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THE C^1-INVARIANCE OF THE GODBILLON-VEY MAP IN ANALYTICAL K-THEORY

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1. Introduction. An action α of a discrete group Γ on the circle S^1 as orientation preserving C^∞-diffeomorphisms gives rise to a foliation on the homotopy quotient $S^1\Gamma$, and its Godbillon-Vey invariant is, by definition, a cohomology class of $S^1\Gamma$ ([**1**]). This cohomology class naturally defines an additive map from the geometric K-group $K^0(S^1, \Gamma)$ into $\mathbf{C}$, through the Chern character from $K^0(S^1, \Gamma)$ to $H_*(S^1\Gamma; \mathbf{Q})$.

Using cyclic cohomology, Connes constructed in [**2**] an additive map, $GV(\alpha)$, which we shall call the Godbillon-Vey map, from the K_0-group of the reduced crossed product C^*-algebra $C(S^1) \rtimes_\alpha \Gamma$ into $\mathbf{C}$. He showed that $GV(\alpha)$ agrees with the geometric Godbillon-Vey invariant through the index map μ from $K^0(S^1, \Gamma)$ to $K_0(C(S^1) \rtimes_\alpha \Gamma)$. In order to define $K^0(S^1, \Gamma)$ and μ, Connes considered C^∞-actions in [**2**]. However, a close examination of his construction shows that the map $GV(\alpha)$ itself can be defined for an action α of Γ on S^1 as orientation preserving C^2-diffeomorphisms.

Raby showed in [**5**] that, if two codimension one C^∞-foliations are C^1-diffeomorphic, then their geometric Godbillon-Vey invariants coincide. By this fact, together with Connes's description mentioned above, when an action is of class C^∞, the C^1-invariance of the Godbillon-Vey map would follow from the Baum-Connes conjecture that the index map μ is always an isomorphism. Unfortunately, so far we do not know whether this conjecture is true for all actions of discrete groups on S^1. Therefore it is desirable to show the C^1-invariance of the Godbillon-Vey map directly in the analytical framework.

In the present work, we will show that if two C^2-actions α and β of a discrete group Γ are conjugate to each other by a C^1-diffeomorphism φ, then the associated maps $GV(\alpha)$ and $GV(\beta)$ coincide, via the canonical isomorphism between $K_0(C(S^1) \rtimes_\alpha \Gamma)$ and $K_0(C(S^1) \rtimes_\beta \Gamma)$ derived from φ (Theorem 4).

It should be noted that even in the case of C^∞-conjugation, and of C^∞-actions, the invariance of the Godbillon-Vey map is not obvious. For one thing, the construction of the Godbillon-Vey map uses a specified volume form on S^1, which is not necessarily invariant under even a C^∞-diffeomorphism giving a conjugation between two actions. It is as

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a consequence of the theorem that the Godbillon-Vey map is independent of the choice of volume form on S^1.

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2. The Godbillon-Vey map. In this section we give an explicit construction of the Godbillon-Vey map.

Let α be an action of a group Γ on S^1 as orientation preserving diffeomorphisms of class C^2. For $g \in \Gamma$, denote by α_g the corresponding C^2-diffeomorphism. Let dx be the canonical volume form on S^1. Define the Jacobian $J(g)$ of α_g by

$$\alpha_g^*(dx) = J(g)dx,$$

and put $l(g) = \log J(g)$. The function $g \mapsto l(g)$ is a group 1-cocycle on Γ with values in the space $C^1(S^1)$ of C^1-functions, where we consider the action of Γ on the right defined by $f \cdot g = \alpha_g^*(f)$. More precisely, we have the relation

$$l(gh) = \alpha_h^* l(g) + l(h).$$

Put

$$\omega_\alpha(g, h) = dl(gh)l(h) - l(gh)dl(h).$$

LEMMA 1. *The function ω_α is a group 2-cocycle on Γ with values in the space Ω^1 of continuous 1-forms on S^1. Moreover, ω_α is normalized in the sense that if one of g, h, and gh is equal to the neutral element $e \in \Gamma$, then $\omega_\alpha(g, h) = 0$.*

The proof is immediate by a routine computation. Following **[1]**, we shall call ω_α the *Thurston cocycle* of α.

