SEMI-GROUPS AND DIFFERENTIAL EQUATIONS

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In this short note we shall apply the theory of semi-groups of operators, (cf: Hille and Phillips, [2]), to the problem of representing solutions of certain differential equations with non-constant coefficients. When the coefficients are constant, this representation reduces to the usual Laplace transform solution of the relevant equation.

Let $C[0, \infty]$ be the Banach space of continuous, real-valued functions defined on the closed interval $[0, \infty]$, with the usual pointwise algebraic operations, and with the supremum norm. Suppose that $a \in C[0, \infty]$ is a positive function. Define the function A by

$$A(t) = \int^t \frac{ds}{a(s)}.$$

Then certainly A is strictly increasing, so A^{-1} exists. Further, as $A'(t) = 1/a(t) > 0$ for $t \geqq 0$, we see that A^{-1} is in fact continuous. Now, for each $\xi \geqq 0$, define the function q_ξ by

$$(1) \qquad q_\xi(t) = A^{-1}(A(t)+\xi);\, t \geqq 0.$$

Then $\{q_\xi : \xi \geqq 0\}$ is a family of continuous fractional iterates of q_1, and A is the Abel function of this family, ([3]). Furthermore, because of (1), the limit

$$(2) \qquad \lim_{\xi \to 0+} \frac{\partial}{\partial \xi} q_\xi(t)$$

exists for each $t \geqq 0$, and equals $a(t)$. Therefore a is the infinitesimal generator of the iterates $\{q_\xi\}$.

Next, consider the mapping $W(\xi) : C[0, \infty] \to C[0, \infty]$ defined, for each non-negative ξ, by $(W(\xi)x)(t) = x(q_\xi(t))$, for $x \in C[0, \infty]$ and $t \geqq 0$. Obviously $W(\xi)$ is a linear mapping and it is readily seen that $W(\xi)W(\eta) = W(\xi+\eta)$ for all $\xi, \eta \geqq 0$. So the family $\mathfrak{W} = \{W(\xi) : \xi \geqq 0\}$ is a semi-group of linear maps. Now

$$||W(\xi)x|| = \sup_{t \geqq 0}|x(q_\xi(t))| = \sup_{t \in E_\xi}|x(t)|,$$

where $E_\xi = q_\xi([0, \infty])$ is the range of q_ξ, which is contained in $[0, \infty]$.

Thus the above norm is $\leqq \sup_{t\geqq 0}|x(t)| = ||x||$. Hence $||W(\xi)|| \leqq 1$, and by choosing the element $x_0 \in C[0, \infty]$ defined by $x_0(t) = 1$, we see that $||W(\xi)x_0|| = 1$. Thus $W(\xi)$ is of norm 1 for each $\xi \geqq 0$. From this we conclude that the type of the semi-group $\mathfrak{W}$ is $0 = \inf_{\xi>0}\xi^{-1}\log||W(\xi)||$, ([2], 306). Further, as the iterates of q_1 are continuous, we see that the function $\xi \to W(\xi)$ is continuous in the strong topology of operators on $C[0, \infty]$. The example $a(t) = 1$, for which $q_1(t) = t+1$, shows that, in general, $\xi \to W(\xi)$ is not continuous in the uniform topology of operators. All of this goes to show that $\mathfrak{W}$ is a strongly continuous semi-group of operators of class (A) on $C[0, \infty]$, ([2], 321).

Now let D be the set of all functions $x \in C[0, \infty]$ for which the limit

$$Tx = \lim_{\xi\to 0+}(W(\xi)x-x)/\xi \tag{3}$$

exists in the norm topology of $C[0, \infty]$. Equation (3) defines a (closed) linear transformation $T : D \to C[0, \infty]$, which is the infinitesimal generator of the semi-group $\mathfrak{W}$. Because of (2) it is seen that

$$D = \{x \in C[0, \infty] : ax' \text{ exists as an element of } C[0, \infty]\},$$

and so, for $x \in D$,

$$(Tx)(t) = a(t)x'(t), \qquad t \geqq 0. \tag{4}$$

Because $\mathfrak{W}$ is of type 0, the spectrum of T is contained in the half-plane $\{z : \text{Re}\,(z) \leqq 0\}$, and therefore, as a result of (4) and Theorem 11.5.1 of [2], the solution of the first order differential equation

$$-ax'+\lambda x = f \tag{5}$$

which belongs to D, for some given $f \in C[0, \infty]$, may be represented as a Bochner integral. But, by a similar argument to that used in [2] page 532, this may be interpreted as the following Lebesgue integral.

$$x(t) = \int_0^\infty e^{-\lambda\xi} f(q_\xi(t))d\xi \tag{6}$$

for $t \geqq 0$ and for all λ with positive real part. This representation of solutions of differential equations thus extends the theory of [2], page 532, which corresponds to the particular case $a(t) = 1$.

