A RIEMANN-TYPE INTEGRAL OF LEBESGUE POWER

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The introduction of a mathematics student to formal integration theory usually follows the lines laid down by Riemann and Darboux. Later a change of ideas is necessary if he tackles Lebesgue's more powerful theory, and connections between the two are laboriously constructed. On the other hand, the commonest method of *evaluating* an integral is through the operation inverse to differentiation (the indefinite integral taken between limits). We refer to this as the calculus integral; few realize that this can succeed in cases where even the Lebesgue integral does not exist, let alone the Riemann one. An example is given later.

The special Denjoy integral is probably the weakest that contains both the Lebesgue and calculus integrals, but few students study this, because of the complexity of the constructions involved. However, constructions of Lebesgue and Denjoy type can be avoided, and Riemann-type constructions can be used to obtain an integral of Denjoy depth. This is the Riemann-complete integral, defined in passing by Kurzweil **(4)**, and defined independently and named by Henstock **(1, 2, 3)**. A simple Riemann-complete integral includes the calculus, Lebesgue, Perron, and special Denjoy integrals that have domain contained in the real line, and the Burkill integral of interval functions. Slight modifications enable it to include Burkill's approximate Perron integral and the general Denjoy integral. Other extensions for more general spaces enable it to include the Lebesgue integral in all fields where the latter is defined. Thus it is not necessary to consider an exceedingly elaborate theory to obtain a powerful integral. An elementary approach to the theory is given here, while more details can be found in **(2, 3)**.

Another approach to Riemann-type integration can be found in Ridder **(5)**, in which measure theory is used to obtain results. My experience is that although measure theory is a good guide in setting up the new theory, the older theory usually fails to supply the simplest and most revealing proofs Ridder does not give the connection between **(5)** and **(2)**, but J. J. McGrotty is looking into this point.

The definite integral defined in elementary books is the Riemann integral as modified by Darboux. Let $f(x)$ be a real function in the closed interval $[a, b]$, which interval we divide into smaller intervals by points

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

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This last arrangement is a *division* $\mathfrak{D}$ of $[a, b]$ with *norm*

$$\text{norm}(\mathfrak{D}) = \max\{x_j - x_{j-1}: j = 1, 2, \ldots, n\}.$$

When $f(x) \geqslant 0$ in $[a, b]$ we take the smallest rectangle with base the interval $[x_{j-1}, x_j]$ that encloses the graph of $y = f(x)$, and so with height the least upper bound or *supremum* of $f(x)$, in that interval,

$$M_j = \sup\{f(x): x_{j-1} \leqslant x \leqslant x_j\}.$$

The collection of n rectangles, for $j = 1, \ldots, n$, has area

$$\bar{s} = \sum_{j=1}^{n} M_j(x_j - x_{j-1}).$$

This is known as the *upper Darboux sum* for the division. Similarly we can take a rectangle with base $[x_{j-1}, x_j]$ that lies just below the graph, and so with height the greatest lower bound or *infimum* of $f(x)$ in that interval,

$$m_j = \inf\{f(x): x_{j-1} \leqslant x \leqslant x_j\}.$$

Taking $j = 1, \ldots, n$, we have another collection of rectangles, with area

$$\underline{s} = \sum_{j=1}^{n} m_j(x_j - x_{j-1}),$$

called the *lower Darboux sum* for the division. Clearly $\bar{s} \geqslant \underline{s}$. If both tend to the same limit I for arbitrary divisions $\mathfrak{D}$ of $[a, b]$ as *norm* $(\mathfrak{D}) \to 0$, then we say that the *Riemann-Darboux* integral of $f(x)$ exists over $[a, b]$ with value I. More generally, if $f(x)$ is sometimes negative the m_j, M_j, $\underline{s}$, $\bar{s}$, and so the integral, are still definable. Also, when the integral exists, $\underline{s}$ and $\bar{s}$ are finite for some division $\mathfrak{D}$, and $f(x)$ is bounded in $[a, b]$.

