J. Austral. Math. Soc. (*Series A*) **63** (1997), 289–296

AMENABILITY AND SEMISIMPLICITY FOR SECOND DUALS OF QUOTIENTS OF THE FOURIER ALGEBRA $A(G)$

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(Received 23 April 1996; revised 8 April 1997)

Communicated by G. Robertson

Abstract

Let $F \subset G$ be closed and $A(F) = A(G)/I_F$. If F is a Helson set then $A(F)^{**}$ is an amenable (semisimple) Banach algebra. Our main result implies the following theorem: Let G be a locally compact group, $F \subset G$ closed, $a \in G$. Assume either (a) For some non-discrete closed subgroup H, the interior of $F \cap aH$ in aH is non-empty, or (b) $R \subset G$, $S \subset R$ is a symmetric set and $aS \subset F$. Then $A(F)^{**}$ is a non-amenable non-semisimple Banach algebra. This raises the question: How 'thin' can F be for $A(F)^{**}$ to remain a non-amenable Banach algebra?

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 46H20, 43A30; secondary 43A20.

1. Introduction

Let G be a locally compact group and $J \subset A(G)$ a closed ideal with zero set $Z(J) = \{x \in G; u(x) = 0 \text{ for all } u \in J\} = F$. Consider the second dual $(A(G)/J)^{**}$, of the quotient algebra $A(G)/J$, equipped with Arens multiplication.

If $F \subset G$ is a Helson set (thus $A(F) = C(F)$) then $A(F)^{**}$ is a commutative C^* algebra and is hence amenable, by a result of Sheinberg (see [CL] for amenability of Banach algebras and references).

The subset S of the real line R is symmetric if there are $t_n > 0$ such that $t_n > \sum_{n+1}^{\infty} t_i$ for all $n \geq 1$ and $S = \left\{\sum_0^{\infty} \varepsilon_i t_i;\ \varepsilon_i = 0, 1\right\}$. The Cantor 1/3 set is such (see [GMc, p. 88]). Analogously for $S \subset T$ (the unit circle). This is the only sense in which 'symmetric' is used in this paper.

Our main result implies the

THEOREM. *Let G be a locally compact group $J \subset A(G)$ a closed ideal with*

https://doi.org/10.1017/S1446788700000938 Published online by Cambridge University Press

$Z(J) = F$ and $a, b \in G$. Assume one of

(a) *For some non-discrete closed subgroup of $H \subset G$, $\mathrm{int}_{aHb} F \neq \emptyset$ or*

(b) *R is a closed subgroup of G, $S \subset R$ is a symmetric set with $aSb \subset F$.*

Then $(A(G)/J)^{**}$ is a non-amenable non-semisimple Banach algebra.

Furthermore, in case (a) *the result holds for the algebras $A_p(G)/J$, $1 < p < \infty$, where $A_2(G) = A(G)$ is the Fourier algebra of G; see* [Hz1].

As a very mild consequence one gets that $A(G)^{**}$ (or $B(G)^{**}$) is an amenable Banach algebra if and only if G is finite. This is a dual result of a theorem of Ghahramani, Loy and Willis [GLW] who have shown that $L^1(G)^{**}$ (or $M(G)^{**}$) is amenable if and only if G is finite. And yet, if $E \subset Z$ is a Sidon set $A(E)^{**}$ is amenable (see the sequel).

2. Notation and definitions

We follow the notation in [Ey] for the Fourier algebra $A(G)$, except that we denote its dual module $A(G)^*$, where $(u \cdot \Phi, v) = (\Phi, uv)$ for $u, v \in A(G)$, $\Phi \in A(G)^*$, by $PM(G)$, while $VN(G)$ is used in [Ey]. Thus $B(G)$ is the linear span of the continuous positive definite functions on G.

If $\mu \in M(G)$, the bounded Borel measures on G, let $\lambda\mu \in PM(G)$ be given by $(\lambda\mu, v) = \int v \, d\mu$ where $v \in A(G)$. Thus $(\lambda\delta_x, v) = v(x)$ if $x \in G, v \in A(G)$. If $\Phi \in PM(G)$, denote its support by $\mathrm{supp}\,\Phi$. Thus $x \in \mathrm{supp}\,\Phi$ if and only if for each neighborhood V of x there is some $v \in A(G)$ such that $\mathrm{supp}\, v \subset V$ and $(\lambda\mu, v) \neq 0$.

