circ m)$.

8. On the number of primary components of an HN ideal

We keep the notation of the former section, i.e. $A = k[x_1, x_2, x_3]$ is the polynomial ring in three variables $x = x_1, x_2, x_3$ over a field k and $\mathfrak{m}$ is the maximal ideal of A generated by x_1, x_2, x_3. The purpose of this section is to give bounds for the number of associated (i.e. minimal) primes of an HN ideal of A. We begin with the following observation.

Remark 8.1. Let I be an HN ideal and let $m(I)$ be its associated integer vector. Then, for each $i = 1, 2, 3$,

$$m_i(I) = \sum_{\mathfrak{q} \in \mathrm{Min}(A/I)} e(x_i \cdot A_{\mathfrak{m}}/\mathfrak{q}_{\mathfrak{m}}) \,\mathrm{length}((A/I)_{\mathfrak{q}}).$$

In particular, cardinal $\mathrm{Min}(A/I) \leqslant \min\{m_1(I), m_2(I), m_3(I)\}$.

Proof. By Remark 7.2, $m_1(I) = e(x_1 \cdot A_{\mathfrak{m}}/I_{\mathfrak{m}})$ and, by the Associativity Formula [**18**, Theorem 11.2.4], this can be calculated as

$$e(x_1 \cdot A_{\mathfrak{m}}/I_{\mathfrak{m}}) = \sum_{\mathfrak{a} \in \operatorname{Min}(A_{\mathfrak{m}}/I_{\mathfrak{m}})} e(x_1 \cdot A_{\mathfrak{m}}/I_{\mathfrak{m}}; A_{\mathfrak{m}}/\mathfrak{a}) \operatorname{length}((A_{\mathfrak{m}}/I_{\mathfrak{m}})_{\mathfrak{a}}).$$

Endowing A with the grading obtained by giving x_i weight $m_i(I)$, $i = 1, 2, 3$, I is then homogeneous in this grading and hence any associated prime $\mathfrak{q}$ of I sits inside $\mathfrak{m}$. Thus, any $\mathfrak{a}$ in $\operatorname{Min}(A_{\mathfrak{m}}/I_{\mathfrak{m}})$ is of the form $\mathfrak{a} = \mathfrak{q}A_{\mathfrak{m}}$ with $\mathfrak{q}$ in $\operatorname{Min}(A/I)$ and vice versa. Hence, the equality follows. In particular, the sum has as many non-zero terms as I has minimal primes. □

To improve on this observation, we need the following lemma, which was inspired by, and in turn generalizes, [**6**, Lemma 10.15].

Lemma 8.2. *Let $m_2, m_3 \in \mathbb{N}$ with $\gcd(m_2, m_3) = e$. Let $m_2 = ep_2$ and $m_3 = ep_3$ with $\gcd(p_2, p_3) = 1$. Let f be a factor of $x_2^{m_3} - x_3^{m_2}$ which is not a unit. Then f is of the form*

$$a_{rp_3} x_2^{rp_3} + a_{(r-1)p_3} x_2^{(r-1)p_3} x_3^{p_2} + \cdots + a_{p_3} x_2^{p_3} x_3^{(r-1)p_2} + a_0 x_3^{rp_2},$$

with $r \in \mathbb{N}$ and $a_i \in k$, $a_{rp_3}, a_0 \neq 0$.

Proof. Set the weight of x_i equal to p_i, $i = 2, 3$, so that $x_2^{m_3} - x_3^{m_2}$ is homogeneous of degree ep_2p_3. Suppose that f is a factor of $x_2^{m_3} - x_3^{m_2} \in k[x_2, x_3]$ which is not a unit. Then f is homogeneous of degree p, say, where $p > 0$. Write

$$f = a_t x_2^t + a_{t-1} x_2^{t-1} x_3^{l_{t-1}} + \cdots + a_0 x_3^{l_0},$$

$a_t, a_0 \neq 0$, with typical term $a_i x_2^i x_3^{l_i}$, for $0 \leqslant i \leqslant t$. So $ip_2 + l_i p_3 = p = tp_2$. Because $\gcd(p_2, p_3) = 1$, p_2 divides l_i and p_3 divides $t - i$. Since $tp_2 = p < p + q = (e-1)p_2p_3$, $0 \leqslant i \leqslant t < (e-1)p_3$. So $0 \leqslant t - i < (e-1)p_3$. Thus, $t - i = jp_3$, and hence $i = t - jp_3$, with $j \in \mathbb{Z}$, $0 \leqslant j \leqslant e - 2$. Therefore,

$$f = a_t x_2^t + a_{t-p_3} x_2^{t-p_3} x_3^{l_{t-p_3}} + a_{t-2p_3} x_2^{t-2p_3} x_3^{l_{t-2p_3}} + \cdots + a_{t-(e-2)p_3} x_2^{t-(e-2)p_3} x_3^{l_{t-(e-2)p_3}}.$$

Since f has a pure term in x_3, $t = rp_3$ for some $r \in \mathbb{N}$, $1 \leqslant r \leqslant e - 2$, and

$$f = a_{rp_3} x_2^{rp_3} + a_{(r-1)p_3} x_2^{(r-1)p_3} x_3^{l_{(r-1)p_3}} + \cdots + a_{p_3} x_2^{p_3} x_3^{l_{p_3}} + a_0 x_3^{l_0}.$$