For $f^0, f^1, f^2 \in C_c(S^1 \rtimes {}_\alpha\Gamma)$, put

$$\tau_\alpha(f^0, f^1, f^2) = \sum_{g_0 g_1 g_2 = e} \int_{S^1} f^0_{g_0} \alpha^*_{g_0^{-1}}(f'_{g_1}) \alpha^*_{(g_0 g_1)^{-1}}(f^2_{g_2}) \omega_\alpha(g_1, g_2)$$

to get a cyclic 2-cocycle on $C_c(S^1 \rtimes {}_\alpha\Gamma)$. We study the cocycle τ_α. For $f^0, f^1 \in C_c(S^1 \rtimes {}_\alpha\Gamma)$, put

$$\tau_2(f^0, f^1) = \sum_{g_0 g_1 = e} \int_{S^1} f^0_{g_0} \alpha^*_{g_0^{-1}}(f^1_{g_1}) dl(g_1).$$

Since l is a normalized 1-cocycle, τ_2 is a cyclic 1-cocycle. For any $f^1 \in C_c(S^1 \rtimes {}_\alpha\Gamma)$, there exists a constant C such that

$$|\tau_2(f^0, f^1)| \leqq C\|f^0\|_A$$

for all $f^0 \in C_c(S^1 \rtimes {}_\alpha\Gamma)$, where $\| \ \|_A$ is the C^*-norm on the reduced crossed product $A = C(S^1) \rtimes {}_\alpha\Gamma$. This enables us to define a densely defined linear map $\delta:A \to A^*$ by

$$\delta(f^1)(f^0) = \tau_2(f^0, f^1).$$

Since τ_2 is a cyclic cocycle, δ is a closable derivation ([**2**, Lemma 4]). Modifying the proof of [**2**, Lemma 2], we get the following.

LEMMA 2. *Let B be a Banach space endowed with an A-bimodule structure. Let $\delta:A \to B$ be a densely defined closed derivation. Then the domain of δ is stable under the holomorphic functional calculus.*

Now let (σ_t) be the modular automorphism group of the state on A associated to the 1-form dx. Then (σ_t) preserves A. Let D_α be the generator of (σ_t). We have the relations

(a) $D_\alpha(f) = 0$ for $f \in C(S^1)$,

(b) $D_\alpha(U_g) = U_g l(g)$ for $g \in \Gamma$.

By a straightforward computation, we get the next lemma.

LEMMA 3. *For $f^0, f^1, f^2 \in C_c(S^1 \rtimes {}_\alpha\Gamma)$, we have*

$$\tau_\alpha(f^0, f^1, f^2) = \tau_2(D_\alpha(f^2)f^0, f^1) - \tau_2(f^0D_\alpha(f^1), f^2).$$

Let B be the direct sum $A \oplus A^*$ of Banach spaces with A-bimodule structure given by

$$a(a_1 \oplus \varphi)b = (aa_1b) \oplus (a\varphi b).$$

Define an unbounded operator $\delta':A \to B$ by

$$\delta'(a) = (D_\alpha(a), \delta(a))$$

for $a \in C_c(S^1 \rtimes {}_\alpha\Gamma)$, where δ is the derivation associated to τ_2 constructed above. Then δ' is a closable derivation. Let $\mathscr{B}$ be the domain of the closure $\bar{\delta}'$ of δ', equipped with the graph norm associated to δ'. Then $\mathscr{B}$ is a Banach algebra embedded in A as a dense subalgebra, stable under the holomorphic functional calculus by Lemma 2. Lemma 3 says that the cyclic 2-cocycle τ_α is continuous with respect to the graph norm on $C_c(S^1 \rtimes {}_\alpha\Gamma)$ induced from that on $\mathscr{B}$. Therefore τ_α extends to a cyclic 2-cocycle on $\mathscr{B}$. Since $K_0(\mathscr{B}) \cong K_0(A)$, we obtain a map

$$GV(\alpha):K_0(A) \to \mathbf{C}$$

by [**2**, Theorem 7]. We call $GV(\alpha)$ the *Godbillon-Vey map* associated to the action α.

For the use in the later sections, let us study the algebra $\mathscr{B}$ more thoroughly.

Let δ'' be the restriction of δ' to $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$. Then δ'' is also closable. Let $\mathscr{B}'$ be the domain of the closure of δ''. The algebra $\mathscr{B}'$ is the completion of $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ with respect to the graph norm. Obviously, $\mathscr{B}' \subset \mathscr{B}$, and this inclusion is continuous. Since $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ is dense in $C_c(S^1 \rtimes {}_\alpha\Gamma)$ with respect to the inductive limit topology on $C_c(S^1 \rtimes {}_\alpha\Gamma)$, actually we have $\mathscr{B}' = \mathscr{B}$. Thus, $\mathscr{B}$ is the completion of $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ with respect to the graph norm given by δ'.

