PRINCIPAL RIGHT IDEAL RINGS

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This paper is concerned with the recent work of A. W. Goldie **(1)** on principal right ideal rings (p.r.i. rings). We shall prove some of his main structure theorems using the methods of **(3)** and **(4)**, and in so doing shall weaken some of his hypotheses.

1. Basic lemmas. A p.r.i. ring is a ring with unity having the property that every right ideal is principal. If R is a p.r.i. ring and A is an ideal of R, then R/A also is a p.r.i. ring. A finite direct sum of p.r.i. rings clearly is a p.r.i. ring. One of the reasons that the structure of a p.r.i. ring R can be determined is that its lattice $L_r(R)$ of right ideals satisfies the ascending chain condition (a.c.c.). Of course, the lattice $L(R)$ of ideals of R also satisfies the a.c.c. On the other hand, the lattice $L_l(R)$ of left ideals of R need not satisfy the a.c.c.

Some of the following lemmas may be found in Goldie's notes **(1)**. We have included them here so that our paper will be essentially self-contained.

1.1. LEMMA. *If R is a p.r.i. ring and aR, $bR \in L(R)$, then $aR\cdot bR = abR$.*

Proof. Evidently $Rb \subset bR$, so that $aR\cdot bR \subset abR$. On the other hand, $ab \in aR\cdot bR$ and therefore $abR \subset aR\cdot bR$.

1.2. COROLLARY. *If R is a p.r.i. ring and $aR \in L(R)$, then $(aR)^n = a^nR$ for every positive integer n.*

1.3. LEMMA. *If the a.c.c. holds for the set of annihilating right ideals of a ring R, then for every $a \in R$ there exists a positive integer k such that*

$$(a^n)^r \cap a^mR = 0$$

for all integers $n > 0$ and $m \geqslant k$.

Proof. Let us select k so that $(a^k)^r = (a^n)^r$ for every $n \geqslant k$. If $x \in (a^k)^r \cap a^kR$, then $x = a^ky$ for some $y \in R$ and $a^kx = 0$. Hence, $a^{2k}y = 0$ and $a^ky = 0$ by the choice of k. Therefore, $(a^k)^r \cap a^kR = 0$. Since $(a^n)^r \subset (a^k)^r$ for every $n > 0$, and $a^mR \subset a^kR$ if $m \geqslant k$, we have proved 1.3.

We shall use the notation A^r, as in the proof above, to designate the right annihilator of set A in ring R; and A^l for the left annihilator of A. We shall let $L_r^{\Delta}(R)$ designate the lattice of all large right ideals of R, and R_r^{Δ} designate

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the right singular ideal of R. Thus, $A \in L_r^{\Delta}(R)$ if and only if $A \in L_r(R)$ and $A \cap B \neq 0$ for every non-zero $B \in L_r(R)$, and $a \in R_r^{\Delta}$ if and only if $a^r \in L_r^{\Delta}(R)$. The following lemma is a partial converse of **(5**, 3.5**)**.

1.4. LEMMA. *If the ring R is not semi-prime, then there exists some* $C \in L_r^{\Delta}(R)$ *such that* $C^r \neq 0$.

Proof. If R is not semi-prime, there exists some non-zero $A \in L(R)$ such that $A^2 = 0$. If B is a complement of A in $L_r(R)$ (that is, B is a maximal element of the set $\{D \in L_r(R) \mid D \cap A = 0\}$), then $C \in L_r^{\Delta}(R)$, where $C = A + B$. Clearly $BA \subset B \cap A = 0$, and therefore $CA = 0$.

1.5. LEMMA. *If R is a p.r.i. ring, then* $R_r^{\Delta} = 0$ *if and only if R is semi-prime.*

Proof. If R is semi-prime and the a.c.c. holds for $L_r(R)$, then $R_r^{\Delta} = 0$ by **(2**, 3.2**)**. Conversely, if R is a p.r.i. ring for which $R_r^{\Delta} = 0$ and if $C \in L_r^{\Delta}(R)$, then $C = aR$ for some $a \in R$. Hence, $a \in B(R)$ in the notation of **(5)** and $a^r = 0$ by **(5**, 3.4**)**. Therefore, $C^r = 0$ and R is semi-prime by 1.4.