If $\boldsymbol{P} \subset PM(G)$ is a w^*closed $A(G)$ module and $a \in G$, let $E_{\mathbf{P}}(a) = \mathrm{ncl}\,\{\Phi \in \boldsymbol{P} : a \notin \mathrm{supp}\,\Phi\}$ (where ncl denotes norm-closure), $\mathrm{TI}_{\mathbf{P}}(a) = \{\Psi \in \boldsymbol{P}^* : \Psi = 0$ on $E_{\mathbf{P}}(a)\}$ and, if $\lambda\delta_a \in \boldsymbol{P}$, $\mathrm{TIM}_{\mathbf{P}}(a) = \{\Psi \in \boldsymbol{P}^* : (\Psi, \lambda\delta_a) = 1 = \|\Psi\|, \Psi = 0$ on $E_{\mathbf{P}}(a)\}$ (this being the set of topologically invariant means on $\boldsymbol{P}$ at a).

Let $J \subset A(G)$ be a closed ideal with zero set $Z(J) = F$. Let $\boldsymbol{P} = (A(G)/J)^*$ where $A(G)/J$ is taken with the quotient norm. If $F \subset G$ is closed then $I_F = \{v \in A(G) : v = 0 \text{ on } F\}$ and $A(F) = A(G)/I_F$. We consider $\boldsymbol{P}^* = (A(G)/J)^{**}$ equipped with the Arens multiplication given by $(\Psi_1 \Box \Psi_2, \Phi) = (\Psi_1, \Psi_2 \Box \Phi)$ for $\Psi_1, \Psi_2 \in \boldsymbol{P}^*$, $\Phi \in \boldsymbol{P}$ where $\Psi_2 \Box \Phi \in \boldsymbol{P}$ is given by $(\Psi_2 \Box \Phi, u) = (\Psi_2, u \cdot \Phi)$ for $u \in A(G)/J$; see [DH]. Thus $(\boldsymbol{P}^*, \Box)$ is a Banach algebra in which $A(G)/J$ is embedded and $\Box$ extends the multiplication in $A(G)/J$.

The Banach algebras $A_p(G)$, $1 < p < \infty$, are as defined in [Hz1] and are such that $A_2(G) = A(G)$ is the Fourier algebra of G. Thus $PM_p(G) = A_p(G)^*$. Let $B_p^M(G) = B_p^M = \{v \in C(G): vu \in A_p \text{ for all } u \in A_p\}$, where $C(G)$ $[C_c(G)]$ are the bounded continuous [with compact support] functions on G. The reader not interested in these may assume that $A_p(G)$ is $A(G)$ and proceed.

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In any case, as in [Gr2, Gr3], the above notation makes sense for $\boldsymbol{P} \subset PM_p(G)$. If $p \neq 2$ then $A_p(G)$ is very different from $A(G)$; see [Gr2, p. 49].

We denote for simplicity $A(G)$ by A, $A_p(G)$ by A_p, and $A_p(F) = A_p/I_F$.

If F, H are subsets of G then $\operatorname{int}_H F$ is the interior of $F \cap H$ in H (with the relative topology from G). If $u \in C(G)$, let $\operatorname{supp} u = \operatorname{cl}\{x : u(x) \neq 0\}$.

3. The main results

In what follows, let $J \subset A_p$ be a closed ideal with $F = Z(J)$ and $\boldsymbol{P} = (A_p/J)^*$.

LEMMA 1. (a) $\Psi \in \mathrm{TI}_{\mathbf{P}}(a)$ *if and only if* $(*)(\Psi, u \cdot \Phi) = u(a)(\Psi, \Phi)$ *for all* $u \in B_p^M$ *and* $\Phi \in \boldsymbol{P}$. (b) $\mathrm{TIM}_{\mathbf{P}}(a) \neq \emptyset$ *for all* $a \in F$.