(1) *If $f(x)$ is Riemann-Darboux integrable, it is bounded.*

Riemann's definition is as follows. For each $j = 1, \ldots, n$ let z_j be an arbitrary point of $[x_{j-1}, x_j]$, and consider the sum

$$s = \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}).$$

If every such s tends to I as *norm*$(\mathfrak{D}) \to 0$, we say that the *Riemann* integral of $f(x)$ exists over $[a, b]$ with value I. More exactly,

(2) *given $\epsilon > 0$, there is a $\delta > 0$ such that if* norm $(\mathfrak{D}) < \delta$ *then*

$$\left| \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - I \right| < \epsilon.$$

This definition holds for complex values of $f(x)$, but if $f(x)$ is real,

$$\underline{s} \leqslant s \leqslant \bar{s}.$$

It follows that if $\underline{s}$, $\bar{s}$ tend to I as norm $(\mathfrak{D}) \to 0$, then also $s \to I$. Conversely, in (2) with a fixed division $\mathfrak{D}$, we can chooze $z_1, \ldots, z_n$ so that

$$f(z_j) > M_j - \epsilon, \quad (j = 1, \ldots, n), \qquad \bar{s} \geqslant s > \bar{s} - \epsilon(b - a),$$

and for another choice of $z_1, \ldots, z_n$,

$$f(z_j) < m_j + \epsilon \quad (j = 1, \ldots, n), \qquad \underline{s} \leqslant s < \underline{s} + \epsilon(b - a).$$

Hence as $\epsilon > 0$ is arbitrary, we also have

$$\underline{s} \to I, \quad \bar{s} \to I, \quad as \ \mathrm{norm}(\mathfrak{D}) \to 0.$$

(3) *If $f(x)$ is real, the Riemann and Riemann–Darboux integrals are equivalent.*

In the Riemann theory we choose $[x_{j-1}, x_j]$ first, and then we choose z_j in the closed interval. The new outlook on Riemann integration begins by taking z_j first, and $[x_{j-1}, x_j]$ secondly, as any closed interval of length less than $\delta > 0$ that contains z_j, calling it an *associated interval* of z_j. Correspondingly, z_j is the *associated point* of the interval in the division $\mathfrak{D}$, which division we say is *compatible* with the fixed number $\delta > 0$.

We now turn to the calculus integral of $f(x)$. This is a function $F(x)$ that is continuous to the right at a, and to the left at b, and with $f(x)$ as derivative at all points of the open interval (a, b). A derivative has one property of continuous functions, for Darboux has shown that if $F'(x)$ exists at all points of $[u, v]$, and if $F'(u) < z < F'(v)$ or $F'(u) > z > F'(v)$, then there is an x in the open interval $u < x < v$ with $F'(x) = z$. Thus the Lebesgue-integrable function that is 1 at the rationals and 0 elsewhere, is not a derivative. However, continuous functions are also bounded, but not all derivatives are bounded For example,

$$F(x) = 2x^{\frac{1}{2}}, \quad F'(x) = x^{-\frac{1}{2}} \qquad (x \neq 0),$$

and $F'(x)$ has a calculus and a Lebesgue integral over $[0, 1]$. Further, $F'(x)$ can exist everywhere, with $|F'(x)|$ not integrable, so that $F'(x)$ is not Lebesgue integrable. It need not even be Cauchy–Lebesgue integrable, the ordinary extension of Lebesgue integration by the continuous expansion of intervals over which the function is Lebesgue integrable. For example,

$$G(x) = x^2 \sin(1/x^2) \quad (x \neq 0), \qquad\qquad G(0) = 0,$$

$$G'(x) = 2x \sin(1/x^2) - (2/x)\cos(1/x^2) \quad (x \neq 0), \qquad G'(0) = 0,$$

$$H(x) = G(x) \quad (|x| \leqslant \tfrac{1}{2}), \qquad H(x) = 0 \quad (|x| \geqslant 1),$$

