BULL. AUSTRAL. MATH. SOC. 39A10, 92A15
VOL. 26 (1982), 263-284.

SOLUTIONS OF PERIOD SEVEN FOR A LOGISTIC DIFFERENCE EQUATION*

A. Brown

Communicated by James M. Hill

The paper extends earlier results for periodic solutions of the difference equation $w_{n+1} = w_n^2 - A$, where A is a constant and $A > -1/4$. Exact equations are given for determining solutions of period seven. The method involves finding roots of two polynomial equations, one of degree 18 and the other of degree 7 . For a given value of A , each real root of the equation of degree 18 corresponds to a cyclic solution and the equation of degree 7 gives the seven values of w_n in this cyclic solution. The equations are valid whether the periodic solution is stable or unstable, although information about the stability emerges as a by-product. Thus it is possible to tabulate precise intervals of stability in the cases where stable solutions occur.

1. Introduction

The difference equation

$$u_{n+1} = 2au_n - bu_n^2 , \tag{1.1}$$

Received 29 March 1982.

* This paper is based on a talk given at the Australian Mathematical Society Applied Mathematics Conference held in Bundanoon, February 7-11, 1982. Other papers delivered at this Conference appear in Volumes 25 and 26.

https://doi.org/10.1017/S000497270000575X Published online by Cambridge University Press

where a and b are positive constants, with $a > 1/2$, has been used freely in population dynamics as a finite analogue of the logistic differential equation [5, Chapters 2, 3]. Its properties have been discussed in detail in a review article by May [4], who mentions that an equation of this kind is particularly appropriate in populations where the generations are distinct and u_n gives the population size in the nth generation. Apart from the zero solution $u_n = 0$, the equation has an equilibrium solution at $u_n = (2a-1)/b$ and for sufficiently large values of a solutions can oscillate about this non-zero equilibrium level. This leads to the possibility of periodic solutions and to questions of the stability of these periodic solutions. The review article by May [4] includes a good deal of information on these matters, not only for equation (1.1) but also for other equations of a similar character.

In examining solutions of equation (1.1) the parameter b is simply a scale factor which can be eliminated by substituting $v_n = bu_n$. We can take this a step further by using $w_n = a - bu_n$, which gives the simpler equation

$$w_{n+1} = w_n^2 - A , \tag{1.2}$$

with $A = a^2 - a$. For $a > 1/2$, A increases monotonically with a and can be used as an appropriate parameter instead of a in discussing periodic solutions. This was done in a previous paper [1] in establishing equations for periodic solutions with periods 2, 3, 4, 5 and 6 and the present paper extends these results to solutions with period 7. The method used is essentially the same and the notation follows that of the previous paper.

For a solution of period 7, we want to have $w_{n+7} = w_n$ for $n = 0, 1, 2, \ldots$ and it is easy to see that

w_{n+2} is a polynomial of degree 2^2 in w_n ,

w_{n+3} is a polynomial of degree 2^3 in w_n ,

....

w_{n+7} is a polynomial of degree 2^7 in w_n .

Thus the condition $w_{n+7} = w_n$ gives

(1.3) $$G_7(w_n) = w_{n+7} - w_n = 0 ,$$

a polynomial equation of degree 128 in w_n . Since the condition $w_{n+7} = w_n$ is satisfied by the two equilibrium solutions, we can write

(1.4) $$G_7(w_n) = (w_n - a)(w_n - 1 + a)H_7(w_n)$$

where H_7 is a polynomial of degree 126 in w_n . (The solution $w_n = a$ corresponds to $u_n = 0$ and the solution $w_n = 1 - a$ corresponds to $u_n = (2a-1)/b$.) Now if $b_1, b_2, b_3, \ldots, b_7$ are the values of w_n for a solution of period 7 , with distinct elements, this solution must contribute a factor

(1.5) $$h_7(x) = \prod_{i=1}^{7} (x-b_i) = x^7 - \alpha x^6 + \beta x^5 - \gamma x^4 + \delta x^3 - \varepsilon x^2 + \theta x - \zeta$$

to the polynomial $H_7(x)$ and at most there will be 18 factors of this type. Equation (1.5) defines $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \zeta$ as symmetrical functions of b_1 to b_7 and the functions $\beta, \gamma, \delta, \varepsilon, \theta, \zeta$ can be determined when α and A are specified. If we refer to a solution of this type as a $C7$ solution, once a suitable value of α is known (for a given A) the polynomial $h_7(x)$ can be constructed and solving $h_7(x) = 0$ gives the b_i values for this $C7$ solution. The problem of finding suitable values of α is dealt with by showing that α satisfies a polynomial equation of degree 18 .

Section 2 deals with the problem of finding $\beta, \gamma, \delta, \varepsilon, \theta, \zeta$ when α and A are specified and Section 3 establishes the polynomial equation for

α . In obtaining these equations it is assumed that $\alpha^2 + \alpha - A \neq 0$ and the main problem in Section 4 is to see what modifications are required in the special case where $\alpha^2 + \alpha - A = 0$. Finally, some numerical results are given in Section 5.

2. Relationships for $C7$ solutions

The equations for the elements b_i in a $C7$ solution are

(2.1) $$b_{i+1} = b_i^2 - A \ , \quad i = 1, 2, 3, \ldots, 7 \ ,$$

where $b_8 = b_1$. From the theory of equations, any symmetrical function of the b's can be expressed in terms of $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \zeta$ and we try to use equation (2.1) to obtain an alternative form. In doing this it is convenient to introduce the symbol Σ_0 to denote cyclic summation over the indices 1 to 7 (because equations (2.1) have cyclic symmetry). For example, we can write

(2.2) $$\alpha^2 - 2\beta = \Sigma_0 b_1^2 = \Sigma_0 \left(b_2 + A\right) = \alpha + 7A \ ,$$

since $\Sigma_0 b_2 = \Sigma_0 b_1 = \alpha$. Similar working gives

(2.3) $$\alpha^3 - 3\alpha\beta + 3\gamma = \Sigma_0 b_1^3 = \Sigma_0 b_1 \left(b_2 + A\right) = \beta_1 + A\alpha \ ,$$

where

(2.4) $$\beta_1 = \Sigma_0 b_1 b_2 = b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_5 + b_5 b_6 + b_6 b_7 + b_7 b_1 \ .$$