For $0 \leqslant j \leqslant r$, the term with coefficient $a_{(r-j)p_3}$ has degree $(r-j)p_3p_2 + l_{(r-j)p_3}p_3$, which must be equal to rp_2p_3, the degree of f. Thus, $l_{(r-j)p_3} = jp_2$. Therefore,

$$f = a_{rp_3} x_2^{rp_3} + a_{(r-1)p_3} x_2^{(r-1)p_3} x_3^{p_2} + \cdots + a_{p_3} x_2^{p_3} x_3^{(r-1)p_2} + a_0 x_3^{rp_2},$$

as desired. □

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Theorem 8.3. *Let I be an HN ideal and let $m(I)$ be its associated integer vector. Let $\mathfrak{q} \in \operatorname{Min}(A/I)$. Then, for all $i, j \in \{1,2,3\}$, $i \neq j$,*

$$e(x_i; A_\mathfrak{m}/\mathfrak{q}_\mathfrak{m}) \geqslant m_i(I)/\gcd(m_i(I), m_j(I)).$$

In particular, setting $d = \gcd(m(I))$ and $s = \min\{\gcd(m_i(I), m_j(I)) \mid 1 \leqslant i < j \leqslant 3\}$,

$$\operatorname{cardinal} \operatorname{Min}(A/I) \leqslant 1 + (s \cdot (d-1)/d).$$

Proof. Take $\mathfrak{q}$ a minimal prime of I in A, and set $B = k[x_2, x_3]$ and $\mathfrak{n} = (x_2, x_3)$ the maximal ideal of B generated by x_2, x_3. Clearly, $\mathfrak{q} \cap B \subset \mathfrak{n}$. Note that $A/\mathfrak{q}$ is a finite $(B/\mathfrak{q} \cap B)$-module. In particular, $(A/\mathfrak{q})_\mathfrak{n}$ is a finite $(B/\mathfrak{q} \cap B)_\mathfrak{n}$-module. But $\mathfrak{m}$ is the unique maximal (indeed prime) ideal $\mathfrak{p} \supseteq \mathfrak{q}$ in A such that $\mathfrak{p}/\mathfrak{q} \cap (B/\mathfrak{q} \cap B) = \mathfrak{n}/\mathfrak{q} \cap B$ (since in $A/\mathfrak{q}$, $x_1^{c_1} = x_2^{b_2} x_3^{a_3}$), so that $(A/\mathfrak{q})_\mathfrak{n} = (A/\mathfrak{q})_\mathfrak{m}$. Therefore, $(A/\mathfrak{q})_\mathfrak{m}$ is a finite $(B/\mathfrak{q} \cap B)_\mathfrak{n}$-module.

Moreover, in A/I, $x_2^{c_2} = x_1^{a_1} x_3^{b_3}$ and $x_3^{c_3} = x_1^{b_1} x_2^{a_2}$. Write $m(I) = (m_1, m_2, m_3)$. Since $m_2 = a_1 c_3 + b_1 b_3$ and $m_3 = a_1 a_2 + b_1 c_2$, $x_2^{m_3} = x_2^{a_1 a_2} x_2^{b_1 c_2} = x_1^{a_1 b_1} x_2^{a_1 a_2} x_3^{b_1 b_3}$ and $x_3^{m_2} = x_3^{a_1 c_3} x_3^{b_1 b_3} = x_1^{a_1 b_1} x_2^{a_1 a_2} x_3^{b_1 b_3}$. Thus, $x_2^{m_3} = x_3^{m_2}$ in A/I and $x_2^{m_3} - x_3^{m_2} \in I$. In particular, $x_2^{m_3} - x_3^{m_2} \in I \cap B \subseteq \mathfrak{q} \cap B$. Thus, $(x_2, x_3^{m_2}) \subseteq x_2 B + (\mathfrak{q} \cap B)$ and $x_2 \cdot B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}$ is $\mathfrak{n} B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}$-primary. Therefore, by [**18**, Corollary 11.2.6],

$$e(x_2; A_\mathfrak{m}/\mathfrak{q}_\mathfrak{m}) = e(x_2; B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}) \cdot \operatorname{rank}_{B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}}(A_\mathfrak{m}/\mathfrak{q}_\mathfrak{m}) \geqslant e(x_2; B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}).$$

Since $B/\mathfrak{q} \cap B \hookrightarrow A/\mathfrak{q}$ is an integral extension, $\mathfrak{q} \cap B$ is a prime ideal of height 1 of B (thus principal) and $B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}$ is a one-dimensional Noetherian local domain, thus Cohen–Macaulay. By [**18**, Proposition 11.1.10], $e(x_2; B_\mathfrak{n}/(\mathfrak{q} \cap B)_\mathfrak{n}) = \operatorname{length}(B_\mathfrak{n}/x_2 B_\mathfrak{n} + (\mathfrak{q} \cap B)_\mathfrak{n})$. Since $x_2 B + \mathfrak{q} \cap B$ is $\mathfrak{n}$-primary, the latter length is equal to $\operatorname{length}(B/x_2 B + \mathfrak{q} \cap B)$.