Besides our $\mathscr{B}$, there might exist a dense Banach subalgebra $\mathscr{B}_2$ of A which is stable under the holomorphic functional calculus and contains $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ as a dense subalgebra, and on which

$$\tau_\alpha|_{C_c^1(S^1 \rtimes_\alpha \Gamma)}$$

extends to a cyclic cocycle τ'. Then τ' also induces an additive map

$$\tau'_*:K_0(A) \rightarrow \mathbf{C}.$$

However, we do not know whether τ'_* coincides with $GV(\alpha)$, because there are no relations between $\mathscr{B}$ and $\mathscr{B}_2$, in general. (Notice that we are dealing with unbounded operators.) For this reason, when we talk about the Godbillon-Vey map, we will keep in mind the algebra $\mathscr{B}$ constructed above as the domain.

3. C^1-conjugation. Let α, β be actions of a group Γ on S^1 as orientation preserving diffeomorphisms of class C^2. Assume that α, β are conjugate to each other by a C^1-diffeomorphism φ of S^1; that is, for any $g \in \Gamma$,

$$\varphi^{-1}\beta_g\varphi = \alpha_g.$$

The diffeomorphism φ^{-1} induces an isomorphism Φ of $C(S^1) \rtimes {}_\alpha\Gamma$ onto $C(S^1) \rtimes {}_\beta\Gamma$ in an obvious way. Consequently we have an isomorphism

$$\Phi_*:K_0(C(S^1) \rtimes {}_\alpha\Gamma) \rightarrow K_0(C(S^1) \rtimes {}_\beta\Gamma).$$

Our main result is the following.

THEOREM 4. *In the above situation, we have the relation*

$$GV(\beta) \circ \Phi_* = GV(\alpha).$$

Proof. We will prove Theorem 4 in a sequence of lemmata. To begin with, we study the 2-cocycle $\tau_\beta \circ \Phi$ which is associated to the group 2-cocycle $\varphi^*\omega_\beta$.

Since φ is of class C^1, the pullback $\varphi^*(dx)$ of dx by φ is defined, and $\varphi^*(dx) = kdx$ for some nowhere-vanishing continuous function k. For simplicity, assume that φ is orientation preserving. Then k is positive.

By easy computations,

$$\varphi^*(l'(g)) = \log(\alpha_g^*k/k) + l(g)$$

for all $g \in \Gamma$, where l, l' are the logarithms of the Jacobians of α, β, respectively. The above formula says, in particular, that (α_g^*k/k) is a C^1-function. Let

$$K(g) = \log(\alpha_g^* k/k).$$

We find that

$$\varphi^*(\omega_\beta(g, h)) - \omega_\alpha(g, h)$$
$$= dK(gh)K(h) - K(gh)dK(h) + dl(gh)K(h) - K(gh)dl(h)$$
$$+ dK(gh)l(h) - l(gh)dK(h).$$

For $g, h \in \Gamma$, put

$$p_1(g, h) = dK(gh)K(h) - K(gh)dK(h),$$
$$p_2(g, h) = dl(gh)K(h) - K(gh)dl(h), \text{ and}$$
$$p_3(g, h) = dK(gh)l(h) - l(gh)dK(h).$$

Then p_1, p_2, and p_3 are Ω^1-valued normalized 2-cocycles on Γ.

For $g \in \Gamma$, let

$$\sigma_1(g) = \log(k\alpha_g^* k)\, dK(g),$$
$$\sigma_2(g) = \log(k\alpha_g^* k)\, dl(g).$$

The functions σ_1, σ_2 are Ω^1-valued normalized 1-cochains on Γ. Let ∂^* be the coboundary operator of the cochain complex $C^*(\Gamma; \Omega^1)$ of the group Γ with coefficients in the right Γ-module Ω^1. By straightforward computations, we get the next lemma.

LEMMA 5. *We have the relations* $\partial^*\sigma_1 = p_1$ *and* $\partial^*\sigma_2 = p_2$.

For $g \in \Gamma$, let $\sigma_3(g)$ be the distribution on S^1 defined by

$$\langle \sigma_3(g), f \rangle = \int_{S^1} \log(k\alpha_g^* k)\, d(l(g)f)$$

for $f \in C^1(S^1)$. Obviously $\sigma_3(g) = 0$ if $g = e$.

Remark. To define $\sigma_3(g)$ we used the fact that $l(g)$ is of class C^1.