with $H(x)$ monotone and $H'(x)$ continuous in $\frac{1}{2} \leqslant x \leqslant 1$ and in

$$-\tfrac{1}{2} \geqslant x \geqslant -1.$$

If

$$\{(a_n - 2.3^{-n}, a_n + 2.3^{-n})\}$$

is a suitable sequence of disjoint intervals, it can be shown that

$$F(x) \equiv \sum_{n=1}^{\infty} 4^{-n} H((x - a_n)\cdot 3^n)$$

has the desired properties. This $F'(x)$ needs the Denjoy extension and so the special Denjoy integral. Thus the calculus and Lebesgue integrals do not either of them include the other. However, a derivative is integrable by the extended type of Riemann integration considered here.

The calculus integral $F(x)$ of $f(x)$ in $[a, b]$ has the following properties. Given $\epsilon > 0$, $a \leqslant z \leqslant b$, there is a function $\delta(z) > 0$ such that, for all x satisfying $0 < |z - x| < \delta(z)$, $a \leqslant x \leqslant b$, we have

$$(4) \qquad \begin{gathered} |f(z) - (F(x) - F(z))/(x - z)| < \epsilon, \\ |f(z)(x - z) - (F(x) - F(z))| < \epsilon\,|x - z| \quad (a < z < b), \end{gathered}$$

$$(5) \qquad |F(x) - F(z)| < \epsilon, \qquad |f(z)(x - z)| < \epsilon \qquad (z = a, b).$$

Continuity gives (5). The z is at one end of the interval, but by (4), when

$a \leqslant u < v \leqslant b$, $z - \delta(z) < u \leqslant z \leqslant v < z + \delta(z)$, $a < z < b$,

$$(6) \quad \begin{aligned} &|f(z)(v - u) - (F(v) - F(u))| \\ &\quad = |f(z)(v - z) - (F(v) - F(z)) + f(z)(z - u) - (F(z) - F(u))| \\ &\quad < \epsilon(v - z) + \epsilon(z - u) = \epsilon(v - u). \end{aligned}$$

A division $\mathfrak{D}$, given by $a = x_0 < x_1 < \ldots < x_n = b$, z_j being the associated point of $[x_{j-1}, x_j]$ $(j = 1, \ldots, n)$, can be said to be *compatible* with the function $\delta(z) > 0$, if, for each $j = 1, \ldots, n$,

$$|x_j - z_j| < \delta(z_j), \qquad |z_j - x_{j-1}| < \delta(z_j).$$

If $\mathfrak{D}$ is compatible with the function $\delta(z)$ of (4, 5, 6), then

$$\left| \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - (F(b) - F(a)) \right| \leqslant \sum_{j=1}^{n} |f(z_j)(x_j - x_{j-1}) \\ - (F(x_j) - F(x_{j-1}))| < \sum_{j=1}^{n} \epsilon(x_j - x_{j-1}) + 4\epsilon = \epsilon(4 + b - a).$$

Thus the sum over the division $\mathfrak{D}$ can be made as near as we please to $F(b) - F(a)$, by choice of $\epsilon > 0$. This suggests that, by analogy with (2), we can define a Riemann-type integral using $\delta(z)$ instead of constants δ.

Let $f(z)$ be defined in the interval $[a, b]$. If there is a number I such that to each $\epsilon > 0$ there corresponds a function $\delta(z) > 0$ in $[a, b]$ with

$$\left| \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - I \right| < \epsilon$$

for all sums over divisions $\mathfrak{D}$ of $[a, b]$ compatible with $\delta(z)$, there being at least one such $\mathfrak{D}$, then we say that $f(x)$ is *Riemann-complete* integrable in

$[a, b]$ with integral I. For simplicity we can replace the words "sum over a division of $[a, b]$ compatible with $\delta(z)$" for "sum for $[a, b]$ and $\delta(z)$."