But $\mathfrak{q} \cap B = (f)$ for some irreducible polynomial $f \in B$ and $x_2^{m_3} - x_3^{m_2} \in \mathfrak{q} \cap B = (f)$. Thus, $x_2^{m_3} - x_3^{m_2} = fg$ for some $g \in B$. By Lemma 8.2, and following its notation, f is a polynomial in $x_2^{p_3}$ and $x_3^{p_2}$ with pure non-zero terms in each of $x_2^{p_3}$ and $x_3^{p_2}$, where $\gcd(m_2, m_3) = e$ and $m_2 = e p_2$ and $m_3 = e p_3$, with $\gcd(p_2, p_3) = 1$. Thus, $\mathfrak{q} \cap B \subseteq (x_2, x_3^{p_2})$. Hence, $e(x_2; A_\mathfrak{m}/\mathfrak{q}_\mathfrak{m}) \geqslant \operatorname{length}(B/x_2 B + \mathfrak{q} \cap B) \geqslant \operatorname{length}(B/(x_2, x_3^{p_2})) = p_2 = m_2/e$.

By Remark 8.1 and Lemma 7.5,

$$\begin{aligned} m_2 &\geqslant e(x_2; A_\mathfrak{m}/(\mathfrak{p}_{m(I)})_\mathfrak{m}) + \sum_{\mathfrak{q} \in \operatorname{Min}(A/I) \setminus \{\mathfrak{p}_{m(I)}\}} e(x_2; A_\mathfrak{m}/\mathfrak{q}_\mathfrak{m}) \\ &\geqslant (m_2/d) + (m_2/e) \cdot (\operatorname{cardinal} \operatorname{Min}(A/I) - 1). \end{aligned}$$

So $\operatorname{cardinal} \operatorname{Min}(A/I) \leqslant 1 + (e \cdot (d-1)/d)$. □

Example 8.4. Let $I = (x_1^{15} - x_2^8 x_3^3, x_2^{10} - x_1^5 x_3^6, x_3^9 - x_1^{10} x_2^2)$ be the HN ideal associated to

$$\mathcal{M}_r = \begin{pmatrix} x_1^5 & x_2^2 & x_3^3 \\ x_2^8 & x_3^6 & x_1^{10} \end{pmatrix}.$$

Then $m(I) = (78, 105, 110)$, whose greatest common divisor is 1, thus $I = \mathfrak{p}_{m(I)}$ is prime by Theorem 7.8. Note that for HN ideals which are prime, the bound given in Theorem 8.3 is precisely 1. However, it may be that $s = \min\{\gcd(m_i(I), m_j(I)) \mid 1 \leqslant i < j \leqslant 3\} \neq 1$. For instance, in this case, $s = 2$.

Example 8.5. Let $I = (x_1^5 - x_2^2 x_3, x_2^4 - x_1 x_3^3, x_3^4 - x_1^4 x_2^2)$ be the HN ideal associated to

$$\mathcal{M} = \begin{pmatrix} x_1 & x_2^2 & x_3 \\ x_2^2 & x_3^3 & x_1^4 \end{pmatrix}.$$

Then $m(I) = (10, 16, 18)$, whose greatest common divisor d equals 2. Therefore, by Theorem 7.8, I is not prime. Moreover, $\min\{\gcd(m_i(I), m_j(I)) \mid 1 \leqslant i < j \leqslant 3\} = 2$. Thus, by Theorem 8.3, I has at most two minimal primes. By Lemma 7.7, the Herzog ideal $\mathfrak{p}_n$ associated to $n = m(I)/d = (5, 8, 9)$ is a minimal prime of I. To calculate $\mathfrak{p}_n$, we use [**14**, pp. 137–139]: since 5 is the least integer number c_1 such that $c_1 n_1 \in \mathbb{N}n_2 + \mathbb{N}n_3$ $(5 \cdot 5 = 2 \cdot 8 + 1 \cdot 9)$, 3 is the least integer number c_2 such that $c_2 n_2 \in \mathbb{N}n_1 + \mathbb{N}n_3$ $(3 \cdot 8 = 3 \cdot 5 + 1 \cdot 9)$ and 2 is the least integer number c_3 such that $c_3 n_3 \in \mathbb{N}n_1 + \mathbb{N}n_2$ $(2 \cdot 9 = 2 \cdot 5 + 1 \cdot 8)$, we have $\mathfrak{p}_n = (x_1^5 - x_2^2 x_3, x_2^3 - x_1^3 x_3, x_3^2 - x_1^2 x_2)$. Observe that $\mathfrak{p}_n$ is the HN ideal associated to the matrix

$$\mathcal{M}_1 = \begin{pmatrix} x_1^3 & x_2 & x_3 \\ x_2^2 & x_3 & x_1^2 \end{pmatrix}.$$

Moreover, if $\operatorname{char}(k) \neq 2$, the k-algebra automorphism $\psi\colon A \to A$ defined by $\psi(x_1) = x_1$, $\psi(x_2) = -x_2$ and $\psi(x_3) = x_3$ leaves I invariant, whereas it takes $\mathfrak{p}_n$ to the prime ideal $\mathfrak{q} = (x_1^5 - x_2^2 x_3, x_2^3 + x_1^3 x_3, x_3^2 + x_1^2 x_2)$. In other words, $I = \psi(I) \subset \psi(\mathfrak{p}_n) = \mathfrak{q}$, and $\mathfrak{q}$ is also a minimal prime of I. Thus, the bound in Theorem 8.3 is attained. In fact, $e(x_1; A_{\mathfrak{m}}/\mathfrak{a}_{\mathfrak{m}}) \geqslant m_1(I)/\gcd(m_1(I), m_2(I)) = 5$ for any minimal prime $\mathfrak{a}$ of I. Thus, by Remark 8.1, $\operatorname{length}((A/I)_{\mathfrak{a}}) = 1$ for each such $\mathfrak{a}$, and $I = \mathfrak{p}_n \cap \mathfrak{q}$ is radical. We will see in the next section that this fact holds more generally.