Let $\mathscr{E}'$ denote the dual of $C^1(S^1)$ with respect to the C^1-topology. A right Γ-action on $\mathscr{E}'$ is defined by

$$\langle T \cdot g, f \rangle = \langle T, \alpha_{g^{-1}}^*(f) \rangle$$

for $T \in \mathscr{E}'$, $f \in C^1(S^1)$, and $g \in \Gamma$. Let $C^*(\Gamma; \mathscr{E}')$ be the cochain complex of Γ with coefficients in $\mathscr{E}'$, and let ∂^* be its coboundary operator. The canonical inclusion $\Omega^1 \subset \mathscr{E}'$ induces an inclusion of cochain complexes,

$$C^*(\Gamma; \Omega^1) \subset C^*(\Gamma, \mathscr{E}').$$

LEMMA 6. *In* $C^2(\Gamma; \mathscr{E}')$ *we have* $\partial^*\sigma_3 = p_3$. *In particular*,

$$\partial^*\sigma_3 \in C^2(\Gamma; \Omega^1).$$

Proof. By definition,

$$(\partial^*\sigma_3)(g, h) = \sigma_3(h) - \sigma_3(gh) + \sigma_3(g)\cdot h.$$

Let $f \in C^1(S^1)$ be fixed. We see that

$$\langle \sigma_3(h), f \rangle = \int_{S^1} \log(k\alpha_h^* k)\, d(fl(h)),$$

$$\langle \sigma_3(gh), f \rangle = \int_{S^1} \log(k\alpha_{gh}^* k)\, d(fl(gh)),$$

and, furthermore,

$$\langle \sigma_3(g)\cdot h, f \rangle = \langle \sigma_3(g), \alpha_{h^{-1}}^* f \rangle$$
$$= \int_{S^1} \log(\alpha_h^* k(\alpha_{gh}^* k))\, d(f(l(gh) - l(h))).$$

From these equations the conclusion follows.

By Lemmas 5 and 6, we know that in $C^2(\Gamma; \mathcal{E}')$ we have the relation

$$\varphi^*\omega_\beta - \omega_\alpha = \partial^*(\sigma_1 + \sigma_2 + \sigma_3).$$

Using the cochains σ_j, we construct cyclic cochains.

LEMMA 7. *The $\mathcal{E}'$-valued* 1*-cochain* $\sigma_j (j = 1, 2, 3)$ *enjoys the following relation*:

$$\sigma_j(g) \cdot g^{-1} = -\sigma_j(g^{-1})$$

for all $g \in \Gamma$. *In particular*, $\sigma_j(e) = 0$.

Proof. We give a proof for σ_1. Let $f \in C^1(S^1)$. By definition,

$$\langle \sigma_1(g) \cdot g^{-1}, f \rangle = \langle \sigma_1(g), \alpha_g^*(f) \rangle$$
$$= \int_{S^1} \log(k\alpha_g^* k)\, \alpha_g^*(f)\, dK(g)$$
$$= \int_{S^1} \alpha_g^*\{\log((\alpha_{g^{-1}}^* k)k)\, f\alpha_{g^{-1}}^*(dK(g))\}$$
$$= -\int_{S^1} \log(k\alpha_{g^{-1}}^* k)\, fdK(g^{-1})$$
$$= -\langle \sigma_1(g^{-1}), f \rangle.$$

Similarly we get relations for σ_2 and σ_3.

Remark. If p is a normalized 1-cocycle with values in $\mathcal{E}'$, then the relation stated in Lemma 7 is automatic.

LEMMA 8. *Let* $p \in C^1(\Gamma; \mathcal{E}')$. *Suppose that*

$$p(g) \cdot g^{-1} = -p(g^{-1}) \quad \textit{for all } g \in \Gamma.$$

Then the following formula defines a cyclic 1*-cochain* τ:

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$$\tau(f^0, f^1) = \sum_{gh=e} \langle p(h), f^0_g \alpha^*_{g^{-1}}(f'_h) \rangle$$

for $f^0, f^1 \in C^1_c(S^1 \rtimes_\alpha \Gamma)$.

Proof. By definition,

$$\begin{aligned}\tau(f^1, f^0) &= \Sigma\langle p(h), f'_g \alpha^*_{g^{-1}}(f^0_h) \rangle \\ &= \Sigma\langle p(h), \alpha^*_{g^{-1}}(\alpha^*_g(f'_g) f^0_h) \rangle \\ &= \Sigma\langle p(h) \cdot h^{-1}, f^0_h \alpha^*_{h^{-1}}(f'_g) \rangle \\ &= -\tau(f^0, f^1).\end{aligned}$$

Let p be as in Lemma 8. Then $C = \partial^* p$ is a normalized 2-cocycle. In fact, as

$$c(g, h) = p(h) - p(gh) + p(g) \cdot h,$$

if $gh = e$, then

$$c(g, h) = p(g^{-1}) + p(g) \cdot g^{-1} = 0,$$

and if $g = e$ or $h = e$, $c(g, h) = 0$ is obvious.

For $f^0, f^1, f^2 \in C^1_c(S^1 \rtimes_\alpha \Gamma)$, let

$$\psi'(f^0, f^1, f^2) = \sum_{g_0 g_1 g_2 = e} \langle c(g_1, g_2), f^0_{g_0} \alpha^*_{g_0^{-1}}(f'_{g_1}) \alpha^*_{(g_0 g_1)^{-1}}(f^2_{g_2}) \rangle.$$

Then we have the following.