PROOF. (a) The proof of Lemma 8′ of [Gr3] holds for all locally compact groups G with $A(G)$, $B(G)$ replaced by $A_p(G)$, $B_p^M(G)$, since only results in [Hz1] (which are valid in this context) were used in its proof. Thus $E_{\mathbf{P}}(a) = \operatorname{ncl}\{\Phi - v \cdot \Phi : \Phi \in \boldsymbol{P}, v \in S_3(a)\}$ where $S_3(a) = \{v \in B_p^M : v(a) = 1\}$. Thus $\Psi \in \mathrm{TI}_{\mathbf{P}}(a)$ if and only if $(\Psi, u \cdot \Phi) = (\Psi, \Phi)$ for all $\Phi \in \boldsymbol{P}$ and $u \in S_3(a)$. If $\Psi \in \mathrm{TI}_{\mathbf{P}}(a)$ and $u \in B_p^M$ with $u(a) \neq 0$, then $(\Psi, u \cdot \Phi) = u(a)(\Psi, \Phi)$ for all $\Phi \in \boldsymbol{P}$. If now $u \in B_p^M$ and $u(a) = 0$, then since $1 \in B_p^M$ we have $(\Psi, (1-u) \cdot \Phi) = (1-u(a))(\Psi, \Phi) = (\Psi, \Phi)$. Thus $(\Psi, u \cdot \Phi) = 0$ and $(*)$ holds for Ψ. If now $\Psi \in \boldsymbol{P}^*$ and $(*)$ holds for Ψ then $(\Psi, \Phi - v \cdot \Phi) = 0$ if $\Phi \in \boldsymbol{P}$ and $v \in S_3(a)$. Thus $\Psi = 0$ on $E_{\mathbf{P}}(a)$.

(b) This is shown for example as in [Gr2, p. 122] with e replaced by $a \in F$.

LEMMA 2. (a) *If* $a \in F$ *then* $\Psi \to (\Psi, \lambda\delta_a)$ *is a multiplicative* w^**-continuous non-zero linear functional on* $\boldsymbol{P}^*$.

(b) *If* $\Psi \in \boldsymbol{P}^*$, $\Psi_1 \in \mathrm{TI}_{\mathbf{P}}(a)$ *then* $\Psi \Box \Psi_1 = (\Psi, \lambda\delta_a)\Psi_1$.

(c) *If* $I_a = \{\Psi \in \boldsymbol{P}^* : (\Psi, \lambda\delta_a) = 0\}$ *then* $I_a \Box \mathrm{TI}_{\mathbf{P}}(a) = \{0\}$.

(d) *For any* $u \in A_p/J$, $\Phi \in \boldsymbol{P}$, $\Psi \in \boldsymbol{P}^*$, $\Psi \Box (u \cdot \Phi) = u \cdot (\Psi \Box \Phi)$.

PROOF. (a) holds by [DH, p. 316]. Alternatively, note that $\Psi \Box \lambda\delta_a = (\Psi, \lambda\delta_a)\lambda\delta_a$.

(b) If $u \in A_p$, $\Phi \in \boldsymbol{P}$ then $(\Psi_1 \Box \Phi, u) = (\Psi_1, u \cdot \Phi) = u(a)(\Psi_1, \Phi) = ((\Psi_1, \Phi)\lambda\delta_a, u)$. Hence $(\Psi \Box \Psi_1, \Phi) = (\Psi, \Psi_1(\Phi)\lambda\delta_a) = ((\Psi, \lambda\delta_a)\Psi_1, \Phi)$.

(c) Immediate from (b).

(d) If $v \in A_p/J$ then $(\Psi \Box (u \cdot \Phi), v) = (\Psi, (uv) \cdot \Phi) = (\Psi \Box \Phi, uv) = (u \cdot (\Psi \Box \Phi)v)$.

PROPOSITION 3. *For any* $a, b \in F$, $a \neq b$:

(a) $\mathrm{TI}_{\mathbf{P}}(a)$ *is a non-zero* w^**-closed two-sided ideal of* $(\boldsymbol{P}^*, \Box)$ *such that* $I_a \Box \mathrm{TI}_{\mathbf{P}}(a) = \{0\}$.