Note that if $\operatorname{char}(k) = 2$, then $(x_2^3 - x_1^3 x_3)^2 = -x_1 x_3^2 (x_1^5 - x_2^2 x_3) + x_2^2 (x_2^4 - x_1 x_3^3)$ and $(x_3^2 - x_1^2 x_2)^2 = x_3^4 - x_1^4 x_2^2$. Therefore, $\mathfrak{p}_n^2 \subseteq I \subsetneq \mathfrak{p}_n$ and $\operatorname{rad}(I) = \mathfrak{p}_n$. In particular, I has only one minimal prime and is not radical (because it is not prime). Note that $\mathfrak{p}_n^2 \subsetneq I$.

Remark 8.6. Let I be an HN ideal and let $m(I) = m = (m_1, m_2, m_3) \in \mathbb{N}^3$ be its associated integer vector. Let $d = \gcd(m)$ and $n = m/d = (n_1, n_2, n_3) \in \mathbb{N}^3$, where $\gcd(n) = 1$. Then $\gcd(m_i, m_j) = d \cdot \gcd(n_i, n_j)$ for all $1 \leqslant i < j \leqslant 3$. Thus,

$$s = \min\{\gcd(m_i, m_j) \mid 1 \leqslant i < j \leqslant 3\} = d \cdot \min\{\gcd(n_i, n_j) \mid 1 \leqslant i < j \leqslant 3\} = d \cdot r,$$

say. Therefore, $1 + (s(d-1)/d) = 1 + r(d-1)$. In particular, if some $\gcd(n_i, n_j) = 1$, then $\operatorname{cardinal} \operatorname{Min}(A/I) \leqslant d$. Does this bound hold in complete generality?

Remark 8.7. We have established two other estimates for the number of minimal primes in A/I. In a number of cases that we have looked at, these estimates are weaker than the one given in the statement of Theorem 8.3. However, the methods used to obtain them are of interest and the estimates themselves may prove to be of worth in other situations. So we confine ourselves to sketching some brief details concerning them.

(1) By the comment right at the end of § 5, the number of minimal primes in A/I is one less than the number of minimal primes in $A/(v)$. Set $n = m(I)/\gcd(m(I))$. We can then use the argument of the proof of Remark 8.1, only this time applied to $e(x_i \cdot A_{\mathfrak{m}}/(v)A_{\mathfrak{m}})$, $i = 1, 2, 3$, to get the following estimate:

$$\operatorname{cardinal} \operatorname{Min}(A/I) \leqslant \min\{c_i c_j - n_k \mid \{i, j, k\} = \{1, 2, 3\}\}.$$

Here we have used symmetry and the fact that $e(x_i \cdot A_{\mathfrak{m}}/(\mathfrak{p}_n)_{\mathfrak{m}}) = n_i$ (see Lemma 7.5).

(2) A more delicate argument using minimal reductions and the criterion of multiplicity 1 establishes the following result. Suppose, possibly after relabelling the suffixes, that $a_3 \geqslant c_1$, and that $b_3 \geqslant c_2$. Then

$$\operatorname{cardinal} \operatorname{Min}(A/I) \leqslant (c_1 c_2 - n_3)/2.$$

9. HN ideals are usually radical

As in the previous section, $A = k[x_1, x_2, x_3]$ is the polynomial ring in three variables over a field k. We start with the following result.

Theorem 9.1. *Let I be an HN ideal. If k has characteristic zero (or large enough), then* $\operatorname{rad}(I) = I$.

Proof. By Example 4.3, v_1, v_2 is a regular sequence. Thus, $\operatorname{rad}(v) = (v) : \operatorname{Jac}(v)$, where $\operatorname{Jac}(v)$ is the Jacobian ideal of (v), i.e. the ideal generated by the 2×2 minors of the Jacobian matrix $\partial(v_1, v_2)/\partial(x_1, x_2, x_3)$, provided that k has characteristic zero or sufficiently large (see [**21**, Theorem 5.4.2, p. 131 and comments on p. 130]). Concretely, setting

$$\begin{aligned} J_1 &= c_1 c_2 x_1^{c_1 - 1} x_2^{c_2 - 1} - a_1 b_2 x_1^{a_1 - 1} x_2^{b_2 - 1} x_3^{c_3}, \\ J_2 &= b_3 c_1 x_1^{a_1 + c_1 - 1} x_3^{b_3 - 1} + a_1 a_3 x_1^{a_1 - 1} x_2^{b_2} x_3^{c_3 - 1}, \\ J_3 &= b_2 b_3 x_1^{a_1} x_2^{b_2 - 1} x_3^{c_3 - 1} + a_3 c_2 x_2^{b_2 + c_2 - 1} x_3^{a_3 - 1}, \end{aligned}$$

these being the three generators of $\operatorname{Jac}(v)$,

$$\operatorname{rad}(v) = (v) : \operatorname{Jac}(v) = (v) : (J_1, J_2, J_3) = [(v) : J_1] \cap [(v) : J_2] \cap [(v) : J_3] \subseteq (v) : J_1.$$

Write $J_1 = x_1^{a_1 - 1} x_2^{b_2 - 1} h$, with $h = -a_1 b_2 D + s x_1^{b_1} x_2^{a_2} \in A$, $D = x_3^{c_3} - x_1^{b_1} x_2^{a_2}$ and $s = c_1 c_2 - a_1 b_2 \in \mathbb{Z}$. Now, by Proposition 2.2 and Corollary 2.3, and using the general