LEMMA 9. *The functional* ψ' *is a cyclic 2-cocycle, and* $\psi' = b\tau$, *where* b *is the Hochschild coboundary.*

Proof. The first statement is a modification of Lemma 1 of [**2**, p. 86]. The second statement follows from a routine computation.

Let us return to the proof of Theorem 4. By Lemmas 5, 6, 7, and 9, we know that on $C^1_c(S^1 \rtimes_\alpha \Gamma)$ there exists a cyclic cochain ψ such that

$$\varphi^* \tau_\beta - \tau_\alpha = b\psi.$$

To complete the proof, it remains to prove that $\varphi^*\tau_\beta$, τ_α, $b\psi$, and ψ extend to cyclic cochains on a dense subalgebra stable under the holomorphic functional calculus, on which we have the relation

$$\varphi^* \tau_\beta - \tau_\alpha = b\psi.$$

For $f^0, f^1 \in C_c(S^1 \rtimes_\alpha \Gamma)$, put

$$\tau_1(f^0, f^1) = \sum_{g_0 g_1 = e} \int_{S^1} f^0_{g_0} \alpha^*_{g_0^{-1}}(f^1_{g_1}) dK(g_1).$$

Similarly, define τ_2 by

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$$\tau_2(f^0, f^1) = \sum_{g_0 g_1 = e} \int_{S^1} f^0_{g_0} \alpha^*_{g_0^{-1}}(f^1_{g_1}) dl(g_1).$$

Then τ_1, τ_2 are 1-traces in Connes's terminology [**2**, Definition 3]. In other words, τ_1 and τ_2 give rise to densely defined closed derivations δ_1 and δ_2 from A into A^*, respectively.

Let c_1, c_2, and c_3 be the cyclic 1-cochains associated to σ_1, σ_2, and σ_3 respectively, by Lemma 8. By brute force we get the next lemma.

LEMMA 10. *On* $C_c(S^1 \rtimes_\alpha \Gamma)$ *we have the relations*

(1) $C_1(f^0, f^1) = \tau_1((\log k)f^0, f^1) + \tau_1(f^0, (\log k)f^1)$,

(2) $C_2(f^0, f^1) = \tau_2((\log k)f^0, f^1) + \tau_2(f^0, (\log k)f^1)$.

Let D be the inner derivation of A defined by $\log k \in A$.

LEMMA 11. *On* $C_c(S^1 \rtimes_\alpha \Gamma)$, *we have*

(1) $bC_1 = i_D\tau_1$, *and*

(2) $bC_2 = i_D\tau_2$.

Proof. By the definition of the contraction i_D ([**2**, p. 91]),

$$\begin{aligned}&(i_D\tau_j)(f^0, f^1, f^2)\\ &= \tau_j(D(f^2)f^0, f^1) - \tau_j(f^0D(f^1), f^2) \quad (j = 1, 2).\end{aligned}$$

The relations follow by somewhat tedious computations.

Remark. In [**2**], to define the contraction $i_D\pi$, Connes assumed that π is invariant under the automorphism group generated by D. What is precisely needed there is that

$$\pi(DX^0, X^1) + \pi(X^0, DX^1) = 0.$$

In our situation, obviously our cocycles τ_1, τ_2 have this property with respect to the derivation D.

LEMMA 12. *We have the relation*

$$bC_3 = i_{D_\alpha}\tau_1 \quad \text{on } C_c(S^1 \rtimes_\alpha \Gamma).$$

Proof. Let us first check that

$$\tau_1(D_\alpha(f^0), f^1) + \tau_1(f^0, D_\alpha(f^1)) = 0.$$

We have

$$\begin{aligned}\tau_1(D_\alpha(f^0), f^1) &= \tau_1\Big(\sum_g f^0_g \alpha^*_{g^{-1}}(l(g))U_g, f^1\Big)\\ &= \sum_{g_0 g_1 = e} \int_{S^1} f^0_{g_0} \alpha^*_{g_0^{-1}}(l(g_0))\alpha^*_{g_0^{-1}}(f^1_{g_1}) dK(g_1).\end{aligned}$$

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Similarly,

$$\tau_1(f^0, D_\alpha(f^1)) = \sum_{g_0g_1=e} \int_{S^1} f^0_{g_0}\alpha^*_{g_0^{-1}}(f^1_{g_1}\alpha^*_{g_1^{-1}}(l(g_1)))dK(g_1).$$

Since $\alpha^*_{g_0^{-1}}(l(g_0)) = -l(g_1)$, we see that

$$\tau_1(D_\alpha(f^0), f^1) + \tau_1(f^0, D_\alpha(f^1)) = 0.$$

Now the relation $i_{D_\alpha}\tau_1 = bC_3$ follows by straightforward computation.

