Can. J. Math. Vol. **45** (3), 1993 pp. 449–469

MOVING WEIGHTED AVERAGES

M. A. AKCOGLU AND Y. DÉNIEL

ABSTRACT Let $\mathbb{R}$ denote the real line Let $\{T_t\}_{t\in\mathbb{R}}$ be a measure preserving ergodic flow on a non atomic finite measure space $(X, \mathcal{F}, \mu)$ A nonnegative function φ on $\mathbb{R}$ is called a *weight function* if $\int_{\mathbb{R}} \varphi(t)\,dt = 1$ Consider the weighted ergodic averages

$$\mathcal{A}_k f(x) = \int_{\mathbb{R}} f(T_t x)\theta_k(t)\,dt$$

of a function $f\ X \longrightarrow \mathbb{R}$, where $\{\theta_k\}$ is a sequence of weight functions Some sufficient and some necessary and sufficient conditions are given for the a e convergence of $\mathcal{A}_k f$, in particular for a special case in which

$$\theta_k(t) = (1/r_k)\varphi((t - a_k)/r_k),$$

where φ is a fixed weight function and $\{(a_k, r_k)\}$ is a sequence of pairs of real numbers such that $r_k > 0$ for all k These conditions are obtained by a combination of the methods of Bellow-Jones-Rosenblatt, developed to deal with moving ergodic averages, and the methods of Broise-Déniel-Derriennic, developed to deal with unbounded weight functions

A method developed by A. Nagel and E. M. Stein [13], and later by J. Sueiro [15], to investigate the pointwise convergence for general approach regions in harmonic analysis has been modified and generalized by A. Bellow, R. Jones, and J. Rosenblatt [1] to deal with certain pointwise convergence problems in ergodic theory. These techniques, combined with the Hardy-Littlewood Maximal Theorem and the Calderón Transfer Principle, have been very successful (see [1, 2, 3, 11, 12, 14]) in the investigation of the "moving" ergodic averages weighted by certain special weight functions. On the other hand, M. Broise, Y. Déniel, and Y. Derriennic [5] have recently generalized the Hardy-Littlewood Maximal Theorem to obtain convergence results for more general weighted "nonmoving" averages that could not have been examined by the Hardy-Littlewood Theorem alone (see also [6, 8, 9]). Our purpose in this note is to use this new maximal inequality, in a slightly modified form, in combination with the methods of Bellow-Jones-Rosenblatt to obtain a general result for the moving averages, formulated as Theorem D below. This theorem includes, in the one dimensional case, many of the results given in [1] and [5], and covers new cases. Theorem C is an important special case, formulated separately. The initial two results, Theorems A and B, show that a previously formulated condition, usually referred to as the "Cone Condition", is necessary and sufficient for

Research done during a visit of the second author at the University of Toronto and supported in part by an NSERC grant

Received by the editors July 18, 1991

AMS subject classification Primary 28D99, secondary 47A35

Key words and phrases maximal ergodic theorems, rearrangements

the pointwise convergence of the moving ergodic averages of bounded functions, with respect to an arbitrary weight function. We formulate this condition in an equivalent way, in the definition of "B-sequences", which seems to be simpler and more natural for our purposes. Most of the results formulated here have multidimensional analogues. These are discussed in a paper under preparation, jointly with D. McIntosh.

We make the present paper fairly self-contained, repeating, in a somewhat different context, some of the arguments already given in the articles mentioned above. One reason for this is the reader's convenience. Another reason, however, is that the reformulation and the modification of the existing results for our present arguments would also have required a substantial amount of work and space and would probably have resulted in a less satisfactory presentation.

1. **Introduction.**

NOTATION 1.1. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}$ the set of integers. The real line is denoted by $\mathbb{R}$, the σ-algebra of its Borel sets by $\mathcal{B}$, and Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ by ℓ. We indicate the integration with respect to Lebesgue measure either by $d\ell$, or by dt, or in a similar way. A nonnegative function φ on $\mathbb{R}$ is called a *weight function* if $\int_{\mathbb{R}} \varphi(t)\,dt = \int_{\mathbb{R}} \varphi\,d\ell = 1$. The characteristic function of set E in any space is denoted by χ_E. Let $(X, \mathcal{F}, \mu)$ be a non atomic finite measure space, $\{T_t\}_{t\in\mathbb{R}}$ a measure preserving aperiodic flow on X, and let f be a function on X. Consider a sequence of weight functions $\{\theta_k\}_{k\in\mathbb{N}}$. We would like to investigate the a.e. convergence of the weighted ergodic averages