Clearly, if there are two values I_1, I_2, of the integral, corresponding to functions $\delta_1(z)$, $\delta_2(z)$, we can put

$$\delta(z) = \min(\delta_1(z), \delta_2(z)).$$

Then a division $\mathfrak{D}$ compatible with $\delta(z)$ is also compatible with $\delta_1(z)$ and $\delta_2(z)$, proving that $|I_1 - I_2| < 2\epsilon$, $I_1 = I_2$, as $\epsilon > 0$ is arbitrary. Thus, if it exists, the integral is uniquely defined.

Clearly we need the following theorem.

THEOREM 1. *If $\delta(z) > 0$ is an arbitrary positive function in $[a, b]$, there is a division of $[a, b]$ compatible with $\delta(z)$.*

The proof uses continued bisection. If no such division of $[a, b]$ exists, then either no such division of $[a, \frac{1}{2}(a + b)]$ exists, or no such division of $[\frac{1}{2}(a+b), b]$ exists, or both. Thus we can obtain a sequence of intervals

$$\{[a_n, b_n]\},\ [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n],\quad b_{n+1} - a_{n+1} = \tfrac{1}{2}(b_n - a_n)\quad (n = 1, 2, \ldots),$$

which therefore contain a single common point z, and, after a certain stage, every interval of the sequence lies in $(z - \delta(z), z + \delta(z))$. The intervals themselves are then their own divisions compatible with $\delta(z)$, giving a contradiction. Hence the theorem is true.

It now follows that every derivative $F'(x)$ is Riemann-complete integrable to $F(b) - F(a)$ in $[a, b]$. Also, by (2), if the Riemann integral of a function exists, then so does the Riemann-complete integral and they are equal. The Riemann integral is the case of the Riemann-complete integral when we can take every $\delta(z)$ constant.

Digressing for a moment, there are several extensions of the simple theory. To integrate over $(-\infty, +\infty)$ we use the ordinary definition with $a = -\infty$ and $b = +\infty$, taking $f(a) = 0 = f(b)$. In place of $\delta(a)$, $\delta(b)$, we use large numbers A, B, so that the end intervals consist of $(-\infty, a']$, $[b', +\infty)$, for some $a' < -A$, $b' > B$. Further, in Lebesgue integration we can integrate over arbitrary measurable sets. Here, if X is a set of real numbers, we integrate $f(x)$ over X by integrating $f(x)\mathrm{ch}(X; x)$ over $(-\infty, +\infty)$, where $\mathrm{ch}(X; x)$ is the characteristic function of X, that is 1 for x in X and 0 for x not in X. It is just as easy to integrate a function $f(x)$ with respect to another function $g(x)$, by replacing $f(z_j)(x_j - x_{j-1})$ by $f(z_j)(g(x_j) - g(x_{j-1}))$. Then we cover Riemann–Stieltjes integration, and Radon integration if $g(x)$ is continuous. Discontinuities of $g(x)$ cause rather trivial differences between the Riemann-complete and Radon integrals. Burkill integration of an interval function $g(u, v)$ is included by writing $g(x_{j-1}, x_j)$ for $f(z_j)(x_j - x_{j-1})$.

A more general result with two interval functions is given in **(2)**, in which we restrict each z_j to lie at an end of its interval. A number of people have

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pointed out that it is more natural (though more complicated) to assume the system of this paper, i.e. z_j unrestricted in its closed interval. In the past I have replied that this is only another special case of **(1)**, but now it provides a simpler approach to Riemann-complete theory. Further, in the case of the integral of $f(x)$, and the integral of $f(x)$ with respect to $g(x)$, the two systems are equivalent. For we can write

$$f(z_j)(g(x_j) - g(x_{j-1})) = f(z_j)(g(x_j) - g(z_j)) + f(z_j)(g(z_j) - g(x_{j-1})),$$

transforming a sum of the unrestricted type to a sum of the restricted type. Conversely, a sum of the restricted type is already one of the sums of the unrestricted type. Compare **(2)**, p. 8, which gives the similar result for Riemann integration.

