SUMMATION OF INFINITE SERIES

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Abstract. In this paper we obtain the sum of some infinite series involving hypergeometric functions of one or more variables.

1. Introduction

In this paper an attempt has been made to seek the aid of fractional derivatives, which has enabled us not only to prove the results already obtained by Carlitz [2] and Halim and Salam [5] but also to generalise quite a few of them.

For the sake of convenience the rules for fractional integration and differentiation are given below. Following Erdélyi [3] we write the rules governing fractional integration by parts in the form

$$\int_a^b U \frac{d^\lambda V}{d(b-x)^\lambda}\,dx = \int_a^b V \frac{d^\lambda U}{d(x-a)^\lambda}\,dx. \tag{1}$$

The fractional derivatives occurring in (1) can be defined by integrals, if the real part of λ is negative.

Thus

$$\left.\begin{aligned} \frac{d^\lambda U}{d(x-a)^\lambda} &= \frac{1}{\Gamma(-\lambda)}\int_a^x (x-y)^{-\lambda-1}U(y)dy \\ \frac{d^\lambda V}{d(b-x)^\lambda} &= \frac{1}{\Gamma(-\lambda)}\int_x^b (y-x)^{-\lambda-1}V(y)dy \end{aligned}\right\} R(\lambda) < 0. \tag{2}$$

If U and V are expressible by means of the series of the types

$$U = \sum A_r(x-a)^{\rho+r-1}, \quad V = \sum B_s(b-x)^{\delta+s-1}, \tag{3}$$

then the fractional derivatives are obtainable by differentiating these series term by term and using the definition

$$\frac{d^\lambda \omega^{\mu-1}}{d\omega^\lambda} = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}\,\omega^{\mu-\lambda-1}, \tag{4}$$

which holds for all values of λ except $\lambda = \mu$.

We denote (4) by the operator notation as

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5) $$D^{\lambda}_{\omega,\mu-1}[\ \ ] = \frac{\Gamma(\mu)}{\Gamma(\mu-\lambda)}[\ \ ],$$

here $D^{\lambda}_{\omega,\mu-1}[\omega^{\alpha}]$ will stand for $\dfrac{d^{\lambda}[\omega^{\alpha+\mu-1}]}{d\omega^{\lambda}}$.

We give below a few elementary results which will be used in our in-estigation.

6) $$D^{\mu-\lambda}_{x,\mu-1}[(1-x)^{-\alpha}] = \frac{\Gamma(\mu)}{\Gamma(\lambda)}x^{\lambda-1}{}_2F_1(\alpha,\mu;\lambda;x)$$

7) $$D^{\mu-\lambda}_{x,\mu-1}[(1-xy)^{-\alpha}(1-xz)^{-\beta}] = \frac{\Gamma(\mu)}{\Gamma(\lambda)}x^{\lambda-1}F_1(\mu;\alpha,\beta;\lambda;xy,xz)$$

8) $$D^{\mu-\lambda}_{x,\mu-1}[e^{xy}(1-xz)^{-\beta}] = \frac{\Gamma(\mu)}{\Gamma(\lambda)}x^{\lambda-1}\phi_1(\mu;\beta;\lambda;xy,xz)$$

9) $$D^{\mu-\lambda}_{x,\mu-1}[e^{xy}(1-xz)^{-\alpha}(1-xU)^{-\beta}] = \frac{\Gamma(\mu)x^{\lambda-1}}{\Gamma(\lambda)}\phi_1(\mu;\alpha,\beta;\lambda;xy,xz,xU).$$

'or the definitions of hypergeometric series of two or more variables, ee [1].