1.6. LEMMA. *Let R be a p.r.i. ring and* $cR \in L(R)$. *If* $c^l = 0$, *then* $cR \in L_r^{\Delta}(R)$ *and* $c^r = 0$.

Proof. If $cR \cap A = 0$ for some $A \in L_r(R)$, then $AcR = 0$, $Ac = 0$, and $A = 0$. Hence, $cR \in L_r^{\Delta}(R)$. Actually, since $(c^n)^l = 0$ for every integer $n > 0$, we have $(c^n)R \in L_r^{\Delta}(R)$ for every n. Therefore, by 1.3, $c^r = 0$.

2. Structure theory. A ring R is called (left) faithful in **(4)** if $R^l = 0$. Clearly every ring with unity is (left and right) faithful. For each (left) faithful ring R, we may define the set $F''(R)$ of ideals of R as in **(4**, p. 524**)**. Thus, $A \in F''(R)$ if and only if $A \in L(R)$ and $A \cap A^l = 0$, $A = A^{ll}$. A similar set $F_r''(R)$ of ideals of R may be defined for a right faithful ring. Although $F''(R)$ and $F_r''(R)$ are not in general equal, it is clear that $F''(R) = F_r''(R)$ in case R is semi-prime **(3)**, since $A^r = A^l$ for every $A \in L(R)$ in this case.

If R is a faithful ring, then $F''(R)$ can be made into a lattice by taking $A \cap B$ to be the usual set-theoretic intersection of A and B, and defining $A \cup B = (A + B)^{ll}$ for $A, B \in F''(R)$. It is proved in **(4**, 1.4**)** that $F''(R)$ is a Boolean algebra, with A^l being the unique complement of each $A \in F''(R)$. By the very definition of $F''(R)$, $(A + A^l)^l = 0$ for every $A \in F''(R)$.

2.1. THEOREM. *If R is a p.r.i. ring, then* $R = A + A^l$ *for every* $A \in F''(R)$.

Proof. Let $A = aR$, $A^l = bR$, and $A + A^l = cR$. Since $c^l = 0$, also $c^r = 0$ by 1.6. Now $a = cu$, $b = cv$, and $c = aa' + bb'$ for some $u, v, a', b' \in R$. Hence, $c = cua' + cvb'$ and $1 = ua' + vb'$. Therefore , $R = uR + vR$. Since $AA^l = 0$, we have $ab = 0$ and $cub = 0$. Hence, $ub = 0$ and $u \in A$ because $b^l = A^{ll} = A$. Consequently, $ur \subset A$ and, similarly, $vR \subset A^l$. It follows that $R = A + A^l$.

The lattice $F''(R)$ is finite in case the a.c.c. holds. For if A is a maximal element of $F''(R)$, $A \neq R$, then A^l is an atom of $F''(R)$. Evidently each B of $F''(R)$ is contained in a maximal element, and consequently each B contains an atom of $F''(R)$. By the a.c.c. and the fact that $F''(R)$ is a Boolean algebra, $F''(R)$ contains only a finite number of atoms. Hence, $F''(R)$ contains only a finite number of elements.

If R is a p.r.i. ring and $A \in F''(R)$, $A \neq 0$, then A is a p.r.i. ring in view of 2.1. By **(4**, 1.7**)**, $F''(A) = \{B \mid B \in F''(R), B \subset A\}$. Thus, we may extend 2.1. by mathematical induction as follows.

2.2. THEOREM. *If R is a p.r.i. ring, then $R = A_1 + \ldots + A_n$ where $\{A_1, \ldots, A_n\}$ is the set of atoms of $F''(R)$.*

In case the ring R is semi-prime, then $F''(R) = \{A^{rl} \mid A \in L(R)\}$ according to **(3**, p. 376**)**. Also, each atom of $F''(R)$ is a prime ring by **(3**, 2.7**)**. Thus, we have the following corollary of 2.2.