In the same way,

(2.5) $$\alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta = \Sigma_0 b_1^4 = \Sigma_0 \left(b_2 + A\right)^2 = (\alpha + 7A) + 2A\alpha + 7A^2 \ ,$$

(2.6) $$\alpha^5 - 5\alpha^3\beta - 5\alpha\delta + 5\alpha^2\gamma + 5\alpha\beta^2 - 5\beta\gamma + 5\varepsilon = \Sigma_0 b_1^5$$
$$= \Sigma_0 b_1 \left(b_2 + A\right)^2 = \Sigma_0 b_1 \left[b_3 + A + 2Ab_2 + A^2\right] = \beta_2 + 2A\beta_1 + \left(A + A^2\right)\alpha \ ,$$

where $\beta_2 = \Sigma_0 b_1 b_3$. Note that we can write $\beta = \beta_1 + \beta_2 + \beta_3$, where $\beta_3 = \Sigma_0 b_1 b_4$.

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Two similar equations which bring in θ and ζ are

$$(2.7)\quad \alpha^6 - 6\alpha^4\beta + 6\alpha^3\gamma - 6\alpha^2\delta - 12\alpha\beta\gamma + 6\beta\delta + 6\alpha\varepsilon - 6\theta + 9\alpha^2\beta^2 - 2\beta^3 + 3\gamma^2$$
$$= \Sigma_0 b_1^6 = \Sigma_0 (b_2+A)^3 = \beta_1 + 4A\alpha + 3A^2\alpha + 21A^2 + 7A^3 ,$$

$$(2.8)\quad \alpha^7 - 7\alpha^5\beta + 7\alpha^4\gamma - 7\alpha^3\delta + 14\alpha^3\beta^2 - 21\alpha^2\beta\gamma + 7\alpha^2\varepsilon + 14\alpha\beta\delta - 7\alpha\beta^3$$
$$+ 7\alpha\gamma^2 - 7\alpha\theta + 7\zeta + 7\beta^2\gamma - 7\beta\varepsilon - 7\gamma\delta$$
$$= \Sigma_0 b_1^7 = \gamma_1 + (A+3A^2)\beta_1 + 3A\beta_2 + (3A^2+A^3)\alpha ,$$

where $\gamma_1 = \Sigma_0 b_1 b_2 b_3$. It will be seen that equations (2.2), (2.3), (2.5), (2.6), (2.7) and (2.8) can be solved to give, in turn, expressions for $\beta, \gamma, \delta, \varepsilon, \theta, \zeta$, although they involve β_1, β_2 and γ_1 as well as α and A . Thus we require equations which allow β_1, β_2 and γ_1 to be evaluated from α and A .

The method used to isolate β_1, β_2 and γ_1 (described below) involved a fair amount of detailed algebra and it is possible that a shorter method could be devised. As defined by equation (1.5), γ and δ consist of 35 terms while β and ε have 21 terms. They can be subdivided into groups of 7 terms, each with cyclic symmetry, by writing

$$(2.9)\quad \gamma_2 = \Sigma_0 b_1 b_2 b_4 ,\quad \gamma_3 = \Sigma_0 b_1 b_2 b_5 ,\quad \gamma_4 = \Sigma_0 b_1 b_2 b_6 ,\quad \gamma_5 = \Sigma_0 b_1 b_3 b_5 ,$$

$$(2.10)\quad \delta_1 = \Sigma_0 b_1 b_2 b_3 b_4 ,\quad \delta_2 = \Sigma_0 b_1 b_2 b_3 b_5 ,\quad \delta_3 = \Sigma_0 b_1 b_2 b_3 b_6 ,$$
$$\delta_4 = \Sigma_0 b_1 b_2 b_4 b_5 ,\quad \delta_5 = \Sigma_0 b_1 b_2 b_4 b_6 ,$$

$$(2.11)\quad \varepsilon_1 = \Sigma_0 b_1 b_2 b_3 b_4 b_5 ,\quad \varepsilon_2 = \Sigma_0 b_1 b_2 b_3 b_4 b_6 ,\quad \varepsilon_3 = \Sigma_0 b_1 b_2 b_3 b_5 b_6 .$$

Then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5$, $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5$, and $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. (The sub-division of β has already been described.) If we form the products $\alpha\beta_i, \alpha\gamma_i, \alpha\delta_i$ and $\alpha\varepsilon_i$ and make use of equation (2.1), we get a total of 16 equations relating the 16 quantities $\beta_i, \gamma_i, \delta_i$ and ε_i and the solution of these equations gives expressions for $\beta_1, \beta_2, \beta_3$; $\gamma_1, \ldots, \gamma_5$; $\delta_1, \ldots, \delta_5$; $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Typical equations are:

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(2.12) $\alpha\beta_1 = \alpha(1+2A) + 7A + \beta_2 + \gamma + \gamma_1 - \gamma_5$,

(2.13) $\alpha\gamma_2 = A(\alpha+\beta) + \beta_1 + \gamma_3 + \gamma_4 - \delta_3 + \delta$,

(2.14) $\alpha\delta_4 = A(2\beta_1+\gamma_2+2\gamma_3+\gamma_4) + \gamma_1 + \gamma_4 + \delta_3 + \delta_5 + \varepsilon_1 + 2\varepsilon_3$,

(2.15) $\alpha\varepsilon_1 = A(7+2\alpha+2\beta_1+\beta_3+2\gamma_1+2\gamma_3+\delta_1-\delta_5+\delta) + \alpha + \beta_2 + \gamma_4 + \delta_3 + \varepsilon_2 + 2\theta$.

The equations are linear and we can

(i) use the $\alpha\gamma_i$ equations to solve for the δ's (in terms of γ_i, β_i, α, A),

(ii) use the $\alpha\delta_1$, $\alpha\delta_4$ and $\alpha\delta_5$ equations to solve for ε_1, ε_2 and ε_3 ,

(iii) use the $\alpha\delta_2$, $\alpha\delta_3$ and $\alpha\beta_i$ equations to solve for the γ's ,

(iv) use the $\alpha\varepsilon_i$ equations to give two equations for β_1 and β_3 .