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(b) $\mathrm{TI}^0_{\mathbf{P}}(a) = I_a \cap \mathrm{TI}_{\mathbf{P}}(a)$ *is a* w^**-closed two-sided ideal such that* $\mathrm{TI}^0_{\mathbf{P}}(a) \Box \mathrm{TI}^0_{\mathbf{P}}(a) = \{0\}$.

If card $\mathrm{TIM}_{P}(a) \geq 2$ *then* $\mathrm{TI}^0_{\mathbf{P}}(a) \neq \{0\}$.

(c) $\mathrm{TI}_{\mathbf{P}}(a) \cap \mathrm{TI}_{\mathbf{P}}(b) = \{0\}$.

PROOF. (a) $\mathrm{TI}_{\mathbf{P}}(a)$ is a left ideal by Lemma 2(b). If $\Psi_1 \in \mathrm{TI}_{\mathbf{P}}(a)$, $\Psi \in \boldsymbol{P}^*$, $u \in A_p/J$, and $\Phi \in \boldsymbol{P}$, then $(\Psi_1 \Box \Psi, u \cdot \Phi) = (\Psi_1, \Psi \Box (u \cdot \Phi)) = (\Psi_1, u \cdot (\Psi \Box \Phi))$ (by Lemma 2(d)) $= u(a)(\Psi_1 \Box \Psi, \Phi)$ (by Lemma 1). Hence, again by Lemma 1, $\Psi_1 \Box \Psi \in \mathrm{TI}_{\mathbf{P}}(a)$. Now $\Psi \Box \Psi_1 = (\Psi, \lambda\delta_a)\Psi_1$ by Lemma 2(b). Thus $I_a \Box \mathrm{TI}_{\mathbf{P}}(a) = 0$.

(b) Since I_a and $\mathrm{TI}_{\mathbf{P}}(a)$ are w^*-closed two-sided ideals, $\mathrm{TI}^0_{\mathbf{P}}(a)$ is such and even $I_a \Box \mathrm{TI}_{\mathbf{P}}(a) = \{0\}$. If $\Psi_1 \neq \Psi_2$ are in $\mathrm{TIM}_{\mathbf{P}}(a)$ then $0 \neq \Psi_1 - \Psi_2 \in I_a \cap \mathrm{TI}_{\mathbf{P}}(a) = \mathrm{TI}^0_{\mathbf{P}}(a)$.

(c) If $\Psi \in \mathrm{TI}_{\mathbf{P}}(a) \cap \mathrm{TI}_{\mathbf{P}}(b)$ then $(\Psi, u \cdot \Phi) = u(a)(\Psi, \Phi) = u(b)(\Psi, \Phi)$ for $u \in A_p/J$, $\Phi \in \boldsymbol{P}$. If we choose $u \in A_p/J$ with $u(a) = 0$, $u(b) \neq 0$, we get $(\Psi, \Phi) = 0$; thus $\Psi = 0$.

THEOREM 4. (a) *If* $\Psi_0 \in \mathrm{TIM}_{\mathbf{P}}(a)$ *then* $\Psi \to \Psi_0 \Box \Psi$ *is a projection operator from* $\boldsymbol{P}^*$ *onto the two-sided ideal* $\mathrm{TI}_{\mathbf{P}}(a)$. *Thus* $\boldsymbol{P}^* = \mathrm{TI}_{\mathbf{P}}(a) \oplus \{\Psi - \Psi_0 \Box \Psi : \Psi \in \boldsymbol{P}^*\}$.

(b) *If card* $\mathrm{TIM}_{\mathbf{P}}(a) \geq 2$ *then* $\mathrm{TI}_{\mathbf{P}}(a)$ *has no (even unbounded) right approximate identity, and is hence a non-amenable* w^**-closed ideal of* $\boldsymbol{P}^*$.