2.3. THEOREM. *A semi-prime ring is a p.r.i. ring if and only if it is a finite direct sum of p.r.i. prime rings.*

Next, let us assume that R is a p.r.i. ring and that N is the prime radical of R. Since the a.c.c. holds in $L_r(R)$, N is a nilpotent ideal of R. Let i_N designate the index of nilpotency of N. The p.r.i. ring $R' = R/N$ is semi-prime, and for each $A' \in F''(R')$, $R' = A' + B'$, $A' = B'^r = B'^l$, $B' = A'^r = A'^l$. If A and B are the ideals of R corresponding to A' and B', then we must have

$$R = A + B, \quad A \cap B = N, \quad NA^{-1} = A^{-1}N = B, \quad NB^{-1} = B^{-1}N = A.$$

Let $A = aR$ and $B = bR$, so that $A' = a'R'$ and $B' = b'R'$, where $a' = a + N$ and $b' = b + N$. Since A' is a p.r.i. ring and $A'^l = a'^l = 0$ in A', we have $a'^r = 0$ in A' by 1.6. Therefore, $a'^r = B'$ in R'. It follows that $ax \in N$ for $x \in R$ if and only if $x \in B$. Since $aR \supset N$, we must have $aB = N$. Hence, $abR = N$ and, similarly, $baR = N$. This means, in view of 1.1, that

$$A \cap B = AB = BA = N.$$

Consequently, $A^nB^n = (AB)^n = 0$ for every integer $n \geqslant i_N$.

2.4. THEOREM. *If R is a p.r.i. ring with prime radical N and if $A \in L(R)$, $A \supset N$, then $A/N \in F''(R/N)$ if and only if $A^n \in F''(R)$ for some positive integer n.*

Proof. We first assume that $A' = A/N \in F''(R/N)$ and that $B' = A'^l$. If B is the ideal of R corresponding to B', then the ideals A and B of R have the properties stated above. Let us prove that $R = A^n + B^n$ for every positive integer n. If k is an integer for which $R = A^k + B^k$, then $1 = a^ku + b^kv$ for some $u, v \in R$ and $N = a^kuN + b^kvN \subset A^{k+1} + B^{k+1}$. Since

$$R = (A^k + B^k)(A + B) \subset A^{k+1} + B^{k+1} + N = A^{k+1} + B^{k+1},$$
$$R = A^{k+1} + B^{k+1}.$$

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It follows by mathematical induction that $R = A^n + B^n$ for every positive integer n.

According to 1.3, there exists an integer k such that $(a^n)^r \cap a^n R = 0$ for every positive integer $n \geqslant k$. If we select $n \geqslant i_N$, then $a^n b^n = 0$ and $B^n \subset (a^n)^r$. For such a choice of n, $R = A^n + B^n$ and $A^n \cap B^n = 0$. Thus, $1 = e + f$ for orthogonal idempotents e and f, where $e \in A^n$ and $f \in B^n$, and $A^n = eR$, $B^n = fR$. Consequently, $(eR)^l = Rf = fR$, $(fR)^l = Re = eR$, and $A^n \in F''(R)$.

Conversely, if $A \in L(R)$, $A \supset N$, and if $A^n \in F''(R)$ for some positive integer n, then let $A_1 = A^n$ and $B_1 = A_1^l$. By 2.1 and what is given, $R = A_1 + B_1$, $A_1 \cap B_1 = 0$, and $B_1^l = A_1$. If we let $\bar{A} = A_1 + N$ and $\bar{B} = B_1 + N$, then $\bar{A} \cap \bar{B} = N$. Hence, $R' = A' + B'$, $A' \cap B' = 0$, where $R' = R/N$, $A' = \bar{A}/N$, and $B' = \bar{B}/N$. It follows that $A' \in F''(R)$. We need only show that $\bar{A} = A$ to complete the proof. If $1 = e + f$, $e \in A_1$ and $f \in B_1$, then $A = eA + fA$ and $A^n = (eA)^n + (fA)^n$. Hence, $A_1 = A^n = (eA)^n$ and $(fA)^n = 0$. This shows that $fA \subset N$ and therefore that $A \subset A_1 + N$. Consequently, $A = \bar{A}$ and the proof of 2.4 is completed.