$$\mathcal{A}_k f(x) = \int_{\mathbb{R}} f(T_t x)\theta_k(t)\,dt$$

for certain special choices of the sequence $\{\theta_k\}$. The basic case we will consider is the following. Let $\{(a_k, r_k)\}_{k\in\mathbb{N}}$ be a sequence of pairs of real numbers such that $r_k > 0$ for all $k \in \mathbb{N}$. Let φ be a fixed weight function. We then let

$$\theta_k(t) = \psi_k(t) = (1/r_k)\varphi\big((t - a_k)/r_k\big).$$

In this case we will denote the corresponding sequence of weighted ergodic averages by $\mathcal{A}_k^{\varphi}$ and call them the moving weighted ergodic averages. Nonmoving averages correspond to the case where $a_k = 0$ for all k. It will turn out that the convergence of the moving averages is closely connected with a property of the sequence $\{(a_k, r_k)\}$ which we will now define.

B-SEQUENCES 1.2. Let $\{(a_k, r_k)\}$ be a sequence in $\mathbb{R} \times \mathbb{R}$ such that $r_k > 0$ for all k. Then this sequence will be called a B-sequence if there is a constant B such that

$$\ell\Big(\big\{t \mid \exists k, (t + a_k, t + a_k + r_k) \subset I\big\}\Big) \leq B\ell(I)$$

for every interval $I \subset \mathbb{R}$.

THEOREM A 1.3. *Assume that $\{T_t\}_{t\in\mathbb{R}}$ is an aperiodic flow. If $\{(a_k, r_k)\}$ is not a B-sequence then, for each weight function φ there is a bounded function f on X such that $\mathcal{A}_k^{\varphi} f$ diverges on a set of positive measure.*

THEOREM B 1.4. *Assume that $\{(a_k, r_k)\}$ is a B-sequence such that either both a_k and r_k converge to zero or r_k converges to infinity. Then $\mathcal{A}_k^{\varphi} f$ converges a.e. for any bounded function f and for any weight function φ.*

If f is an unbounded function then, in general, the averages $\mathcal{A}_k^{\varphi} f$ do not converge, even if $\{(a_k r_k)\}$ is a B-sequence. The following theorem gives a sufficient condition for convergence. We need a definition to formulate this theorem.

DISTRIBUTIONS AND REARRANGEMENTS 1.5. Let h be a nonnegative function on a general measure space $(Y, \mathcal{G}, \nu)$ such that its support

$$S_h = \{y \mid h(y) > 0\}$$

has finite measure. The distribution of such a function is the finite measure D_h on $(\mathbb{R}, \mathcal{B})$ defined by

$$D_h(B) = \nu(S_h \cap h^{-1}B), \quad B \in \mathcal{B}.$$

The (decreasing) rearrangement of h is the nonnegative function h^* on the measure space $(\mathbb{R}, \mathcal{B}, \ell)$ which is zero on $(-\infty, 0]$, decreasing on $(0, \infty)$, and has the same distribution as h. It is easy to see that h^* is unique in the sense that two rearrangements of h differ only on a set of Lebesgue measure zero.

THEOREM C 1.6. *Assume that $\{(a_k, r_k)\}$ is a B-sequence such that either both a_k and r_k converge to zero or r_k converges to infinity. Let φ be a weight function with a compact support. Then $\mathcal{A}_k^{\varphi} f$ converges a.e. for all functions f on X such that*

$$\int_{\mathbb{R}} |f|^* \varphi^* \, d\ell < \infty.$$

Note that, when the weight function φ is bounded then the last condition on f is satisfied for all integrable functions. Finally we will also consider the following general situation.

THEOREM D 1.7. *Let $\{\varphi_k\}$ be a sequence of weight functions. Assume that these functions are dominated by an $L_1(\mathbb{R}) = L_1(\mathbb{R}, \mathcal{B}, \ell)$ function Φ of compact support and converge ℓ-a.e. to a function φ, which is necessarily another weight function. Assume that $\{(a_k, r_k)\}$ is a B-sequence such that either both a_k and r_k converge to zero or r_k converges to infinity. Then the averages*

$$\mathcal{A}_k f(x) = \int_R f(T_t x)\, \theta_k(t)\, dt$$

formed with the weight functions

$$\theta_k(t) = (1/r_k)\varphi_k\big((t - a_k)/r_k\big)$$

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press

converge a.e. on X if the function f satisfies

$$\int_{\mathbb{R}} |f|^* \Phi^* \, d\ell < \infty.$$

In this case the a.e. limit of these averages is equal to the a.e. limit of the averages $\mathcal{A}_k^{\varphi} f$, which also exists.