Parts (i) and (ii) of this procedure are straightforward. In part (iii) a key equation is

$$(2.16)\quad (\alpha^2+\alpha-A)(\gamma_2-\gamma_4) = (\alpha^2-\alpha-A)(\beta_3-\beta_2) + 2\alpha(\beta_1-2\beta_2) + 4\alpha A + 2\gamma + 4\beta_1 - 2\beta_2 + 2\beta_3 .$$

Provided $\alpha^2 + \alpha - A \neq 0$, this gives $\gamma_2 - \gamma_4$ in terms of α, A, β_1, β_2, β_3 and we can go on to obtain similar solutions for γ_i, δ_i, ε_i and θ . In particular, the solution for γ_1 can be written as

$$(2.17)\quad 90C\gamma_1 = \{3\alpha^5-5\alpha^4-(50+60A)\alpha^3+(15-250A)\alpha^2+(-15-13A+57A^2)\alpha+(105A^2-105A)\} + \beta_1\{30\alpha^3+30\alpha^2-(10+30A)\alpha-10-96A\} + \beta_2(30\alpha-18) + \beta_3\{-30\alpha^3+(60+30A)\alpha-90-30A\} ,$$

where $C = \alpha^2 + \alpha - A$. This solution can be written in different ways, since $\beta_1 + \beta_2 + \beta_3 = \beta = (1/2)(\alpha^2-\alpha-7A)$. For example, the term in β_2 can be eliminated by replacing β_2 by $\beta - \beta_1 - \beta_3$. This is what was done in part (iv) of the procedure to obtain equations relating β_1 and β_3 . The two equations are

$$0 = p_0 + p_1\beta_1 + p_3(3\beta_3) , \tag{2.18}$$

$$0 = q_0 + q_1\beta_1 + q_3(3\beta_3) , \tag{2.19}$$

with

$$\begin{aligned} p_0 = {} & 15\alpha^7 + 15\alpha^6 - (115+597A)\alpha^5 - (635+1200A)\alpha^4 \\ & + (-1300-2285A+3165A^2)\alpha^3 + (420-1594A+4875A^2)\alpha^2 \\ & + (240+5824A+3072A^2-2583A^3)\alpha + (1680A+588A^2+630A^3) , \end{aligned} \tag{2.20}$$

$$\begin{aligned} p_1 = {} & 45\alpha^6 - 45\alpha^5 - (135+495A)\alpha^4 - (195+630A)\alpha^3 \\ & + (1090-675A+855A^2)\alpha^2 + (2840+1620A+675A^2)\alpha \\ & + (2560-792A+2754A^2-405A^3) , \end{aligned} \tag{2.21}$$

$$\begin{aligned} p_3 = {} & -15\alpha^6 + 15\alpha^5 + (-155+165A)\alpha^4 - (95+270A)\alpha^3 \\ & + (290-55A-285A^2)\alpha^2 + (1160+60A+255A^2)\alpha + (376A-150A^2+135A^3) , \end{aligned} \tag{2.22}$$

$$\begin{aligned} q_0 = {} & -45\alpha^8 - 135\alpha^7 + (-405+810A)\alpha^6 + (-793+1863A)\alpha^5 \\ & + (-1430+4350A-4320A^2)\alpha^4 + (-760+9115A-8505A^2)\alpha^3 \\ & + (384+18506A-14115A^2+6390A^3)\alpha^2 + (1536+19432A-13140A^2+6777A^3)\alpha \\ & + (10752A-8232A^2+9450A^3-2835A^4) , \end{aligned} \tag{2.23}$$

$$\begin{aligned} q_1 = {} & 135\alpha^6 + 585\alpha^5 + (1275-1485A)\alpha^4 + (3015-6210A)\alpha^3 \\ & + (8950-10905A+2565A^2)\alpha^2 + (14120-5940A+5625A^2)\alpha \\ & + (9472-2856A+7974A^2-1215A^3) , \end{aligned} \tag{2.24}$$

(2.25) $q_3 = 45\alpha^6 + 195\alpha^5 + (625-495A)\alpha^4 + (1165-1590A)\alpha^3$

$$+ (2330-3355A+855A^2)\alpha^2 + (2600-2580A+1395A^2)\alpha$$

$$+ (2304-1064A+1890A^2-405A^3) .$$

These equations depend on the assumption that $C \neq 0$. For $p_1q_3 - p_3q_1 \neq 0$, the equations can be solved for β_1 and β_3 and then $\beta_2 = \beta - \beta_1 - \beta_3$.

Thus for given values of α and A , we can evaluate $\beta, p_0, p_1, p_3, q_0, q_1, q_3$ and (in general) solve equations (2.18) and (2.19) for β_1 and β_3 . The value of β_2 follows and equation (2.17) gives γ_1 . Equations (2.3), (2.5), (2.6), (2.7) and (2.8) can be used to obtain $\gamma, \delta, \varepsilon, \theta$ and ζ in turn and this defines $h_7(x)$. The roots of $h_7(x) = 0$ correspond to the values of b_i for the solution and equation (2.1) can be used to put them in order. The stability of the solution is determined by $\zeta = b_1b_2b_3 \dots b_7$, since

(2.26) $$\frac{dw_{n+7}}{dw_n} = \prod_{i=0}^{6} \frac{dw_{n+i+1}}{dw_{n+i}} = 2^7 \prod_{i=0}^{6} w_{n+i} = 2^7\zeta$$

for a $C7$ solution. For local stability, we must have $2^7|\zeta| < 1$, that is,

(2.27) $$-1/128 < \zeta < 1/128 .$$

3. Equation for α

To obtain an equation for α we need an additional relationship between $\alpha, \beta, \gamma, \delta, \varepsilon, \theta, \zeta$ and we would like to have it in a form which allows us to eliminate β_1 and β_3 . Thus a suitable form is an equation similar to (2.18) and (2.19), for we can then write down the equation for α in determinantal form. There are a number of relationships that can be used and the eventual choice was somewhat arbitrary.