(c) *If card* $\mathrm{TIM}_{\mathbf{P}}(a) \geq 2$ *for some* $a \in F$ *then* $\boldsymbol{P}^*$ *is a non-amenable non-semisimple Banach algebra.*

PROOF. (a) Let $Q(\Psi) = \Psi_0 \Box \Psi$. If $\Psi \in \mathrm{TI}_{\mathbf{P}}(a)$ then $Q(\Psi) = (\Psi_0, \lambda\delta_a)\Psi = \Psi$ by Lemma 2(b). For any $\Psi \in \boldsymbol{P}^*$, $Q^2(\Psi) = \Psi_0 \Box (\Psi_0 \Box \Psi) = \Psi_0 \Box \Psi = Q(\Psi)$ since $\Psi_0 \Box \Psi \in \mathrm{TI}_{\mathbf{P}}(a)$ by Proposition 3(a).

If now $Q\Psi = \Psi$ then $\Psi_0 \Box \Psi = \Psi$ hence $\Psi \in \mathrm{TI}_{\mathbf{P}}(a)$ by Proposition 3(a). Thus $\boldsymbol{P}^* = Q\boldsymbol{P}^* \oplus (I - Q)\boldsymbol{P}^* = \mathrm{TI}_{\mathbf{P}}(a) \oplus \{\Psi - \Psi_0 \Box \Psi : \Psi \in \boldsymbol{P}^*\}$ where $I : \boldsymbol{P}^* \to \boldsymbol{P}^*$ is the identity.

(b) Let $\Psi_\alpha \subset \mathrm{TI}_{\mathbf{P}}(a)$ be a right approximate identity. Let $\Psi_1 \neq \Psi_2$ be in $\mathrm{TIM}_{\mathbf{P}}(a)$. Thus $\Psi_1 - \Psi_2 \in I_a$. Hence, by Proposition 3(a), $(\Psi_1 - \Psi_2) \Box \Psi_\alpha = 0$. But $\Psi_1 \leftarrow \Psi_1 \Box \Psi_\alpha = \Psi_2 \Box \Psi_\alpha \to \Psi_2$ which cannot be.

(c) If $\Psi_1 \neq \Psi_2$ are in $\mathrm{TIM}_{\mathbf{P}}(a)$ then $0 \neq \Psi_1 - \Psi_2 \in \mathrm{TI}^0_{\mathbf{P}}(a)$ and $\mathrm{TI}^0_{\mathbf{P}}(a) \Box \mathrm{TI}^0_{\mathbf{P}}(a) = \{0\}$. Hence $\{0\} \neq \mathrm{TI}^0_{\mathbf{P}}(a) \subset \mathrm{rad}\, \boldsymbol{P}^*$ and $\boldsymbol{P}^*$ is not semisimple.

If now $\boldsymbol{P}^*$ is an amenable Banach algebra then $\mathrm{TI}_{\mathbf{P}}(a)$ is a w^*- (hence norm-) closed two-sided ideal which is complemented in $\boldsymbol{P}^*$; hence $\mathrm{TI}_{\mathbf{P}}(a)^\perp$ is complemented in $\boldsymbol{P}^{**}$. But then by Khelemskii's Theorem (see [CL, p. 97, Thm 3.7]) $\mathrm{TI}_{\mathbf{P}}(a)$ has a bounded approximate identity, which cannot be the case by (b).

REMARK. Note that if $\Psi_0 \in \mathrm{TIM}_{\mathbf{P}}(a)$ then $\{\Psi - \Psi_0 \colon \Psi \in \mathrm{TIM}_{\mathbf{P}}(a)\} \subset \mathrm{TI}^0_{\mathbf{P}}(a) \subset \mathrm{rad}\, \boldsymbol{P}^*$.

We recall now some results of ours in the next theorem. The set $D_1(J) \subset F$ was defined in [Gr3] by: $a \in D_1(J)$ if there exists a sequence $u_n \in A_p$ with compact supports such that (i) $1 = u_n(a) = \|u_n\|$, (ii) $\{F \cap \operatorname{supp} u_n\}$ is a neighborhood base in F at a, and (iii) there is some $d > 0$ such that $\| \sum_1^n \alpha_k u'_k \| \geq d \sum_1^n |\alpha_k|$, for all n and $\alpha_k \in C$, where $u'_k = u_k + J \in A_p/J$. Note that (iii) can be replaced by: (iii)$'$ $\{u'_k\}$ has no weak Cauchy subsequence (by Rosenthal's Theorem).