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rule of quotient ideals that $L : fg = (L : f) : g$, we have

$$
\begin{aligned}
(v) : J_1 &= ((v) : x_1^{a_1-1}x_2^{b_2-1}) : h \\
&= [(I \cap (u)) : x_1^{a_1-1}x_2^{b_2-1}] : h \\
&= [(I : x_1^{a_1-1}x_2^{b_2-1}) \cap ((u) : x_1^{a_1-1}x_2^{b_2-1})] : h \\
&= (I \cap (x_1, x_2)) : h \\
&= (I : h) \cap ((x_1, x_2) : h) \\
&= (I : x_1^{b_1}x_2^{a_2}) \cap ((x_1, x_2) : x_3^{c_3}) \\
&= I \cap (x_1, x_2) \subseteq I.
\end{aligned}
$$

Therefore, $\operatorname{rad}(v) \subseteq I$. By Lemma 5.1,

$$\operatorname{rad}(I) = \operatorname{rad}((v) : u_1) \subseteq \operatorname{rad}(v) : u_1 \subseteq I : u_1 = I.$$

□

Remark 9.2. In particular,

$$\operatorname{rad}(v) = \operatorname{rad}(I) \cap \operatorname{rad}(u) = I \cap (x_1, x_2) = (v_1, v_2, x_1D, x_2D).$$

Example 9.3. Let I be the HN ideal considered in Example 8.5. By Theorem 9.1, if k has characteristic zero, I is radical. In fact, we have shown that, if $\operatorname{char}(k) \neq 2$, then I is radical, whereas if $\operatorname{char}(k) = 2$, then I is not radical.

This fact holds rather more generally (see the discussion in Example 9.7). Before coming to this, we need the following two results.

Proposition 9.4. *Let I be an HN ideal and let $m(I) = (m_1, m_2, m_3) \in \mathbb{N}^3$ be its associated integer vector. If $a_1 = 1$, then $(A/I)_{x_2x_3}$ is isomorphic to $(k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2}))_{x_2x_3}$. In particular, their total quotient rings $Q(A/I)$ and $Q(k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2}))$ are isomorphic. Furthermore, the cardinality of $\operatorname{Min}(A/I)$ is equal to the cardinality of a maximal complete set of orthogonal idempotents of $Q(k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2}))$.*

Proof. One has $I = (v_1, v_2, D)$, with $v_1 = x_1^{c_1} - x_2^{b_2}x_3^{a_3}$, $v_2 = x_2^{c_2} - x_1^{a_1}x_3^{b_3}$, $D = x_3^{c_3} - x_1^{b_1}x_2^{a_2}$. In $A_{x_2x_3}$, since $a_1 = 1$, $x_1 = x_2^{c_2}x_3^{-b_3} - v_2x_3^{-b_3}$. Thus, in $A_{x_2x_3}$,

$$
\begin{aligned}
v_1 &= (x_2^{c_2}x_3^{-b_3} - v_2x_3^{-b_3})^{c_1} - x_2^{b_2}x_3^{a_3} \\
&= x_2^{c_1c_2}x_3^{-b_3c_1} - x_2^{b_2}x_3^{a_3} + v_2p \\
&= x_2^{b_2}x_3^{-b_3c_1}(x_2^{c_1c_2-b_2} - x_3^{a_3+b_3c_1}) + v_2p \\
&= x_2^{b_2}x_3^{-b_3c_1}(x_2^{m_3} - x_3^{m_2}) + v_2p,
\end{aligned}
$$

and

$$
\begin{aligned}
D &= x_3^{c_3} - (x_2^{c_2}x_3^{-b_3} - v_2x_3^{-b_3})^{b_1}x_2^{a_2} \\
&= x_3^{c_3} - x_2^{a_2+b_1c_2}x_3^{-b_1b_3} + v_2q \\
&= x_3^{-b_1b_3}(x_3^{b_1b_3+c_3} - x_2^{a_2+b_1c_2}) + v_2q \\
&= x_3^{-b_1b_3}(x_3^{m_2} - x_2^{m_3}) + v_2q,
\end{aligned}
$$

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with $p, q \in A_{x_2x_3}$. Therefore,

$$IA_{x_2x_3} = (x_2^{m_3} - x_3^{m_2}, v_2)A_{x_2x_3} = (x_2^{m_3} - x_3^{m_2}, x_1 - x_2^{c_2}x_3^{-b_3})A_{x_2x_3}.$$

So

$$(A/I)_{x_2x_3} \cong (k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2}))_{x_2x_3}.$$

Since $x_2, x_3 \in A$ are regular modulo I and each of $x_2, x_3 \in k[x_2, x_3]$ is regular modulo $(x_2^{m_3} - x_3^{m_2})$, the total quotient ring of A/I is isomorphic to the total quotient ring of $k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2})$.