Let us study the cochain C_3 more carefully. Let τ be the transverse fundamental class for a transformation group (S^1, Γ, α). By definition, for $f^0, f^1 \in C^1_c(S^1 \rtimes_\alpha \Gamma)$ we have

$$\tau(f^0, f^1) = \sum_{g_0g_1=e} \int_{S^1} f^0_{g_0}\alpha^*_{g_0^{-1}}(df^1_{g_1}).$$

LEMMA 13. *For $f^0, f^1 \in C^1_c(S^1 \rtimes_\alpha \Gamma)$, we have the relation*

$$\begin{aligned} C_3(f^0, f^1) = C_2(f^0, f^1) &+ \tau(D_\alpha(f^1)\log k, f^0) \\ &+ \tau(D_\alpha(\log k \cdot f^1), f^0) - \tau(D_\alpha(\log k \cdot f^0), f^1) \\ &- \tau(D_\alpha(f^0)\log k, f^1). \end{aligned}$$

Proof. By definition,

$$\begin{aligned} &C_3(f^0, f^1) \\ &= \sum_{g_0g_1=e} \int_{S^1} d(f^0_{g_0}\alpha^*_{g_0^{-1}}(f^1_{g_1})l(g_1))(\log k + \log \alpha^*_{g_1}(k)) \\ &= \sum \int \{d(f^0_{g_0})\alpha^*_{g_0^{-1}}(f^1_{g_1})l(g_1) \\ &+ f^0_{g_0}d(\alpha^*_{g_0^{-1}}(f^1_{g_1}))l(g_1)\}(\log k(\alpha^*_{g_1}k)) \\ &+ \sum \int f^0_{g_0}\alpha^*_{g_0^{-1}}(f^1_{g_1})\, dl(g_1) \log(k(\alpha^*_{g_1}k)) \\ &= \sum \int d(f^0_{g_0})\alpha^*_{g_0^{-1}}(f^1_{g_1}\alpha^*_{g_0}(l(g_1)) \log \alpha^*_{g_0}k) \\ &+ \sum \int d(f^0_{g_0})\alpha^*_{g_0^{-1}}(f^1_{g_1}\alpha^*_{g_0}(l(g_1)) \log k) \\ &+ \sum \int f^0_{g_0}(d\alpha^*_{g_0^{-1}}(f^1_{g_1}))l(g_1) \log k \\ &+ \sum \int f^0_{g_0}l(g_1) \log(\alpha^*_{g_1}k)\, d(\alpha^*_{g_0^{-1}}(f^1_{g_1})) \\ &+ C_2(f^0, f^1) \end{aligned}$$

$$= \tau(D_\alpha(f^1)\log k, f^0) + \tau((\log k)D_\alpha(f^1), f^0)$$
$$- \tau(\log k(D_\alpha(f^0)), f^1) - \tau(D_\alpha(f^0)\log k, f^1)$$
$$+ C_2(f^0, f^1),$$

as was to be shown.

On $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$, let us consider unbounded derivations δ_1, δ_2, $\delta : A \to A^*$ associated to τ_1, τ_2, τ, respectively. These derivations are closable, because τ_1, τ_2, τ are cyclic. Notice that D_α restricted to $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ is also closable. Let $\bar{\delta}_1$, $\bar{\delta}_2$, $\bar{\delta}$, and $\bar{D}_\alpha$ be the closure of

$$\delta_1, \delta_2, \delta, \quad \text{and} \quad D_{\alpha|_{C_c^1(S^1 \rtimes_\alpha \Gamma)}},$$

respectively. Consider the direct sum of Banach spaces $B = A \oplus A^* \oplus A^* \oplus A^*$, equipped with the A-bimodule structure

$$a(a_0 \oplus \varphi_1 \oplus \varphi_2 \oplus \varphi_3)b = (aa_0b) \oplus (a\varphi_1 b) \oplus \ldots \oplus (a\varphi_3 b).$$

Define a densely defined map $\delta_0 : A \to B$ by

$$\delta'(a) = \bar{D}_\alpha(a) \oplus \bar{\delta}(a) \oplus \bar{\delta}_1(a) \oplus \bar{\delta}_2(a).$$

Then δ' is a closable derivation, since $\bar{D}_\alpha$, $\bar{\delta}$, $\bar{\delta}_1$, and $\bar{\delta}_2$ are closed. Let δ_0 be the closure of

$$\delta'|_{C_c^1(S^1 \rtimes_\alpha \Gamma)}.$$

Then, by Lemma 2, the domain of δ_0 is a Banach algebra $\mathscr{B}$ stable under the holomorphic functional calculus. The algebra $\mathscr{B}$ is the completion of $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ with respect to the graph norm $|||\cdot|||$ associated to δ_0.