One neat-looking set of results comes from eliminating A in equation (2.1). For example, (3.1) $$b_3 - b_2 = \left(b_2^2-A\right) - \left(b_1^2-A\right) = b_2^2 - b_1^2$$

and hence

(3.2) $$\Pi_0\left(b_3-b_2\right) = \Pi_0\left(b_2^2-b_1^2\right) = \Pi_0\left(b_2-b_1\right)\Pi_0\left(b_2+b_1\right) ,$$

where Π_0 is used to denote a cyclic product, that is

$$\Pi_0\left(b_2-b_1\right) = \left(b_2-b_1\right)\left(b_3-b_2\right)\left(b_4-b_3\right) \ldots \left(b_1-b_7\right) = \Pi_0\left(b_3-b_2\right) .$$

For a $C7$ solution the b_i are distinct and hence $\Pi_0\left(b_2-b_1\right) \neq 0$. Thus equation (3.2) gives

(3.3) $$1 = \Pi_0\left(b_1+b_2\right) .$$

A similar argument leads to

(3.4) $$1 = \Pi_0\left(b_1+b_3\right) ,$$

(3.5) $$1 = \Pi_0\left(b_1+b_4\right) .$$

When the right-hand side of equation (3.3) is expanded, we can replace b_1^2 by $b_2 + A$, and so on, until only first powers of each b_i are left. This gives

(3.6) $$1 = (\alpha+\beta+\gamma+\delta+\varepsilon+\theta+2\zeta) + A(7+6\alpha+5\beta+4\gamma+3\delta+2\varepsilon) + A^2(14+9\alpha+5\beta+2\gamma) + A^3(7+2\alpha)$$

and equations (3.4) and (3.5) give the same result.

Equation (3.6) is clearly a useful equation, for example, for checking the numerical results obtained from the equations in Section 2, but it is less convenient to use it directly to give a relation between β_1 and β_3 . The inconvenience stems from equations (2.7) and (2.8), which involve γ^2 . There is no difficulty in determining θ and ζ numerically from these equations but in the algebraic solution the γ^2 term includes β_1^2 in its expansion and we would prefer linear terms only. (This difficulty can be circumvented but there was no need to to so in this case.)

Another useful set of equations comes from forming the products

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$\alpha\theta$, $\alpha\varepsilon$, $\alpha\delta$, ..., $\alpha\beta$. Indeed some of these equations were used in checking the results for $\alpha\beta_i$, $\alpha\gamma_i$, $\alpha\delta_i$ and $\alpha\varepsilon_i$. Using equation (2.1), these products become

$$(3.7)\quad \alpha\beta = 3\gamma + 2\beta + \alpha - \beta_1 + A(6\alpha+7) ,$$

$$(3.8)\quad \alpha\gamma = 4\delta + 2\gamma + \beta + \alpha + \gamma_5 - \gamma_1 + \beta_3 + A(5\beta+5\alpha+7) ,$$

$$(3.9)\quad \alpha\delta = 5\varepsilon + 2\delta + \gamma + \beta + \alpha + \delta_5 - \delta_1 + \gamma_3 + A(4\gamma+3\beta+4\alpha+7+\beta_1) ,$$

$$(3.10)\quad \alpha\varepsilon = 6\theta + 2\varepsilon + \delta + \gamma + \beta + \alpha - \varepsilon_1 - \delta_5 - \gamma_3 - \gamma_5 - \beta_3 + A(3\delta+2\gamma+2\beta+3\alpha+7+\gamma_1-\gamma_5+\beta_1-\beta_3) ,$$

$$(3.11)\quad \alpha\theta = 7\zeta + \theta + \alpha + \varepsilon_1 + \delta_1 + \gamma_1 + \beta_1 + A(2\varepsilon+\delta+\gamma+\beta+2\alpha+7) + A(\delta_1-\delta_5+\gamma_1-\gamma_3-\gamma_5+\beta_1-\beta_3) .$$

From equation (2.2), $\alpha^2 = 2\beta + \alpha + 7A$ and hence

$$(3.12)\quad \alpha(\theta+\varepsilon+\delta+\gamma+\beta+\alpha+1) = 7(\zeta+\theta+\varepsilon+\delta+\gamma+\beta+\alpha) + A(2\varepsilon+4\delta+7\gamma+11\beta+20\alpha+42) + A(\delta_1-\delta_5+2\gamma_1-\gamma_3-2\gamma_5+3\beta_1-2\beta_3) .$$

It can be shown that

$$(3.13)\quad \delta_1 - \delta_5 + 2\gamma_1 - \gamma_3 - 2\gamma_5 + 3\beta_1 - 2\beta_3 = (1/12)\left[\beta_1\left(6\alpha^2+22\alpha+44-18A\right)-12\beta_3-\alpha^4-2\alpha^3+(-3+10A)\alpha^2 -(18+22A)\alpha-126A-21A^2\right] ,$$

which means that the final term in equation (3.12) causes no difficulties. What was done, in effect, was to eliminate ζ between equations (3.6) and (3.12) and replace θ, ε, δ, γ and β by their expansions in terms of α, A, β_1 and β_3 . This gave

$$(3.14)\quad 0 = (2\alpha-7)\left\{r_0+r_1\beta_1+r_3\left(3\beta_3\right)\right\} ,$$

with

$$(3.15)\quad r_0 = 9\alpha^8 - 72\alpha^7 + (6-324A)\alpha^6 + (-4+576A)\alpha^5 + \left(-347-2424A+2646A^2\right)\alpha^4 + \left(476-8432A+792A^2\right)\alpha^3 + \left(7404-10108A+7278A^2-4356A^3\right)\alpha^2 + \left(6864-3224A+11772A^2-1296A^3\right)\alpha + \left(-3792A+4476A^2-4860A^3+2025A^4\right) ,$$

(3.16) $r_1 = 8\{45\alpha^5+75\alpha^4+(33-306A)\alpha^3+(101-552A)\alpha^2$

$+(202-72A+261A^2)\alpha+(296+30A+261A^2)\}$,

(3.17) $r_3 = 8\{5\alpha^4-34\alpha^3-(53+50A)\alpha^2+(34-6A)\alpha+(72+14A+45A^2)\}$.