Let V be the affine k-variety defined by I and let $R(V)$ be the ring of rational functions of V, i.e. the total quotient ring $Q(A/I)$ of A/I (see, for example, [**14**, Chapter III, Proposition 3.4]). So

$$R(V) \cong Q(k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2})).$$

Let $V = V_1 \cup \cdots \cup V_r$ be the decomposition of V into irreducible components, which induces an isomorphism $R(V) \cong R(V_1) \times \cdots \times R(V_r)$ (see, for example, [**14**, Chapter III, Proposition 2.8]). So the number of minimal primes of I is equal to the number of elements in a maximal complete set of orthogonal idempotents of

$$R(V) \cong Q(k[x_2, x_3]/(x_2^{m_3} - x_3^{m_2})).$$

$\square$

Proposition 9.5. *Let I be an HN ideal and let $m(I) \in \mathbb{N}^3$ be its associated integer vector. If $a_1 = 1$ and* $\gcd(m_2(I), m_3(I)) = d$ *is prime, then* $\gcd(m(I)) = d$ *as well and I is not a prime ideal. Moreover,* cardinal $\operatorname{Min}(A/I) \leqslant d$*. Furthermore, if* $\operatorname{char}(k) = d$*, then I is primary, is not radical and $\mathfrak{p}_{m(I)}^{(d)} \subsetneq I \subsetneq \mathfrak{p}_{m(I)}$.*

Proof. Let us prove first that I is not a prime ideal. Note that, once this is established, $\gcd(m(I)) \neq 1$ by Theorem 7.8; but, since $\gcd(m_2(I), m_3(I)) = d$ is prime, $\gcd(m(I))$ must be equal to d as well. In particular, by Theorem 8.3, cardinal $\operatorname{Min}(A/I) \leqslant 1 + (d(d-1)/d) = d$.

Write $m(I) = m = (m_1, m_2, m_3) \in \mathbb{N}^3$ and $m_2 = dn_2$, $m_3 = dn_3$ with $\gcd(n_2, n_3) = 1$. Then

$$x_2^{m_3} - x_3^{m_2} = (x_2^{n_3} - x_3^{n_2})(x_2^{(d-1)n_3} + x_2^{(d-2)n_3}x_3^{n_2} + \cdots + x_2^{n_3}x_3^{(d-2)n_2} + x_3^{(d-1)n_2}),$$

where $x_2^{n_3} - x_3^{n_2}$ is irreducible (by, for example, [**6**, Lemma 10.15] or Lemma 8.2 above).

Set

$$B = k[x_2, x_3], \qquad \mathfrak{a} = (x_2^{n_3} - x_3^{n_2}), \qquad \mathfrak{b} = (x_2^{(d-1)n_3} + \cdots + x_3^{(d-1)n_2})$$

(the corresponding ideals generated in B) and $C = B/\mathfrak{a}$. In C, $x_2^{n_3} = x_3^{n_2}$ so that, in C,

$$x_2^{(d-1)n_3} + \cdots + x_3^{(d-1)n_2} = dx_2^{(d-1)n_3}.$$

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Hence, if $\mathrm{char}(k) \neq d$, $\mathfrak{a}+\mathfrak{b}$ contains the element $x_2^{(d-1)n_3}$ and so $(B/(\mathfrak{a}+\mathfrak{b}))_{x_2}$ becomes the zero ring. In other words, $\mathfrak{a}B_{x_2}$ and $\mathfrak{b}B_{x_2}$ are relatively prime ideals of B_{x_2}. By the Chinese Remainder Theorem (see, for example, [**14**, Chapter II, Proposition 1.7]),

$$B_{x_2}/(x_2^{m_3}-x_3^{m_2})B_{x_2} \cong (B_{x_2}/\mathfrak{a}B_{x_2}) \times (B_{x_2}/\mathfrak{b}B_{x_2}).$$

In particular, by Proposition 9.4,

$$(A/I)_{x_2x_3} \cong (k[x_2,x_3]/(x_2^{m_3}-x_3^{m_2}))_{x_2x_3} \cong (B/\mathfrak{a})_{x_2x_3} \times (B/\mathfrak{b})_{x_2x_3},$$

which is not a domain. In particular, I is not prime.

If $\mathrm{char}(k) = d$, then $x_2^{m_3}-x_3^{m_2} = (x_2^{n_3}-x_3^{n_2})^d$. Keeping the same notation as above, by Proposition 9.4 again,

$$(A/I)_{x_2x_3} \cong (k[x_2,x_3]/(x_2^{m_3}-x_3^{m_2}))_{x_2x_3} \cong (B/\mathfrak{a}^d)_{x_2x_3},$$

where $\mathfrak{a}$ is a prime ideal in B and a complete intersection (in fact principal), so $\mathfrak{a}^d = \mathfrak{a}^{(d)}$, the dth symbolic power, and $\mathfrak{a}^d$ is $\mathfrak{a}$-primary. Hence, the nilradical of $B/\mathfrak{a}^d$ is the (non-zero) prime ideal $\mathfrak{a}/\mathfrak{a}^d$, whose dth power is zero; in fact the co-length of $\mathfrak{a}^d$ at $\mathfrak{a}$ is precisely d. Since x_2x_3 lies outside $\mathfrak{a}$, this structure is preserved when we localize at the element x_2x_3. In the light of the isomorphism established above, we deduce that $(A/I)_{x_2x_3}$ has a (non-zero) prime nilradical with dth power equal to zero, so this prime radical must therefore be $(\mathfrak{p}_m/I)_{x_2x_3}$. Since I is unmixed and x_2x_3 is regular modulo I, we must have that I is $\mathfrak{p}_m$-primary and $\mathfrak{p}_m^d \subseteq I \subsetneq \mathfrak{p}_m$. In particular, $\mathfrak{p}_m^{(d)} \subseteq I$. Furthermore, $\mathfrak{p}_m^{(d)}$ equals I if and only if they have the same local co-length at $\mathfrak{p}_m$. Now $A_{\mathfrak{p}_m}$ is a regular local ring of dimension 2, so the local co-length of $A/\mathfrak{p}_m^{(d)}$ at $\mathfrak{p}_m$ is $d(d+1)/2$. Hence, the co-lengths agree if and only if $d(d+1)/2 = d$, i.e. $d = 1$. So for $d > 1$, I properly contains $\mathfrak{p}_m^{(d)}$. □

For the concrete case $d = 2$, we obtain the following result.