Returning to the main theme of the paper, it is interesting to take a typical simple Lebesgue integrable function and integrate it by the new method. Let $\{S_n\}$ be a sequence of sets of real points, each set being of "measure zero," i.e. for each n and each $\epsilon > 0$, there is a countable union G of non-overlapping open intervals I with $G \supseteq S_n$, $mG \leqslant \epsilon$, where mG is the sum of lengths of the intervals I of G. If S is the union of sets S_n, and if

$$|f(x)| \leqslant 2^n \quad (x \in S_n; n = 1, 2, \ldots), \qquad f(x) = 0 \quad (x \notin S),$$

we call $f(x)$ a *null function*. For example, let $f(x)$ be 1 when x is rational, and 0 when x is irrational. The rationals can be put as a sequence $\{r_n\}$, and we enclose r_n in an open interval

$$(r_n - \epsilon.2^{-1-n}, r_n + \epsilon.2^{-1-n}) \qquad (n = 1, 2, \ldots).$$

The union of such partially overlapping intervals is a set G with $mG \leqslant \epsilon$ that encloses all rationals, and $f(x)$ is a null function.

THEOREM 2. *The Riemann-complete integral of a null function over a finite (or infinite) interval is zero.*

Given $\epsilon > 0$, we choose unions G_n of open intervals, and $\delta(z) > 0$, with

$$G_n \supseteq S_n, \quad mG_n < 4^{-n}\,\epsilon, \quad \delta(z) = 1 \qquad (z \notin S),$$

$$(z - \delta(z), z + \delta(z)) \subseteq G_n \qquad (z \in S_n, z \notin S_j \quad (j < n)) \quad (n = 1, 2, \ldots).$$

If $\mathfrak{D}$ is a division over $[a, b]$ compatible with $\delta(z)$, the only non-zero $f(z_j)$ occur in the sets S_n, and hence the result, since

$$\left| \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - 0 \right| \leqslant \sum_{n=1}^{\infty} 2^n \cdot mG_n < \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n} = \epsilon.$$

To prove that the Riemann-complete integral includes the Lebesgue, we need Lebesgue's monotone convergence theorem for Riemann-complete integration. Some preliminary results are useful in themselves, and for simplicity we use finite intervals, and integrals of $f(x)$.

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THEOREM 3. *Let $a < c < b$. If $f(x)$ is Riemann-complete integrable over $[a, c]$ and $[c, b]$, then it is Riemann-complete integrable over $[a, b]$, to the sum of the two other integrals.*

If $\delta_1(z)$ is given for $[a, c]$, and $\delta_2(z)$ for $[c, b]$, let

$$\delta(z) = \begin{cases} \min(\delta_1(z), \frac{1}{2}(c - z)) & (a \leqslant z < c), \\ \min(\delta_2(z), \frac{1}{2}(z - c)) & (c < z \leqslant b), \\ \min(\delta_1(c), \delta_2(c)) & (z = c). \end{cases}$$

Then a division $\mathfrak{D}$ over $[a, b]$ and compatible with $\delta(z)$ contains either one interval with $x_{j-1} < z_j = c < x_j$ or two intervals with associated points c. In the former case we divide $[x_{j-1}, x_j]$ at c without altering the sum, obtaining the latter case. Thus we can separate the sum for $\mathfrak{D}$ into two parts, a sum for $[a, c]$ and $\delta_1(z)$, and a sum for $[c, b]$ and $\delta_2(z)$. By choice of $\delta_1(z)$ and $\delta_2(z)$ we can ensure that the latter two sums are within $\frac{1}{2}\epsilon$ of the corresponding integrals. Then the sum for $\mathfrak{D}$ is within ϵ of the sum of the integrals over $[a, c]$, $[c, b]$, giving the result.