The factor $2\alpha - 7$ in equation (3.14) corresponds to the case where $b_i = (1/2)$ for $i = 1, 2, \ldots, 7$. This is an equilibrium solution corresponding to $a = 1/2$ and $A = 1/4$ and it is easy to check that equations (3.3), (3.4), (3.5) and (3.12) are satisfied in this case. Thus we can expect it to emerge as a possible solution of the equations but since we have stipulated that $a > 1/2$ we can ignore this solution. This leaves

(3.18) $$0 = r_0 + r_1\beta_1 + r_3(3\beta_3) .$$

As with equations (2.18) and (2.19), equation (3.18) depends on the condition that $C = \alpha^2 + \alpha - A \neq 0$. For the three equations to be consistent, we must have

(3.19) $$0 = E(\alpha, A) = \begin{vmatrix} p_0 & p_1 & p_3 \\ q_0 & q_1 & q_3 \\ r_0 & r_1 & r_3 \end{vmatrix}$$

and this is (almost) the equation we want for α . An examination of the leading terms shows that $E(\alpha, A)$ is a polynomial of degree 20 in α rather than a polynomial of degree 18 . Numerical checks gave $E(0, 0) = 0 = E(-1, 0)$ and $E(1, 2) = 0 = E(-2, 2)$, which suggests that $\alpha^2 + \alpha - A$ might be a factor and this was verified by expressing p_0 as

(3.20) $$p_0 = Cp_0^{(1)} + p_0^{(2)} ,$$

where $p_0^{(2)}$ is linear in α . Similar expressions were obtained for p_1, p_3 ; q_0, q_1, q_3 ; r_0, r_1, r_3 and the determinant obtained from the remainder terms, that is,

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$$\begin{vmatrix} p_0^{(2)} & p_1^{(2)} & p_3^{(2)} \\ q_0^{(2)} & q_1^{(2)} & q_3^{(2)} \\ r_0^{(2)} & r_1^{(2)} & r_3^{(2)} \end{vmatrix} ,$$

proved to have a factor C . With this assurance that C is a factor of $E(\alpha, A)$ brute force was used to expand $E(\alpha, A)$ and write it in the form

(3.21) $$E(\alpha, A) = (36450)\left(\alpha^2+\alpha-A\right)D(\alpha, A) ,$$

where

(3.22) $$D(\alpha, A) = \sum_{n=0}^{18} k_n\alpha^n ,$$

with

(3.23) $$k_{18} = k_{17} = 1 , \quad k_{16} = 4 - 57A , \quad k_{15} = 20 - 136A ,$$

$$k_{14} = 110 - 380A + 1188A^2 , \quad k_{13} = 638 - 1048A + 3740A^2 ,$$

$$k_{12} = 3828 - 2054A + 9652A^2 - 11924A^3 ,$$

$$k_{11} = 10452 - 824A + 23660A^2 - 42168A^3 ,$$

$$k_{10} = 27225 - 42796A + 53404A^2 - 100588A^3 + 61950A^4 ,$$

$$k_9 = 60665 - 113352A - 22652A^2 - 187504A^3 + 228070A^4 ,$$

$$k_8 = 120032 - 272825A + 186604A^2 - 468700A^3 + 490988A^4 - 168606A^5 ,$$

$$k_7 = 195632 - 602304A + 439044A^2 + 148168A^3 + 537772A^4 - 531896A^5 ,$$

$$k_6 = 494368 - 1254960A + 311856A^2 + 381644A^3 + 1402822A^4 - 1211764A^5 + 254932A^6 ,$$

$$k_5 = 886384 - 389280A - 1462000A^2 + 477600A^3 + 644702A^4 - 1176536A^5 + 638876A^6 ,$$

$$k_4 = 698944 + 1180624A + 110448A^2 - 2616528A^3 + 2721088A^4 - 3248366A^5 + 1558332A^6 - 217188A^7 ,$$

$$k_3 = -424704 + 976896A + 2347360A^2 - 3500032A^3 + 4361008A^4 - 3148288A^5 + 1377620A^6 - 376200A^7 ,$$

$$k_2 = -566272 - 2844416A + 6158720A^2 - 9309792A^3 + 9790784A - 6466416A^5 + 3379216A^6 - 997668A^7 + 97929A^8 ,$$

$$k_1 = 573440 - 3377152A + 5289728A^2 - 7828480A^3 + 6725680A^4 - 4708576A^5 + 2193936A^6 - 573984A^7 + 86913A^8 ,$$

$$k_0 = 1048576 - 2277376A + 4746240A^2 - 5809408A^3 + 6054976A^4 - 4572528A^5 + 2629296A^6 - 1116432A^7 + 251424A^8 - 18225A^9 .$$

For a given value of A the coefficients k_0 to k_{18} can be evaluated and the appropriate values of α are the real roots of $D(\alpha, A) = 0$.

4. Special case $C = 0$

It might appear that the special case $\alpha = 0$ requires attention because of the use of equations such as (2.12) to (2.15) or (3.7) to (3.11). However when a check was made this case produced no difficulties. For example, in the derivation of equation (2.12),

$$\begin{aligned}\alpha\beta_1 &= \left(\Sigma_0 b_1\right)\left(\Sigma_0 b_1 b_2\right) = \Sigma_0\left\{b_1\left(b_1 b_2 + b_2 b_3 + b_3 b_4 + b_4 b_5 + \dots b_7 b_1\right)\right\} \\ &= \Sigma_0 b_1^2\left(b_2+b_7\right) + \Sigma_0\left(b_1b_2b_3+b_3b_4b_1+b_4b_5b_1+b_5b_6b_1+b_6b_7b_1\right) \\ &= \Sigma_0\left(b_2+A\right)\left(b_2+b_7\right) + \gamma_1 + \gamma_4 + \gamma_3 + \gamma_2 + \gamma_1 \\ &= (\alpha+7A) + 2A\alpha + \beta_2 + \gamma + \gamma_1 - \gamma_5 .\end{aligned}$$

This expansion is still valid when $\Sigma_0 b_1 = 0$; the equation reduces to $0 = 7A + \beta_2 + \gamma + \gamma_1 - \gamma_5$. Checking the details in this way gave $k_0 = 0$ as the equation for A , in agreement with $D(0, A) = 0$.