THEOREM 5. *Let G be a locally compact group, J a closed ideal of $A_p(G)$, $1 < p < \infty$, with $Z(J) = F$, $a, b \in G$ and $\mathbf{P} = (A_p(G)/J)^*$.*

(i) *If $D_1(J) \neq \emptyset$ and $x \in D_1(J)$ then* $\operatorname{card} \mathrm{TIM}_{\mathbf{P}}(x) \geq 2^c$.

(ii) *If for some non-discrete closed subgroup H of G, $\operatorname{int}_{aHb} F \neq \emptyset$, then* $\operatorname{card} \mathrm{TIM}_{\mathbf{P}}(x) \geq 2^c$ *for all $x \in \operatorname{int}_{aHb} F$.*

(iii) *If R is a closed subgroup of G, $S \subset R$ is a symmetric set (such as the Cantor $1/3$ set) and $aSb \subset F$, then* $\operatorname{card} \mathrm{TIM}_{\mathbf{P}}(x) \geq 2^c$ *for all $x \in aSb$, provided $p = 2$.*

For (i) see [Gr3, Theorem 4] and for (ii), (iii) see [Gr4, Theorems 6 ,7].

THEOREM 6. *Let G be a locally compact group, $J \subset A_p$ a closed ideal with $Z(J) = F$, $a, b \in G$, $1 < p < \infty$. Assume that* (i) *or* (ii) [*or* (iii)] *of the above theorem holds. Then $(A_p/J)^{**}$ $[(A/J)^{**}]$ is a non-amenable and non-semisimple Banach algebra.*

PROOF. By Proposition 3, Theorem 4, and Theorem 5.

REMARKS. (a) In fact (the w^*-closed two-sided ideal) $\mathrm{TI}^0_{\mathbf{P}}(x) \subset \operatorname{rad} \mathbf{P}^*$ (since $\mathrm{TI}^0_{\mathbf{P}}(x) \Box \mathrm{TI}^0_{\mathbf{P}}(x) = \{0\}$) and $\operatorname{card} \mathrm{TI}^0_{\mathbf{P}}(x) \geq 2^c$. In cases (ii) or (iii) there are at least c such ideals, by Proposition 3(c).

(b) In case (ii) [(iii)] $A_p(F)^{**}$ $[A(F)^{**}]$ is not amenable.

(c) Theorem 6 says nothing about the amenability of the algebras A/J or $A(F)$. For example, if G is abelian then $A(G)$ is amenable, hence so are all the quotient algebras A/J [or $A(F)$] for any closed ideal $J \subset A(G)$ [set $F \subset G$]. Yet for sets $F \subset G$ as in Theorem 6, $A(F)^{**}$ and $(A/J)^{**}$ are not amenable.

The following is folklore.

PROPOSITION 7. *Let $J_1 \subset J_2$ be closed ideals in A_p with $F_i = Z(J_i)$; thus $F_2 \subset F_1$. If $(A_p/J_2)^{**}[A_p(F_2)^{**}]$ is not amenable, then $(A_p/J_1)^{**}[A_p(F_1)^{**}]$ is not amenable.*

PROOF. Let $q : A_p/J_1 \to A_p/J_2$ be the canonical quotient onto map. Then $q^{**} : (A_p/J_1)^{**} \to (A_p/J_2)^{**}$ is a homomorphism whose image B contains A_p/J_2 (considered as embedded in $(A_p/J_2)^{**}$). But by [Ru2, 4.14] B has to be w^*-closed.

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Hence q^{**} is an onto continuous homomorphism and amenability is preserved by such ([CL]: see Lemma 1.1 in [LL]).