Corollary 9.6. *Let I be an HN ideal and let $m(I) \in \mathbb{N}^3$ be its associated integer vector. Suppose that $a_1 = 1$ and $\gcd(m_2(I), m_3(I)) = 2$. In this case, if $\mathrm{char}(k) \neq 2$, then I is radical and equal to the intersection of exactly two prime ideals; on the other hand, if $\mathrm{char}(k) = 2$, then I is primary, is not radical and $\mathfrak{p}_{m(I)}^{(2)} \subsetneq I \subsetneq \mathfrak{p}_{m(I)}$.*

Proof. By Proposition 9.5, we have only to show that I is radical whenever $\mathrm{char}(k) \neq 2$. Thus, suppose that $\mathrm{char}(k) \neq 2$. Write $m(I) = (m_1, m_2, m_3) \in \mathbb{N}^3$ and $m_2 = 2n_2$, $m_3 = 2n_3$ with $\gcd(n_2, n_3) = 1$. Then $x_2^{m_3}-x_3^{m_2} = (x_2^{n_3}-x_3^{n_2})(x_2^{n_3}+x_3^{n_2})$ is a decomposition into prime factors in $B = k[x_2,x_3]$ (see, for example, [**6**, Lemma 10.15] or Lemma 8.2 above). Let $\mathfrak{a}_1 = (x_2^{n_3}-x_3^{n_2})$ and $\mathfrak{a}_2 = (x_2^{n_3}+x_3^{n_2})$ be the corresponding prime ideals generated in B. Then $\mathfrak{a}_1 \neq \mathfrak{a}_2$ and, localizing at x_2, $\mathfrak{a}_1B_{x_2}$ and $\mathfrak{a}_2B_{x_2}$ are two relatively prime ideals of B_{x_2}. By the Chinese Remainder Theorem (see, for example, [**14**, Chapter II, Proposition 1.7]),

$$B_{x_2}/(x_2^{m_3}-x_3^{m_2})B_{x_2} \cong (B_{x_2}/(x_2^{n_3}-x_3^{n_2})B_{x_2}) \times (B_{x_2}/(x_2^{n_3}+x_3^{n_2})B_{x_2}).$$

In particular, by Proposition 9.4,

$$(A/I)_{x_2x_3} \cong (k[x_2,x_3]/(x_2^{m_3} - x_3^{m_2}))_{x_2x_3} \cong (B/\mathfrak{a}_1)_{x_2x_3} \times (B/\mathfrak{a}_2)_{x_2x_3},$$

which is a reduced ring. Since x_2x_3 is regular modulo I, A/I is reduced and I is radical. $\square$

Example 9.7. Consider (again) the HN ideal I of Example 8.5, which satisfies the hypotheses of Corollary 9.6, i.e. $a_1 = 1$ and $\gcd(m_2(I), m_3(I)) = 2$. Thus, one can conclude that if $\mathrm{char}(k) \neq 2$, I is radical and equal to the intersection of two primes, whereas if $\mathrm{char}(k) = 2$, then I is primary, is not radical and $\mathfrak{p}_{m(I)}^{(2)} \subsetneq I \subsetneq \mathfrak{p}_{m(I)}$.

For the concrete case $d = 3$, we have the following result.

Corollary 9.8. *Let I be an HN ideal and let $m(I) \in \mathbb{N}^3$ be its associated integer vector. Suppose that $a_1 = 1$ and $\gcd(m_2(I), m_3(I)) = 3$. In this case, if $\mathrm{char}(k) \neq 2, 3$ and k contains a square root of -3, then I is radical and equal to the intersection of exactly three prime ideals; if $\mathrm{char}(k) \neq 3$ and either $\mathrm{char}(k) = 2$ or else k does not contain a square root of -3, then I is radical and equal to the intersection of exactly two prime ideals; finally, if $\mathrm{char}(k) = 3$, then I is primary, is not radical and $\mathfrak{p}_{m(I)}^{(3)} \subsetneq I \subsetneq \mathfrak{p}_{m(I)}$.*

Proof. Write $m(I) = (m_1, m_2, m_3) \in \mathbb{N}^3$ and $m_2 = 3n_2$, $m_3 = 3n_3$ with $\gcd(n_2, n_3) = 1$. Then $x_2^{m_3} - x_3^{m_2}$ decomposes as $(x_2^{n_3} - x_3^{n_2})(x_2^{2n_3} + x_2^{n_3}x_3^{n_2} + x_3^{2n_2})$, where $x_2^{n_3} - x_3^{n_2}$ is irreducible (by, for example, [**6**, Lemma 10.15] or Lemma 8.2 above). On the other hand, any proper factor of $x_2^{2n_3} + x_2^{n_3}x_3^{n_2} + x_3^{2n_2}$ is a proper factor of $x_2^{m_3} - x_3^{m_2}$. Hence, by Lemma 8.2 again, any decomposition of $x_2^{2n_3} + x_2^{n_3}x_3^{n_2} + x_3^{2n_2}$ must be of the form $(x_2^{n_3} + \lambda x_3^{n_2})(x_2^{n_3} + \lambda^{-1}x_3^{n_2})$, $\lambda \in k \setminus \{0\}$. Therefore, $\lambda + \lambda^{-1} = 1$ and hence $\lambda^2 - \lambda + 1 = 0$.