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When $C = 0$, equation (2.16) is still valid but it no longer gives an equation for $\gamma_2 - \gamma_4$ and the solution for the γ_i breaks down. This invalidates most of the subsequent equations in Section 2 and it is better to make a fresh start, replacing α^2 by $A - \alpha$ wherever it occurs. Equations (2.2), (2.5) and (2.6) become

$$\beta = -\alpha - 3A \ , \quad 3\gamma = \alpha(2-9A) - 2A + \beta_1 \ ,$$

$$(4.1) \qquad 3\delta = \alpha(7A-2) + 9A^2 - 4A + \alpha\beta_1 \ ,$$

$$15\varepsilon = \alpha\left(7-41A+45A^2\right) + \left(21A^2-7A\right) + 3\beta_2 - (5\alpha+9A)\beta_1 \ .$$

The equations for $\alpha\beta_i$, $\alpha\gamma_i$, $\alpha\delta_i$ and $\alpha\varepsilon_i$ are still valid, so equation (2.16) can be used; hence

$$(4.2) \qquad 0 = 2\alpha\left(\beta_2-\beta_3\right) + 2\alpha\left(\beta_1-2\beta_2\right) + 4\alpha A + 2\gamma + 4\beta_1 - 2\beta_2 + 2\beta_3 \ .$$

If we replace γ from equation (4.1) and put $\beta_3 = \beta - \beta_1 - \beta_2$, we get

$$(4.3) \qquad 3\beta_2 = (3\alpha+2)\beta_1 + (3A-2)\alpha - 4A \ ,$$

and from this

$$(4.4) \qquad 3\beta_3 = -(3\alpha+5)\beta_1 - (3A+1)\alpha - 5A \ .$$

Thus we can put β_2 and β_3 in terms of β_1, α, A whenever they appear.

From the $\alpha\beta_1$ and $\alpha\beta_3$ equations it is straightforward to obtain

$$(4.5) \qquad \gamma_5 = \gamma_1 + \beta_1 + \alpha + 5A \ ,$$

$$\gamma_3 = \gamma_1 + (1/3)\left\{\beta_1(2-2\alpha-3A)+(A+2)\alpha+10A-3A^2\right\} ,$$

and hence $\gamma_2 + \gamma_4 = \gamma - \gamma_1 - \gamma_3 - \gamma_5$ can also be expressed in terms of γ_1, β_1, α, A. The main problem is to find an equation for $\gamma_2 - \gamma_4$ as a replacement for equation (2.16).

An unexpected bonus came from the equation for $\alpha\delta_3$, which is

$$(4.6) \qquad \alpha\delta_3 = A\left(\alpha+\beta+\gamma-\beta_1-\gamma_2\right) + \varepsilon + \delta_1 + \delta_5 + \gamma_5 + \beta_2 \ .$$

As mentioned in Section 2, it is easy to solve for the δ's in terms of γ_i, β_i, α, A and these solutions were substituted for δ_1, δ_3 and δ_5 in equation (4.6). The results in equations (4.1), (4.3), (4.4) and (4.5) were used to obtain an equation for β_1 and this gave

(4.7) $$\beta_1(4-3A+\alpha) = 3A^2 - 3A - 4\alpha A ,$$

a surprisingly neat result. Note that if the co-factor of β_1 is zero in this equation, then the right-hand side of the equation must be zero also and we either have

$$3A = 4 + \alpha , \text{ with } A = 0 ,$$

or

$$3A = 4 + \alpha , \text{ with } 3A = 4\alpha + 3 .$$

Neither of these is compatible with the assumption that $C = 0$. Thus we can take $Q = 4 - 3A + \alpha \neq 0$ and regard equation (4.7) as an equation for β_1, in terms of α and A. From equations (4.3), (4.4) and (4.1),

(4.8) $$Q\beta_2 = 3A^2 - 8A + (2A-2)\alpha , \quad Q\beta_3 = 3A^2 - 2A + (2A-1)\alpha ,$$

$$Q\gamma = -3A + \left(2-13A+9A^2\right)\alpha ,$$

$$Q\delta = -6A + 17A^2 - 9A^3 + \left(-2+8A-3A^2\right)\alpha ,$$

$$Q\varepsilon = -3A + 8A^2 - 3A^3 + \left(1-10A+20A^2-9A^3\right)\alpha .$$

To obtain similar equations for γ_1 to γ_5, the equations used were

(4.9) $$\beta_1^2 = 7A^2 + 4A\alpha + \beta_1 + 2A\beta_2 + 2\beta_3 + 2\left(\delta_1+\delta_4\right) ,$$

$$\beta_2^2 = 7A^2 + 2A\alpha + \beta_2 + 2A\beta_3 + 2\gamma_3 + 2\left(\delta_1+\delta_5\right) ,$$

$$\beta_3^2 = 7A^2 + 2A\alpha + \beta_3 + 2A\beta_1 + 2\gamma_4 + 2\left(\delta_4+\delta_5\right) .$$

These equations are valid in the general case and the β_1^2 equation can be used in evaluating γ^2, which occurs in equations (2.7) and (2.8). If we

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take $\beta_2^2 - \beta_3^2$ and substitute for δ_1 and δ_4 , the resulting equation is

$$(4.10)\quad \beta_2^2 - \beta_3^2 = 2(\beta_2-\beta_3) + \alpha(\gamma_1-\gamma_3) + (2\gamma_3+\gamma_5-\gamma_2-\gamma_4) - (1+A)\alpha - 7A + A(4\beta_3-3\beta_1-\beta_2) .$$

The essential point about this equation is that γ_3, γ_5 and $\gamma_2 + \gamma_4$ can be expressed in terms of γ_1 (plus terms in α, A, β_i) and in the special case $C = 0$ this leads to

$$(4.11)\qquad 2Q^2\gamma_1 = -44A + 72A^2 - 30A^3 + (-8+23A-40A^2+18A^3)\alpha .$$

Similar expressions for $2Q^2\gamma_3$, $2Q^2\gamma_5$ and $2Q^2(\gamma_2+\gamma_4)$ follow immediately and the β_1^2 equation gives $2Q^2(\gamma_2-\gamma_4)$. This gives a set of equations for the γ_i and solutions for δ_i, ε_i follow in a straightforward way.