A necessary condition for q_1 to generate a family of fractional iterates is that q_1 have a fixed point $\alpha \in [0, \infty]$. Thus $q_1(\alpha) = \alpha$, where possibly $\alpha = \infty$. As q_1 has α as a fixed point, so does each iterate q_ξ, $\xi \geqq 0$. We now see that (6) represents that unique solution of equation (5) which belongs to D and which satisfies the boundary condition

$$x(\alpha) = \int_0^\infty e^{-\lambda\xi} f(q_\xi(\alpha))d\xi = f(\alpha)\int_0^\infty e^{-\lambda\xi}d\xi = f(\alpha)/\lambda.$$

We summarize this result as

THEOREM 1. *Suppose that* a *is positive and continuous on* $[0, \infty]$, *then that solution* $x \in D$ *of* (5) *which satisfies* $x(\alpha) = f(\alpha)/\lambda$, *is given by* (6), *at least when* $\mathrm{Re}(\lambda) > 0$.

At the expense of introducing 'almost everywhere' language, we can derive a solution similar to (6) when T acts on the domain $\{x \in L^p[0, \infty]: ax \text{ is absolutely continuous}\}$ in the Lebesgue spaces $L^p[0, \infty]$, for $p \geqq 1$.

We will illustrate the above representation by exhibiting a simple example. Choose $a(t) = e^{-t}$; we then find that $A(t) = e^t$, and, from equation (1), that $q_\xi(t) = \log(\xi + e^t)$. Thus that solution of

$$-e^{-t}x'(t) + \lambda x(t) = f(t)$$

which has a continuous derivative, and which satisfies $x(\infty) = f(\infty)/\lambda$, is given by

$$x(t) = \int_0^\infty e^{-\lambda\xi} f(\log(\xi + e^t))d\xi,$$

for $\mathrm{Re}(\lambda) > 0$, and for each $f \in C[0, \infty]$.

Consider again the linear transformation $T : D \to C[0, \infty]$ as defined by (4). It is readily seen that T^2 is closed on its domain, and, ([1], 639), that it is the infinitesimal generator of a strongly continuous semi-group of operators $\{\Gamma(\xi) : \xi \geqq 0\}$ over $C[0, \infty]$, of type 0. In fact, this semi-group is given explicitly by, ([1], 638),

$$(\Gamma(\xi)x)(t) = \frac{1}{\sqrt{\pi\xi}}\int_0^\infty e^{-s^2/4\xi} x(q_s(t))ds.$$

From this representation, we find that a solution x of

$$-a(ax')' + \lambda x = f \tag{7}$$

which belongs to $D(T^2) = \{x \in C[0, \infty] : a(ax')' \text{ exists and belongs to } C[0, \infty]\}$, is given by

$$x(t) = \frac{1}{\sqrt{\pi}}\int_0^\infty \xi^{-\frac{1}{2}} e^{-\lambda\xi} \int_0^\infty e^{-s^2/4\xi} f(q_s(t))ds\,d\xi, \tag{8}$$

for $\mathrm{Re}(\lambda) > 0$.

Again we note that if α is a fixed point of the iterates $\{q_\xi\}$, then (8) satisfies $x(\alpha) = f(\alpha)/\lambda$, and, after a further laborious computation involving the use of Abel's equation (1), we find that if f is differentiable at α, then $x'(\alpha) = f'(\alpha)/\lambda$. We have therefore proven the

THEOREM 2. *Let* a *be as in Theorem* 1, *and suppose that* f *is differentiable. Then the solution* x *of* (7) *which belongs to* $D(T^2)$, *and which satisfies* $x(\alpha) = f(\alpha)/\lambda$, $x'(\alpha) = f'(\alpha)/\lambda$, *is given by* (8) *when* $\mathrm{Re}(\lambda) > 0$.

In conclusion, we mention that the above method yields representations of solutions of certain initial-value problems for partial differential equations. We refer to [1], page 641, for the relevant details.

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References

[1] Dunford, N. and Schwartz, J. T., '*Linear operators*'. Vol. I (Interscience, 1964, Second edition).

[2] Hille, E. and Phillips, R. S., '*Functional analysis and semigroups*' (Amer. Math. Soc. Coll. XXXI, 1958).

[3] Szekeres G. 'Regular iteration of real and complex functions.' *Acta Math.* 100 (1958), 203—258.

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