2.5. THEOREM. *If R is a p.r.i. ring with radical N, then the Boolean algebras $F''(R)$ and $F''(R/N)$ are isomorphic under the correspondence $A \to (A + N)/N$, $A \in F''(R)$.*

Proof. If $A \in F''(R)$ and $B = A^l$, then $R = A + B$ and $A \cap B = 0$. Hence, $A + N = A + N_1$ for some ideal $N_1 \subset B \cap N$ and

$$(A + N)^k = A^k = A$$

for every $k \geqslant i_N$. Therefore, $(A + N)/N \in F''(R/N)$ by 2.4.

Conversely, if $A' \in F''(R/N)$ and A is the corresponding ideal of R, then $A^n \in F''(R)$ for some positive integer n by 2.4. We may show, as in the proof of 2.4, that $A = A^n + N$. Hence, there exists some $A^n \in F''(R)$ such that $(A^n + N)/N = A'$.

Finally, let us show that the mapping $\theta : \theta(A) = (A + N)/N$, $A \in F''(R)$, of $F''(R)$ onto $F''(R/N)$ is 1–1. If $\theta(A_1) = \theta(A_2)$, then $A_1 + N = A_2 + N$ and $(A_1 + N)^k = (A_2 + N)^k$ for every positive integer k. If we select $k \geqslant i_N$, then we get that $A_1 = A_1^k \subset (A_1 + N)^k \subset A_1$ and hence that $A_1 = (A_1 + N)^k$. Similarly, $A_2 = (A_2 + N)^k$ and $A_1 = A_2$. This completes the proof of 2.5.

A ring R is called (left) *irreducible* if and only if $F''(R) = \{0, R\}$. If R is a p.r.i. ring with radical N, then R is irreducible if and only if R/N is by 2.5. If R is semi-prime, then R is irreducible if and only if it is prime **(3, 2.7)**. Thus, we have the following result.

2.6. THEOREM. *If R is a p.r.i. ring with radical N, then R is irreducible if and only if R/N is a prime p.r.i. ring.*

Let us now look at the irreducible p.r.i. rings. We first prove the following lemma.

2.7. LEMMA. *If R is an irreducible p.r.i. ring with prime radical N and if $aR \in L(R)$, $aR \not\subset N$, then $a^r = 0$ and $N \subset aR$.*

Proof. Let $B = aR + N$ and $B' = B/N$, a non-zero ideal of $R' = R/N$. We know that $B'^r = B'^l = 0$, since R' is a prime ring. Therefore, if $B = bR$ and $b' = b + N$, $b'^l = 0$ in R' and also $b'^r = 0$ by 1.6. Hence, $bx \in N$, for $x \in R$, if and only if $x \in N$. Since $N \subset bR$, evidently $N = bN$. It follows that $N = b^n N$ and that $N \subset B^n$ for every positive integer n. Therefore, there exists a positive integer k by 1.3 such that $b^r \cap B^k = 0$. Since $b^r \subset N \subset B^k$, this means that $b^r = 0$. However, $B^n = (aR + N)^n \subset aR$ if $n \geqslant i_N$ and therefore $N \subset aR$ and $B = aR$. Thus, the lemma is proved by letting $b = a$.

A ring R is called (left) *primary* if and only if whenever $A, B \in L(R)$ are such that $AB = 0$, then either $A^n = 0$ for some integer $n > 0$ or $B = 0$. Of course, each prime ring is also primary.

2.8. THEOREM. *If R is a p.r.i. ring, then R is primary if and only if it is irreducible.*

Proof. If R is irreducible and $A, B \in L(R)$, with $AB = 0$, then either $A \not\subset N$, the radical of R, and $B = 0$ by 2.7, or $A \subset N$ and $A^n = 0$ for $n = i_N$. Therefore, R is primary.

If R is primary and $A \in F''(R)$, $A \neq 0$, then $A^n = A$ for every positive integer n. Since $AA^l = 0$, we must have $A^l = 0$. Therefore, $A = R$ and R is irreducible.

On combining 2.2 and 2.8, we obtain the following result.

2.9. THEOREM. *A ring is a p.r.i. ring if and only if it is a finite direct sum of p.r.i. primary rings.*

Theorem 2.3 was obtained by Goldie in **(1**, 6.3**)**. He obtains Theorem 2.9 also **(1**, 6.20**)**, but only under the added assumption that the a.c.c. holds for the lattice of left ideals of the ring.

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