THEOREM 4. *If $f(x)$ is Riemann-complete integrable in $[a, b]$, and if $[u, v]$ is contained in $[a, b]$, then $f(x)$ is Riemann-complete integrable in $[u, v]$.*

Let $\delta(z)$ be such that each sum for $[a, b]$ and $\delta(z)$ is within $\epsilon > 0$ of the integral over $[a, b]$, Then two such sums differ by at most 2ϵ.

Let s_1, s_2 be sums for $[u, v]$ and $\delta(z)$. As the part of $[a, b]$ that is not in $[u, v]$ forms one or two intervals, we add end points to obtain one or two closed intervals. Then there are one or two sums for the intervals and $\delta(z)$, with sum s, and $s_1 + s$, $s_2 + s$ are sums for $[a, b]$ and $\delta(z)$. Hence

$$|s_1 - s_2| = |(s_1 + s) - (s_2 + s)| \leqslant 2\epsilon. \tag{7}$$

This situation corresponds to that of a fundamental (Cauchy) sequence in the theory of sequences, where we deduce that the sequence is convergent. Here, replacing ϵ by $1/n$ $(n > 2)$, let $\delta(z)$ become $\delta_n(z)$, where by induction we can assume that $\delta_n(z) \leqslant \delta_{n-1}(z)$ $(n > 3)$. Let s_n be a fixed sum, and s_n^* an arbitrary sum, for $[u, v]$ and $\delta_n(z)$, and for each $n > 2$. Then for $m > n$, s_m is an s_n^* and, applying (7),

$$|s_n^* - s_n| \leqslant 2/n \quad (n > 2), \qquad |s_m - s_n| \leqslant 2/n \quad (m > n > 2).$$

The fundamental sequence $\{s_n\}$ is convergent to some limit I, so that if we let $m \to \infty$, we obtain

$$|I - s_n| \leqslant 2/n, \qquad |s_n^* - I| \leqslant 4/n \quad (n > 2).$$

Hence we prove the theorem with I the value of the integral over $[u, v]$.

THEOREM 5. *Let the integral over $[u, v]$ be $I(u, v)$ and, given $\epsilon > 0$, let $\delta(z) > 0$ be defined over $[a, b]$ such that $|s - I(a, b)| \leqslant \epsilon$ for all sums s for $[a, b]$ and $\delta(z)$. Then if p is a (partial) sum of terms $\{f(z)(v - u) - I(u, v)\}$, for any* *number of distinct intervals* $[u, v]$ *of a division of* $[a, b]$ *compatible with* $\delta(z)$, *we have* $|p| \leqslant \epsilon$.

Taking from $[a, b]$ the intervals $[u, v]$ from which p is built, we have a finite number of intervals. Adding end points, the intervals are closed, say, $J_1, \ldots, J_k$. For each $j = 1, \ldots, k$, there is a sum s_j for J_j and $\delta(z)$ that is as near as we please to $I(J_j)$. Then if q is the sum of $f(z)(v - u)$, using the $[u, v]$ of p,

$$q + s_1 + \ldots + s_k$$

is a sum for $[a, b]$ and $\delta(z)$. The result follows from Theorem 3 and

$$|q + s_1 + \ldots + s_k - I(a, b)| \leqslant \epsilon, \quad |q + I(J_1) + \ldots + I(J_k) - I(a, b)| \leqslant \epsilon.$$

THEOREM 6. *Let* $\{f_n(x)\}$ *be a sequence of functions that are Riemann-complete integrable in* $[a, b]$. *For each* x *let* $\{f_n(x)\}$ *be monotone increasing and convergent to a finite limit function* $f(x)$. *If the sequence* $\{I_n(a, b)\}$ *of integrals is convergent to* I, *then* $f(x)$ *is Riemann-complete integrable in* $[a, b]$ *with integral* I.