In our investigation, we require the definition of Laguerre polynomial, ainville [6, p. 200] as follows

10) $$L^{(\alpha)}_n(x) = \frac{(1+\alpha)_n}{n!}{}_1F_1(-n;1+\alpha;x).$$

10) can be expressed in the following forms:

11) $$L^{(\alpha)}_n(x) = \frac{(-x)^n}{n!}{}_2F_0\left(-n,-\alpha-n;-;-\frac{1}{x}\right)$$

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12) $$L^{(\alpha-n)}_n(x) = \frac{(-x)^n}{n!}{}_2F_0\left(-n,-\alpha;-;-\frac{1}{x}\right).$$

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Let us start with the elementary result

$$e^{(1-x)t} = e^{t}e^{-xt}.$$

Iultiplying both the sides by $(1-Ux)^m$ and employing the operator $D^{\alpha-\gamma}_{x,\alpha-1}$, we get

$$(13) \qquad \sum_{n=0}^{\infty} \frac{1}{n!} F_1(\alpha; -n, -m; \gamma; x, Ux)t^n = e^t \phi_1(\alpha; -m; \gamma; -xt, Ux).$$

Assuming $U = 1$ and using the formula [4, p. 239]

$$(14) \qquad F_1(\alpha; \beta, \beta'; \gamma; x, x) = {}_2F_1\,(\alpha, \beta+\beta'; \gamma; x),$$

in (13), it gives

$$(15) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!}\, {}_2F_1\,(-m-n, \alpha; \gamma; x) = e^t \phi_1(\alpha; -m; \gamma; -xt, x).$$

If in (15) we replace x by x/α and let $\alpha \to \infty$ then using (10), we get

$$(16) \qquad \sum_{n=0}^{\infty} \frac{(m+n)!}{n!(1+\gamma)_{m+n}} L_{m+n}^{(\gamma)}(x)t^n = e^t \phi_3(-m; 1+\gamma; x, -xt).$$

In case $m = 0$, (16) reduces to a well known generating function, Rainville [6, p. 201].

But if in (15) we multiply x by γ and let $\gamma \to \infty$, with the help of (12), we obtain

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{m!n!} x^{m+n} L_{m+n}^{(-\alpha-m-n)}\left(-\frac{1}{x}\right) t^n$$
$$= e^t(1+xt)^{-\alpha}\left(\frac{x}{1+xt}\right)^m L_m^{(-\alpha-m)}\left(-\frac{1+xt}{x}\right),$$

which on changing x, t and α into $-1/x$, $-xt$ and $-\alpha-m$ respectively, it becomes the formula due to Carlitz [2]

$$(17) \qquad \sum_{n=0}^{\infty} \binom{m+n}{m} L_{m+n}^{(\alpha-n)}(x)t^n = (1+t)^\alpha\, e^{-xt} L_m^{\alpha}[x(1+t)].$$

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Returning to (13), we replace t by $(1-y)t$, multiply both the sides by $(1-vy)^k$ and apply the operator $D_{\gamma,\beta-1}^{\beta-\delta}$ thus getting

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} F_1(\alpha; -n, -m; \gamma; x, Ux) F_1(\beta; -n, -k; \delta; y, vy)$$
$$(18) \qquad = e^t \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(\delta)_n} \frac{(xyt)^n}{n!} \phi_1(\alpha+n; -m; \gamma+n; -xt, Ux)$$
$$\phi_1(\beta+n; -k; \delta+n; -yt, vy).$$

On putting $U = v = 1$ and using (14), (18) becomes

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$$\sum_{n=0}^{\infty} \frac{t^n}{n!} {}_2F_1(-m-n, \alpha; \gamma; x) {}_2F_1(-k-n, \beta; \delta; y)$$

(19)
$$= e^t \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (\delta)_n} \frac{(xyt)^n}{n!} \phi_1(\alpha+n; -m; \gamma+n; -xt, x)$$
$$\phi_1(\beta+n; -k; \delta+n; -yt, y)$$

which on putting $m = k = 0$ further reduces to

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} {}_2F_1(-n, \alpha; \gamma; x) {}_2F_1(-n, \beta; \delta; y)$$

(20)
$$= e^t \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (\delta)_n} \frac{(xyt)^n}{n!} {}_1F_1(\alpha+n; \gamma+n; -xt)$$
$${}_1F_1(\beta+n; \delta+n; -yt).$$