COROLLARY 8. (a) *If G is non-discrete then $A_p(G)^{**}$ is not amenable and not semisimple.*
(b) *For any locally compact group G, $A(G)^{**}$ [or $B(G)^{**}$] is amenable if and only if G is finite.*

PROOF. (a) Take $H = G = F$ in Theorem 6(b). By Lau's result [La, Proposition 3.2(b)], if $A(G)^{**}$ is amenable then G is compact and by (a) it has to be discrete. Assume now that $B(G)^{**}$ is amenable. Since $A(G)$ is a complemented ideal in $B(G)$, $A(G)^{**}$ is a complemented ideal in $B(G)^{**}$. Thus by [CL], $A(G)^{**}$ is amenable; hence G is finite.

PROPOSITION 9. *Let $A(G)$ be amenable. Then $A_p(G)/J$ is amenable for all $1 < p < \infty$ and for all closed ideals $J \subset A_p(G)$.*

PROOF. G is necessarily amenable; hence, by Herz's Theorem C in [Hz2], $A_2(G) \subset A_p(G)$ for all $1 < p < \infty$, with contraction of norms. Thus the identity embedding $h : A_2 \to A_p$ is a homomorphism such that $\|h\| \leq 1$. But hA_2 contains the functions $f * \tilde{g}$ with $f, g \in C_c(G)$, the linear span of which is norm dense in A_p (see [Hz1]). Thus hA_2 is norm dense in A_p. By a theorem of Johnson ([Jo1, (5.3)]) $A_p = A_p(G)$, hence A_p/J is an amenable Banach algebra.

REMARKS. (a) Proposition 9 improves Theorem 3.10 of [Fo1].

(b) Corollary 8(b) is the dual result to Theorem 1.3 and Corollary 1.4 of [GLW].

(c) It has been proved by Gourdeau [Go] that for any Banach algebra B the amenability of B^{**} implies that of B (see also [GLW]). Thus $A(G)^{**}$ amenable implies that $A(G)$ is so. It has been proved by Johnson in [Jo2] that there exist compact groups for which $A(G)$ is not amenable. If, however, G is infinite and contains an abelian subgroup of finite index then $A(G)$, hence $A_p(G)$, is amenable (see [LLW, Corollary 4.2] and [Fo2]), yet $A(G)^{**}$ is not amenable by Corollary 8(b).

(d) It has been proved by Brown and Moran [BM] that if G is a non-compact abelian locally compact group then $B(G)$, hence $B(G)^{**}$, is not amenable. If G is compact abelian infinite then $A(G) = B(G)$ is amenable yet $A(G)^{**} = B(G)^{**}$ is not amenable by, say, our Corollary 8.

(e) If G is abelian, every perfect compact set F contains a perfect Helson set $E \subset F$ by Varopoulos [V, Ch. 4.3]. Taking $F \subset R = G$ to be the Cantor 1/3 (or any symmetric) set and $E \subset F$ a Helson set we get that $A(E)^{**}$ is amenable while $A(F)^{**}$ is not.

(f) There exist continuous [smooth] curves E in R^2 [in $R^n, n \geq 3$] which are Helson sets as shown by Kahane; see [Mc]. Thus if $G = R^n$ then $A(E)^{**}$, and hence $A(E_1)^{**}$ for all closed $E_1 \subset E$, is an amenable Banach algebra.

(g) Let G be infinite discrete and abelian. Then any infinite set $F \subset G$ contains an infinite Sidon set E; thus $A(E) = c_0(E)$ ([Ru1, (5.7.3) (5.7.6)]). Hence $A(E)^{**} = C(X)$ (for some compact X) is amenable. Yet $A(F)^{**}$ need not be amenable (take $F = G$ and use our Corollary 8(b)).

QUESTIONS. (1) Characterize the closed sets $E \subset R^n$ for which $A(E)^{**}$ is an amenable Banach algebra.
(2) Let $G = \mathbb{Z}$, the integers, or $G = \mathbb{Z}^n$. Characterize all infinite sets $F \subset G$ for which $A(F)^{**}$ is amenable.
(3) The only examples of sets $E \subset G$ for which $A(E)^{**}$ is amenable, given here, are Helson sets or Sidon sets. Do other such exist?

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