Equations (2.7) and (2.8) gave

$$(4.12)\quad 2Q^2\theta = 8A + 56A^2 - 116A^3 + 76A^4 - 18A^5 + (-4+47A-78A^2+44A^3-6A^4)\alpha ,$$

$$(4.13)\quad 14Q^2\zeta = -54A + 505A^2 - 690A^3 + 308A^4 - 42A^5 + (2-76A+586A^2-922A^3+574A^4-126A^5)\alpha$$

and equation (3.11) was used as a check for these expressions. Thus for given values of α and A , if it turns out that C is zero or close to zero we have to switch from the equations in Section 2 and instead use equations (4.7), (4.8) or (4.1), (4.12) and (4.13) to calculate the coefficients in the polynomial $h_7(x)$. As before, the roots of $h_7(x) = 0$ give the elements of the $C7$ solution and the value of ζ tells whether the cycle is stable or unstable.

We also have to check on the equation for α in the case when $C = 0$. When the expressions obtained for β, γ, δ, ε, θ and ζ were substituted in equation (3.6) it was found that the condition

$$(4.14)\qquad 0 = 224 - 508A + 404A^2 - 104A^3 + (80-107A-10A^2+24A^3)\alpha$$

had to be satisfied for equation (3.6) to hold. Another equation that is available in the general case is

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$$(4.15) \qquad \alpha\zeta = \alpha + 7A + A\left(\theta+\varepsilon_1+\delta_1+\gamma_1+\beta_1+\alpha\right) ,$$

which is similar to equations (3.7) to (3.11), and this equation was also used as a check equation in the case $C = 0$. This gave a somewhat different condition

$$(4.16) \qquad 0 = 224 - 380A + 188A^2 - 24A^3 + \left(208-435A+310A^2-72A^3\right)\alpha .$$

Equations (4.14) and (4.16) can obviously be combined in different ways to obtain alternative conditions. For example, if we subtract equation (4.14) from equation (4.16) we eliminate the constant term and obtain, after cancelling a factor 8 ,

$$(4.17) \qquad 0 = 16A - 27A^2 + 10A^3 + \left(16-41A+40A^2-12A^3\right)\alpha .$$

In the same way, we can eliminate the $A^3\alpha$ term by adding three times equation (4.14) to equation (4.16). Again, a numerical factor can be cancelled and the result is

$$(4.18) \qquad 0 = 32 - 68A + 50A^2 - 12A^3 + \left(16-27A+10A^2\right)\alpha .$$

Equations (4.17) and (4.18) can be regarded as an alternative pair to equations (4.14) and (4.16). However, equations (4.17) and (4.18) are related, for if we multiply equation (4.18) by α and use $\alpha^2 = A - \alpha$ we obtain equation (4.17). Thus we can regard equation (4.18) as the basic equation for α , with equation (4.17) as a derived form and with equations (4.16) and (4.14) as linear combinations of equations (4.17) and (4.18).

This is a satisfactory conclusion since it agrees with the equation $D(\alpha, A) = 0$ obtained in Section 3. If we divide $D(\alpha, A)$ by $\alpha^2 + \alpha - A$ and write the remainder as $R(\alpha, A)$, then

$$(4.19) \qquad D(\alpha, A) = \left(\alpha^2+\alpha-A\right)S(\alpha, A) + R(\alpha, A)$$

where $S(\alpha, A)$ is a polynomial of degree 16 in α and of degree 8 in A and

$$(4.20) \qquad R(\alpha, A) = (32768)\left\{\left(32-68A+50A^2-12A^3\right)+\left(16-27A+10A^2\right)\alpha\right\} .$$

Thus when $C = 0$ the equation $D(\alpha, A) = 0$ reduces to equation (4.18).

5. Numerical results

Although periodic solutions exist for $A > 2$ and the equations obtained in Sections 2, 3 and 4 remain valid for $A > 2$, the results obtained by Chaundy and Phillips [2] indicate that the periodic solutions are unstable for $A > 2$. The numerical working has accordingly been limited to $-1/4 < A \leq 2$. For $A = 2$, Lorenz [3] gives an exact solution in terms of trigonometric functions and these trigonometric solutions were of great help in checking the equations of Sections 2 and 3. For $C7$ solutions, the trigonometric solutions are

(5.1) $$b_m = 2 \cos\left(2^m\phi\right) , \quad m = 1, 2, 3, \ldots, 7 ,$$

with

(5.2) $$\phi = N\pi/129 \quad \text{or} \quad \phi = N\pi/127 ,$$

for any integer N. In practice, all 18 of the $C7$ solutions were obtained by using in turn $N = 1, 3, 5, 7, 9, 11, 13, 19, 21$. The 18 values of α were compared with the roots of $D(\alpha, 2) = 0$ and the agreement was excellent. For any particular solution, combinations of the b's such as $\beta_1, \gamma_1, \beta, \gamma, \delta, \varepsilon, \theta, \zeta$ can be calculated and compared with the values given by the formulae in Section 2.

It is fairly easy to show that, with the above values for N, $\zeta = +1$ when $\phi = N\pi/127$ and $\zeta = -1$ when $\phi = N\pi/129$. If $z = \exp(4i\phi)$ then

$$1 + z = (2 \cos 2\phi)\exp(2i\phi) = b_1 \exp(2i\phi)$$

$$1 + z^2 = (2 \cos 4\phi)\exp(4i\phi) = b_2 \exp(4i\phi)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(1+z^{64}\right) = (2 \cos 128\phi)\exp(128i\phi) = b_7 \exp(128i\phi) .$$

Also,

$$1 - z = -2i(\sin 2\phi)\exp(2i\phi) ,$$

$$1 - z^{128} = -2i(\sin 256\phi)\exp(256i\phi) .$$

The identity

$$(1+z)\left(1+z^2\right)\left(1+z^4\right) \ldots \left(1+z^{64}\right) = \left(1-z^{128}\right)/(1-z) ,$$

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which is valid for $z \neq 1$, now gives

$$\zeta \exp(254i\phi) = \{(\sin 256\phi)/(\sin 2\phi)\}\exp(254i\phi)$$

and hence

(5.3) $$\zeta = (\sin 256\phi)/(\sin 2\phi) .$$

For $\phi = N\pi/129$, $256\phi = 2N\pi - 2\phi$ and $\zeta = -1$. For $\phi = N\pi/127$, $256\phi = 2N\pi + 2\phi$ and $\zeta = +1$.