Now if we put $x = 1$, $y = 1$ and $\delta = \gamma - \alpha$ then with the aid of Kummer's transformation [4, p. 253]

$${}_1F_1(a; c; x) = e^x {}_1F_1(c-a; c; -x),$$

(20) yields the most interesting result of the paper

(21)
$${}_1F_1(\alpha+\beta; \gamma; t) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n (\gamma-\alpha)_n} \frac{(-t)^n}{n!} {}_1F_1(\alpha+n; \gamma+n; t)$$
$${}_1F_1(\beta+n; \gamma-\alpha+n; t).$$

Further on putting $\alpha = -m$, $\beta = -k$ and changing γ into $\gamma+1$ in (21) and using the result (10), we obtain a known result due to Carlitz [2]

(22)
$$\binom{m+k}{m} L_{m+k}^{(\gamma)}(t) = \sum_{n=0}^{\min(m,k)} \frac{(-t)^n}{n!} L_{k-n}^{(\gamma+m+n)}(t) L_{(m-n)}^{(\gamma+n)}(t).$$

Turning now to (19), if we multiply x and y by γ and δ respectively and let both $\gamma \to \infty$ and $\delta \to \infty$ and using (11) we get after a little simplification

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \binom{m+n}{m} \binom{k+n}{k} L_{m+n}^{(\alpha-n)}(x) L_{k+n}^{(\beta-n)}(y)$$

(23)
$$= e^{xyt} (1-xt)^{\beta} (1-yt)^{\alpha} \sum_{n=0}^{\infty} \frac{(-\alpha-m)_n (-\beta-k)_n}{n!}$$
$$\times \left[\frac{t}{(1-xt)(1-yt)}\right]^n L_m^{(\alpha-n)}[x(1-yt)] \, L_k^{(\beta-n)}[y(1-xt)].$$

On putting $m = 0$, $k = 0$, (23) yields a known result due to Carlitz [2].

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$$\begin{aligned}(24)\qquad &\sum_{n=0}^{\infty}\frac{t^n}{n!}L_n^{(\alpha-n)}(x)L_n^{(\beta-n)}(y)t^n\\ &=\beta!e^{xyt}(1-yt)^{\alpha-\beta}t^{\beta}L_{\beta}^{(\alpha-\beta)}\left[-\frac{(1-xt)(1-yt)}{t}\right].\end{aligned}$$

Returning to (18), we take $m = 0$ and replace x by xy then taking $\gamma \to \infty$, we obtain with the help of (12) and (11)

$$\begin{aligned}(25)\qquad &\sum_{n=0}^{\infty}L_n^{(\alpha-n)}(x)F_1(\beta; -n, -k; \delta; y, vy)t^n\\ &=\left((1+t)^{\alpha}e^{-xt}\phi_1(\beta; -\alpha, -k; \delta; xyt, \frac{yt}{1+t}, vy\right),\end{aligned}$$

and

$$\begin{aligned}(26)\qquad &\sum_{n=0}^{\infty}\frac{t^n}{n!}L_{\alpha}^{(n-\alpha)}(x)F_1(\beta; -n, -k; \delta; y, vy)t^n\\ &=\frac{(t-x)^{\alpha}e^t}{\alpha!}\phi_1\left(\beta; -\alpha, -k; \delta; -yt, \frac{yt}{t-x}, vy\right).\end{aligned}$$

(25) and (26) reduce to known results due to Halim and Salam [5], in case we take $k = 0$.