For each integer n and each $\epsilon > 0$, there is a $\delta_n(x) > 0$ such that if s_n is an arbitrary sum for $[a, b]$ and $\delta_n(z)$, using $f_n(x)$, then

$$|s_n - I_n(a, b)| < \epsilon.2^{-n} \qquad (n = 1, 2, \ldots).$$

As $I_n(a, b) \to I$, there is an integer n_0 such that

$$|I_n(a, b) - I| \leqslant \epsilon \qquad (n \geqslant n_0).$$

Also, for each z in $[a, b]$ there is an integer $m = m(z, \epsilon) \geqslant n_0$ such that

$$|f_m(z) - f(z)| < \epsilon.$$

Using the $\delta_n(z)$ and $m(z, \epsilon)$ we can define

$$\delta(z) = \delta_{m(z,\epsilon)}(z) > 0.$$

If the division $\mathfrak{D}$ of $[a, b]$ is compatible with $\delta(z)$, then for $m = m(z_j, \epsilon)$,

$$\left| \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - \sum_{j=1}^{n} f_m(z_j)(x_j - x_{j-1}) \right|$$
$$\leqslant \sum_{j=1}^{n} |f(z_j) - f_m(z_j)|\,(x_j - x_{j-1}) < \epsilon(b - a).$$

Grouping the $f_m(z_j)(x_j - x_{j-1})$ into brackets with equal $m = m(z_j, \epsilon)$, and using Theorem 5, we see that

$$\left| \sum_{j=1}^{n} f_m(z_j)(x_j - x_{j-1}) - \sum_{j=1}^{n} I_m(x_{j-1}, x_j) \right| \leqslant \sum_{m=1}^{\infty} \epsilon \cdot 2^{-m} = \epsilon.$$

Further, we have

$$f_n(x) \leqslant f_{n+1}(x) \qquad (n = 1, 2, \ldots).$$

By using sums over divisions of $[u, v]$ we see that $I_n(u, v)$ is also monotone increasing in n. Hence by Theorem 3, if g, h are respectively the least and greatest integers in the finite set

$$m(z_1, \epsilon), \ldots, m(z_n, \epsilon),$$

$$I_g(a, b) = \sum_{j=1}^{n} I_g(x_{j-1}, x_j) \leqslant \sum_{j=1}^{n} I_m(x_{j-1}, x_j) \leqslant \sum_{j=1}^{n} I_h(x_{j-1}, x_j) = I_h(a, b) \leqslant I.$$

By choice of $m(z, \epsilon)$, we have $g \geqslant n_0$. Hence

$$\left| \sum_{j=1}^{n} f(z_j)(x_j - x_{j-1}) - I \right| \leqslant \epsilon(b - a + 2).$$

As $\mathfrak{D}$ is an arbitrary division over $[a, b]$ compatible with $\delta(z)$, and as $\epsilon > 0$ is arbitrary, the theorem is proved.

By using Theorem 2 we can weaken the hypotheses of Theorem 6 to a form in which $\{f_n(x)\}$ is monotone increasing and convergent to a finite $f(x)$ except in a set of measure zero (in the Lebesgue notation). Retaining the "monotone increasing," it is then possible to prove the convergence of $\{f_n(x)\}$ from the other hypotheses. See (**2**, p. 83, Theorem 36.1).

To prove that the Lebesgue integral is included in the Riemann-complete integral we first note that each non-negative Lebesgue-integrable function is the monotone increasing limit of a sequence of functions each of which takes only a finite set of values, being a finite linear combination of characteristic functions of measurable sets. Each such characteristic function is the monotone decreasing limit almost everywhere of a sequence of characteristic functions of open sets, each of which is the monotone increasing limit of a sequence of characteristic functions of finite unions of open intervals. The last functions are Riemann, and so Riemann-complete, integrable. Reversing the process, we show at each stage that the Lebesgue and Riemann-complete integrals coincide. As our functions are real, the general Lebesgue integral over $[a, b]$ is equal to the corresponding Riemann-complete integral. But not all Riemann-complete integrals are Lebesgue integrals since every calculus integral is a Riemann-complete integral.

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