Let us check that C_3 on $C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ is continuous. By Lemmas 10 and 13, for $x^0, x^1 \in C_c^1(S^1 \rtimes {}_\alpha\Gamma)$ we have

$$\begin{aligned} |C_3(x^0, x^1)| &\leqq \|\log k\|_A \|X^0\|_A \|\delta_1(x^1)\|_{A^*} \\ &\quad + 2\|\log k\|_A \|D_\alpha(x^1)\|_A \|\delta(x^0)\|_A \\ &\quad + 2\|\log k\|_A \|D_\alpha(x^0)\|_A \|\delta(x^1)\|_A \\ &\leqq 5\|\log k\|_A |||x^0|||\ |||x^1|||. \end{aligned}$$

This says that C_3 is continuous as a function of two variables with respect to $|||\cdot|||$.

Similarly, the cyclic cochains τ_α, C_1, C_2, bC_1, bC_2, and bC_3 are continuous. Consequently, $\Phi^*\tau_\beta$ is also continuous. Therefore all these cyclic cochains extend to cyclic cochains on $\mathscr{B}$, and satisfy

$$\Phi^*\tau_\beta - \tau_\alpha = b(C_1 + C_2 + C_3)$$

on $\mathscr{B}$.

It remains to analyze the cocycle $\Phi^*\tau_\beta$. Since Φ is an isomorphism from

$C(S^1) \rtimes_\alpha \Gamma$ onto $C(S^1) \rtimes_\beta \Gamma$, the image $\Phi(\mathscr{B})$ of $\mathscr{B}$ is stable under the holomorphic functional calculus. We have to show that τ_β on $C^1_c(S^1 \rtimes_\beta \Gamma)$ extends to a cocycle on $\Phi(\mathscr{B})$. Let $\mathscr{B}_\beta$ be the Banach algebra constructed in Section 1 for the transformation group (S^1, Γ, β). We compare $\mathscr{B}_\beta$ with $\Phi(\mathscr{B})$. Since φ is a C^1-diffeomorphism, obviously

$$\Phi(C^1_c(S^1 \rtimes_\alpha \Gamma)) = C^1_c(S^1 \rtimes_\beta \Gamma).$$

Let δ_α, δ_β be the closed derivation associated to the transverse fundamental classes on $C^1_c(S^1 \rtimes_\alpha \Gamma)$, $C^1_c(S^1 \rtimes_\beta \Gamma)$, respectively.

LEMMA 14. *We have* $\Phi(\mathrm{Dom}(\delta_\alpha)) = \mathrm{Dom}(\delta_\beta)$.

Proof. By a straightforward computation, we see that

$$\delta_\beta(\Phi(a))(x) = \delta_\alpha(a)(\Phi^{-1}(x))$$

for all $a \in C^1_c(S^1 \rtimes_\alpha \Gamma)$ and all $x \in C_c(S^1 \rtimes_\beta \Gamma)$. From this, we find

$$\|\delta_\beta(\Phi(a))\| = \|\delta_\alpha(a)\|,$$

since Φ is an isometry. The conclusion follows immediately.

Let D_β be the generator of the modular automorphism group for the state on $C(S^1) \rtimes_\beta \Gamma$ associated to the 1-form dx.

LEMMA 15. *We have the relation*

$$\Phi^{-1} \circ D_\beta \circ \Phi = D_\alpha + D,$$

where D is the inner derivation determined by

$$\log k \in C(S^1) \rtimes_\alpha \Gamma.$$

Proof. Since $\varphi^*(dx) = kdx$, the conclusion follows (cf. [**4**]).

Lemma 15, together with 14, says that $\Phi(\mathscr{B}) \subset \mathscr{B}_\beta$, and that this inclusion is continuous. Therefore the extension τ'_β of τ_β to $\Phi(\mathscr{B})$ coincides with the restriction of the extension $\bar{\tau}_\beta$ of τ_β to $\mathscr{B}_\beta$. Hence the additive map defined by the pairing with τ'_β from $K_0(\Phi(\mathscr{B}))$ to $\mathbf{C}$ is just equal to $GV(\beta)$.

Let $\mathscr{B}_\alpha$ be the domain of the Godbillon-Vey map $GV(\alpha)$. Then by the definition of $\mathscr{B}$, we see that $\mathscr{B} \subset \mathscr{B}_\alpha$, and that this inclusion is continuous. Hence the extension of τ_α to $\mathscr{B}$ gives us the same map as $GV(\alpha)$.