Suppose now that $\mathrm{char}(k) \neq 2, 3$ and that k contains a square root of -3. Take $\lambda_0 \in k$ a solution of $\lambda^2 - \lambda + 1 = 0$. Set $B = k[x_2, x_3]$ and $\mathfrak{a}_1 = (x_2^{n_3} - x_3^{n_2})$, $\mathfrak{a}_2 = (x_2^{n_3} + \lambda_0 x_3^{n_2})$ and $\mathfrak{a}_3 = (x_2^{n_3} + \lambda_0^{-1}x_3^{n_2})$ the distinct prime ideals generated in B. Localizing at x_2, $\mathfrak{a}_1 B_{x_2}$, $\mathfrak{a}_2 B_{x_2}$ and $\mathfrak{a}_3 B_{x_2}$ become three pairwise relatively prime ideals of B_{x_2}. By the Chinese Remainder Theorem (see, for example, [**14**, Chapter II, Proposition 1.7]),

$$B_{x_2}/(x_2^{m_3} - x_3^{m_2})B_{x_2} \cong (B_{x_2}/\mathfrak{a}_1 B_{x_2}) \times (B_{x_2}/\mathfrak{a}_2 B_{x_2}) \times (B_{x_2}/\mathfrak{a}_2 B_{x_3}).$$

In particular, by Proposition 9.4,

$$(A/I)_{x_2x_3} \cong (k[x_2,x_3]/(x_2^{m_3} - x_3^{m_2}))_{x_2x_3} \cong (B/\mathfrak{a}_1)_{x_2x_3} \times (B/\mathfrak{a}_2)_{x_2x_3} \times (B/\mathfrak{a}_3)_{x_2x_3},$$

which is a reduced ring. By [**14**, Chapter III, Proposition 4.23], $Q(A/I)$ is the product of three fields. So I has exactly three minimal primes. Moreover, since $(A/I)_{x_2x_3}$ is reduced and x_2x_3 is regular modulo I, A/I is reduced and I is radical. Thus, I is the intersection of exactly three prime ideals.

If $\mathrm{char}(k) \neq 3$ and either $\mathrm{char}(k) = 2$ or else k does not contain a square root of -3, then $\mathfrak{a} = (x_2^{n_3} - x_3^{n_2})$ and $\mathfrak{b} = (x_2^{2n_3} + x_2^{n_3}x_3^{n_2} + x_3^{2n_2})$ are two distinct prime ideals of $B = k[x_2, x_3]$. Note that $3x_2^{2n_3} = (2x_2^{n_3} + x_3^{n_2})(x_2^{n_3} - x_3^{n_2}) + (x_2^{2n_3} + x_2^{n_3}x_3^{n_2} + x_3^{2n_2})$.

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Thus, localizing at x_2, $\mathfrak{a}B_{x_2}$ and $\mathfrak{b}B_{x_2}$ become two relatively prime ideals of B_{x_2}. Proceeding as before, one deduces that $(A/I)_{x_2x_3}$ is reduced and that $Q(A/I)$ is the product of two fields. So I is radical and equal to the intersection of exactly two prime ideals.

If $\mathrm{char}(k) = 3$, then finish by applying Proposition 9.5. □

Example 9.9. Let $I = (x_1^4 - x_2^2x_3^3, x_2^5 - x_1x_3^3, x_3^6 - x_1^3x_2^3)$ be the HN ideal associated to

$$\mathcal{M} = \begin{pmatrix} x_1 & x_2^3 & x_3^3 \\ x_2^2 & x_3^3 & x_1^3 \end{pmatrix}.$$

Here $a_1 = 1$ and $m(I) = m = (21, 15, 18)$, so $\gcd(m_2, m_3) = 3$. Thus, one can apply Corollary 9.8. For instance, if $k = \mathbb{C}$, I is radical and equal to the intersection of three prime ideals, whereas if $k = \mathbb{Q}$, I is radical and equal to the intersection of two prime ideals. On the other hand, if $\mathrm{char}(k) = 3$, I is primary, is not radical and $\mathfrak{p}_m^{(3)} \subsetneq I \subsetneq \mathfrak{p}_m$.

In any case, the Herzog ideal $\mathfrak{p}_m$ associated to $m = (21, 15, 18)$ is a minimal prime of I. A simple computation shows that $\mathfrak{p}_m = (x_1^3 - x_2^3x_3, x_2^4 - x_1^2x_3, x_3^2 - x_1x_2)$.

If $k = \mathbb{Q}$, Singular [**8**] gives for the minimal prime of I other than $\mathfrak{p}_m$ the ideal

$$(x_1^4 - x_2^2x_3^3, x_2^5 - x_1x_3^3, x_1^3x_3 + x_1x_2^4 + x_2^3x_3^2, x_1^2x_2^2 + x_1x_2x_3^2 + x_3^4, x_1^3x_2 + x_1^2x_3^2 + x_2^4x_3),$$

which is not a binomial ideal. We recall that, according to the work of Eisenbud and Sturmfels [**5**], if $k = \mathbb{C}$, all the associated primes of I must be binomial.

In this connection, we remark that the results of [**5**], and the rich combinatorial and algorithmic theory of binomial ideals that grew from them, could well throw light on the questions left open in this paper. We intend to pursue this line of enquiry in future work.

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