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In this section, we consider the well known generating function of the Laguarre polynomials

$$(27)\qquad \sum_{n=0}^{\infty}L_n^{(\alpha-n)}(x)t^n = (1-t)^{\alpha}e^{-xt}.$$

Changing t into ut, multiplying both the sides by $(1-uy)^m$ and then employing the operator $D_{u,\beta-1}^{\beta-\gamma}$, we get

$$\begin{aligned}(28)\qquad &\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\gamma)_n}L_n^{(\alpha-n)}(x)\,{}_2F_1(-m, \beta+n; \gamma+n; y)t^n\\ &=\phi_1(\beta; -\alpha, -m; \gamma; -xt, -t, y).\end{aligned}$$

Replacing t and y by t/β and y/β respectively and taking $\beta \to \infty$, we get

$$\begin{aligned}(29)\qquad &\sum_{n=0}^{\infty}\frac{m!}{(1+\gamma)_{m+n}}L_n^{(\alpha-n)}(x)L_m^{(\gamma+n)}(y)t^n\\ &=\phi_3(-\alpha, -m; 1+\gamma; -xt, -t, y).\end{aligned}$$

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In the last, we consider

$$e^{x(1-y)t}\, e^{-y(1-x)t} = e^{xt}\, e^{-yt}$$

$$\sum_{m,n=0}^{\infty} \frac{x^m(-y)^n(1-y)^m(1-x)^n t^{m+n}}{m!\,n!} = \sum_{m,n=0}^{\infty} \frac{(x)^m(-y)^n t^{m+n}}{m!\,n!}.$$

Applying the operator $D_{x,\lambda-1}^{\lambda-\alpha}$ and $D_{y,\mu-1}^{\mu-\beta}$ on both sides, we have after a little simplification

$$\begin{aligned}(30)\qquad &\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(\lambda)_m(\mu)_{n-m}}{m!(n-m)!(\alpha)_m(\beta)_{n-m}}\, x^m(-y)^{n-m}t^n \\ &\quad {}_2F_1[-(n-m), \lambda+m; \alpha+m; x]\, {}_2F_1[-m, \mu+n-m; \beta+n-m; y] \\ &= \sum_{n=0}^{\infty} {}_3F_2\left[-n, \lambda, 1-\beta-n; \alpha, 1-\mu-n; \frac{x}{y}\right] t^n.\end{aligned}$$

On comparing the coefficients of t^n on both sides, we obtain

$$\begin{aligned}(31)\qquad &\sum_{m=0}^{n} \binom{n}{m} \frac{(\lambda)_m(1-\beta-n)_m}{(\alpha)_m(1-\mu-n)_m} \left(-\frac{x}{y}\right)^m {}_2F_1[-(n-m), \lambda+m; \alpha+m; x] \\ &\quad {}_2F_1[-m, \mu+n-m; \beta+n-m; y] \\ &= {}_3F_2\left[[-n, \lambda, 1-\beta-n; \alpha, 1-\mu-n; \frac{x}{y}\right].\end{aligned}$$

We shall obtain some interesting particular cases of (31).

(i) By putting $x = y = 1$, we obtain

$$(32)\qquad {}_3F_2\left[\begin{matrix}-n, \lambda, \beta-\mu; 1\\ 1+\lambda-\alpha-n, 1-\mu-n\end{matrix}\right] = \frac{(\lambda)_n}{(\alpha-\lambda)_n}\, {}_3F_2\left[\begin{matrix}-n, \lambda, 1-\beta-n; 1\\ \alpha, 1-\mu-n\end{matrix}\right].$$

(ii) Dividing x and y by λ and μ respectively and taking $\lambda \to \infty$, $\mu \to \infty$, changing α and β into $1+\alpha$ and $1+\beta$ respectively and using (10), we get

$$\begin{aligned}(33)\qquad &\sum_{m=0}^{n} \frac{(-\beta-n)_m}{(1+\alpha)_m} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(y) \\ &= \frac{(-1)^n(x+y)^n}{(1+\alpha)_n} P_n^{(\alpha,\beta)}\left(\frac{y-x}{y+x}\right).\end{aligned}$$

In particular for $y = -x$,

$$\begin{aligned}(34)\qquad &\sum_{m=0}^{n} \frac{(-\beta-n)_m}{(1+\alpha)_m} L_m^{(\alpha)}(x) L_{n-m}^{(\beta)}(-x) \\ &= \frac{(x)^n(1+\alpha+\beta)_{2n}}{(1+\alpha)_n(1+\beta)_n(1+\alpha+\beta)_n}.\end{aligned}$$

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