Now we know that on the Banach algebra $\mathscr{B}$ the two cyclic cocycles $\Phi^*\tau_\beta$ and τ_α are cohomologous. Consequently, they define the same map from $K_0(C(S^1) \rtimes_\alpha \Gamma)$ to $\mathbf{C}$.

That is the end of the proof of Theorem 4.

The proof of Theorem 4 given above shows also that the definition of the Godbillon-Vey map is independent of the choice of a volume form of class C^2 on S^1, used to get the Jacobians of the diffeomorphisms. This corresponds to the fact that the Godbillon-Vey class in geometry is independent of the choices of certain differential forms involved in the definition.

Finally, let us emphasize that the stability of the Godbillon-Vey map under C^2-conjugation is not quite as simple as one might expect, because we are dealing with unbounded operators.

4. Example. We consider the following example.

The group $SL_2(\mathbf{Z})$ faithfully acts on the space of oriented lines through 0 in $\mathbf{R}^2$, which we identify with the circle S^1, Call this action α.

PROPOSITION 16. *The Godbillon-Vey map $GV(\alpha)$ associated to α is the zero map.*

Before giving a proof, let us make an observation. Since the Thurston cocycle ω_α is zero in $H^2(SL_2(\mathbf{Z}), \Omega^1)$, one might think that Proposition 16 is trivial. If a 1-cochain p with $\partial^* p = \omega_\alpha$ satisfies the assumption of Lemma 8, we get a cyclic 1-cochain ψ such that $b\psi = \tau_\alpha$ on $C_c(S^1 \rtimes {}_\alpha\Gamma)$. Then it is really a trouble that we do not know whether the relation $b\psi = \tau_\alpha$ holds on a suitable subalgebra which has the same K-theory as $C(S^1) \rtimes {}_\alpha SL_2(\mathbf{Z})$. As Connes pointed out in [**2**, p. 26], a cyclic cocycle associated to a group cocycle does not necessarily give rise to an n-trace. Thus Proposition 16 is never trivial. Something that goes below the surface is required.

Proof of Proposition 16. By the six-term exact sequence obtained in [**3**] we see that the canonical map

$$K_0(C(S^1) \rtimes {}_\alpha\mathbf{Z}_4) \oplus K_0(C(S^1) \rtimes {}_\alpha\mathbf{Z}_6) \to K_0(C(S^1) \rtimes {}_\alpha SL^2(\mathbf{Z}))$$

is surjective. Therefore it suffices to show that $GV(\alpha)$ is null on both $K_0(C(S^1) \rtimes {}_\alpha\mathbf{Z}_4)$ and $K_0(C(S^1) \rtimes {}_\alpha\mathbf{Z}_6)$.

Consider

$$GV(\alpha):K_0(C(S^1) \rtimes {}_\alpha\mathbf{Z}_4) \to \mathbf{C}.$$

It is known that the action α of $\mathbf{Z}_4$ is smoothly conjugate to rational rotations. More precisely, there exist an action

$$\alpha':\mathbf{Z}_4 \to SO(2)$$

and $\varphi \in \mathrm{Diff}^\infty_+(S^1)$ such that

$$\alpha'_g = \varphi \circ \alpha_g \circ \varphi^{-1}$$

for all $g \in \mathbf{Z}_4$. Since α'_g is linear, it is obvious that $GV(\alpha') = 0$. Therefore, by Theorem 4,

$$GV(\alpha):K_0(C(S^1) \rtimes {}_\alpha\mathbf{Z}_4) \to \mathbf{C}$$

is the null map.

Similarly,

$$GV(\alpha): K_0(C(S^1) \rtimes {}_\alpha \mathbf{Z}_6) \rightarrow \mathbf{C}$$

is also null. Consequently,

$$GV(\alpha) = 0 \quad \text{on } K_0(C(S^1) \rtimes {}_\alpha SL_2(\mathbf{Z})).$$

References

1. R. Bott, *On some formulas for the characteristic classes of group-actions*, Lecture Notes in Math. *652* (Springer-Verlag, 1978), 25-61.
2. A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation*, Proceedings of the Japan-U.S. Seminar on Operator Algebras, Kyoto (1983), *Geometric methods in operator algebras* (Longman, London, 1986), 52-144.
3. T. Natsume, *On* $K_*(C^*(SL_2(\mathbf{Z})))$, J. Operator Theory *13* (1985), 103-118.
4. G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Math. Soc. Monographs *14* (Academic Press, London, 1979).
5. G. Raby, *L'invariant de Godbillon est stable par C^1-difféomorphisme*, preprint.

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