A corresponding result was noted [1] in discussing the $C6$ solutions for $A = 2$ and the proof above can easily be modified to deal with that case. Indeed it appears that for $A = 2$ the only cyclic solution which does not have $|\zeta| = 1$ is the equilibrium solution $w_n = 2$.

In looking at the solutions of $D(\alpha, A) = 0$ for different values of A it was clear that the real solutions occurred in pairs. For each pair there was a critical value of A, denoted by A^*, at which the pair first appeared. At $A = A^*$, there was a double root $\alpha = \alpha^*$ and the stability criterion, ζ, equalled $1/128$, that is the solution was at the upper limit for local stability. For $A > A^*$ the double root split into two distinct roots, with $\zeta < 1/128$ in one case and $\zeta > 1/128$ in the other. Thus one family of α values gives unstable solutions while the other gives stable solutions until ζ becomes less than $-1/128$, say for $A^* < A < A^{**}$ where A^{**} is the value of A at which $\zeta = -1/128$. Table 1 lists the critical values A^* and A^{**} for each pair of roots, with the roots numbered in order of increasing magnitude. The table also shows which root in each pair gives stable solutions and, in the last column, values of b_i for a typical solution. In each case, the trigonometric solution (for $A = 2$) corresponding to the stable sequence was used to provide these typical values. It will be seen that there is a marked change in the type of solution as α^* increases. A rough figure is given for the interval of stability, $A^{**} - A^*$, and it will be seen that it varies by a factor of about 3000 from the widest to the narrowest interval.

The solutions for the case $A = 2$ can be paired off in the same way as the pairs of roots in Table 1. For example, $\phi = 21\pi/127$ corresponds to α_1 and $\phi = 21\pi/129$ corresponds to α_2. The values of ζ for the

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TABLE 1

Critical values for *C7* solutions

Roots	Stable sequence	α^*	A^* / A^{**}	$A^{**} - A^*$	Values of b_1 to b_7 for typical solution
α_1, α_2	α_2	-4.259071	1.574 715 705 1.575 410 314	6.95*E* - 4	1.042, -0.914, -1.164, -0.646, -1.583, +0.506, -1.744
α_3, α_4	α_3	-3.134947	1.673 954 010 1.674 401 209	4.45*E* - 4	1.203, -0.553, -1.694, +0.871, -1.242, -0.459, -1.790
α_5, α_6	α_5	-2.488669	1.884 792 617 1.884 836 415	4.4*E* - 5	1.720, +0.958, -1.083, -0.827, -1.316, -0.267, -1.929
α_7, α_8	α_8	-2.066388	1.832 290 566 1.832 389 049	9.85*E* - 5	1.612, +0.599, -1.641, +0.692, -1.522, +0.315, -1.901
α_9, α_{10}	α_{10}	+0.363024	1.927 140 613 1.927 169 005	2.85*E* - 5	1.811, +1.279, -0.363, -1.868, +1.490, +0.219, -1.952
α_{11}, α_{12}	α_{12}	+0.652948	1.977 178 092 1.977 184 071	6.0*E* - 6	1.941, +1.767, +1.124, -0.737, -1.457, +0.122, -1.985
α_{13}, α_{14}	α_{13}	1.347039	1.953 701 970 1.953 717 660	1.55*E* - 5	1.885, +1.553, +0.411, -1.831, +1.353, -0.170, -1.971
α_{15}, α_{16}	α_{15}	2.904726	1.991 813 660 1.991 815 710	2.05*E* - 6	1.979, +1.915, +1.668, +0.782, -1.338, -0.073, -1.995
α_{17}, α_{18}	α_{18}	7.210548	1.999 095 627 1.999 095 848	2.2*E* - 7	1.998, +1.991, +1.962, +1.850, +1.423, +0.024, -1.999

trigonometric solutions provide a check on the identification of the stable family of roots, in the sense that for the α_1 family ζ goes from $+1/128$ at $A = A^*$ to $+1$ at $A = 2$, whereas for the α_2 family ζ goes from $1/128$ at $A = A^*$ to -1 at $A = 2$. Thus we can expect the α_2 family to have $|\zeta| < 1/128$ for some range of A-values. A similar argument applies for the other pairs of roots.

As A increases, the first $C7$ solution to occur is for $A = 1.575$ (which corresponds to $a = 1.851$). It looks like a perturbation of the equilibrium solution $w_n = 1 - a$ and, as was true for $C3$, $C4$, $C5$ and $C6$ solutions, the smallest value of α^* is associated with this smallest value of A^*. However, there is numerical evidence that this association does not extend to $C8$ solutions. For the latter, the smallest value of α^* appears to arise from a solution which is a perturbation of the equilibrium solution but the smallest value for A^* is for a $C8$ solution which arises when a $C4$ solution becomes unstable and bifurcates.

For the special case $C = 0$, we have

$$\alpha^2 + \alpha = A = a^2 - a$$

and hence $\alpha = -a$ or $\alpha = a - 1$. If we start with a specified value for the constant a, with $1/2 < a \le 2$, it is easy to calculate A and then

$$\alpha_0 = \left(12A^3-50A^2+68A-32\right)/\left(10A^2-27A+16\right) . \tag{5.4}$$

From equation (4.18), α_0 is the value of α appropriate to the case $C = 0$ and we can compare it with $-a$ and $a - 1$. The calculations gave $\alpha_0 = -a$ for $a = 1.9594473$, corresponding to $A = 1.879986$, so the equations in Section 4 would be appropriate in this case. The other possibility, $\alpha_0 = a - 1$, did not occur for the range of values tested.

In concluding this section I should like to thank Dr B.L. Martin who carried out the calculations which form the basis of Table 1 and who checked a number of other points numerically. Because of the large powers of α required in calculating $D(\alpha, A)$ double precision was needed in the computer programmes and the values for A^* and A^{**} were obtained to an accuracy of at least 10^{-12}, although not all of the figures have been included in Table 1.

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Department of Mathematics,
Faculty of Science,
Australian National University,
PO Box 4,
Canberra,
ACT 2600,
Australia.