REMARK 1.8. We assume that the flow $\{T_t\}_{t\in\mathbb{R}}$ is a full aperiodic flow, rather than a semi-flow $\{T_t\}_{t\geq 0}$, just for the sake of convenience, except in Section 3, in the proof of Theorem A. In none of the remaining arguments is the invertibility of the transformations or the aperiodicity of the flow used. Hence the Theorems B, C, and D are also valid for a semi-flow. When dealing with a semi-flow, however, the support of all the weight functions involved must be in the nonnegative part of the real line.

REMARK 1.9. We also note that all of our results, including Theorem A, are valid for the discrete flow $\{T^n\}_{n\in\mathbb{Z}}$ or the discrete semi-flow $\{T^n\}_{n\geq 0}$, formed by the powers of an invertible or, respectively, a not necessarily invertible ergodic measure preserving transformation. In fact these flows can be imbedded in a continuous standard flow under a constant ceiling function of unit height. In this case, if the weight functions are also step functions that are constant over the intervals of the form $[n, n+1)$, the integrals that define $\mathcal{A}_k f$ can be replaced by sums, giving weighted averages of the powers of a single operator. More explicitly, let $\{a_k\}$ and $\{r_k\}$ be two sequence of integers, where $r_k \geq 1$ and $r_k \longrightarrow \infty$. Let $\{C_k^i\}$, $i = 0, \ldots, r_k$, be a finite sequence of nonnegative numbers with $\sum_{i=0}^{r_k} C_k^i = 1$. To investigate the discrete averages

$$\mathcal{A}_k f = \sum_{i=0}^{r_k} C_k^i T^{a_k+i} f$$

one may, for example, consider the sequence of weight functions

$$\varphi_k = \sum_{i=0}^{r_k} (r_k+1) C_k^i \chi_{[\frac{i}{r_k+1}, \frac{i+1}{r_k+1})}.$$

If these functions converge ℓ-a.e. to a function φ and are dominated by an $L_1(\mathbb{R})$ function Φ of compact support, then Theorem D shows that the discrete averages $\mathcal{A}_k f$ converge a.e. whenever

$$\int_{\mathbb{R}} \Phi^* |f|^* \, d\ell < \infty$$

and (a_k, r_k) is a B-sequence.

EXAMPLE 1.10 α-CESÁRO MEANS. As a special case we consider the moving α-Cesáro averages of a transformation. The corresponding case for the nonmoving $(a_k = 0)$ averages was discussed in [5]. Let $\alpha > 0$ be a fixed number. Let $r_k = k$ and define $C_k^i = A_{k-i}^{\alpha-1} / A_k^{\alpha}$, $i = 0, \ldots, k$, where $A_0^{\beta} = 1$ and

$$A_m^{\beta} = \frac{(\beta+1)(\beta+2)\cdots(\beta+m)}{m!} = \binom{m+\beta}{m}$$

for any real number $\beta > -1$ and for any integer $m \geq 1$. We see [16] that $\sum_{\iota=0}^{k} C_k^\iota = 1$, and

$$\lim_{m\to\infty} \frac{m^\beta}{A_m^\beta} = \Gamma(\beta + 1).$$

From this it follows easily that

$$\begin{aligned}\lim_{k\to\infty} \varphi_k(t) &= \lim_{k\to\infty} \sum_{\iota=0}^{k} (k+1) \frac{A_{k-\iota}^{\alpha-1}}{A_k^\alpha} \chi_{[\frac{\iota}{k+1}, \frac{\iota+1}{k+1})}(t) \\ &= \alpha(1-t)^{\alpha-1} \chi_{[0,1)}(t) \\ &= \varphi(t)\end{aligned}$$

for all $t \in \mathbb{R}$. To see that the sequence $\{\varphi_k\}$ is dominated by an $L_1(\mathbb{R})$ function it will be enough to show that

$$\inf\left\{ \frac{\varphi(t)}{\varphi_k(t)} \;\middle|\; 0 < t < 1, k = 1, 2, \ldots \right\} > 0.$$

In fact, if

$$\frac{i}{k+1} \leq t < \frac{i+1}{k+1},$$

then, with $\beta = \alpha - 1$ and $m = k + 1$,

$$\begin{aligned}\frac{\varphi(t)}{\varphi_k(t)} &\geq \frac{\varphi\big(i/(k+1)\big)}{\varphi_k\big(i/(k+1)\big)} \\ &= \left(1 - \frac{i}{m}\right)^\beta \left(1 + \frac{\beta}{m-i}\right) \cdots \left(1 + \frac{\beta}{m-1}\right)\left(1 + \frac{\beta}{m}\right),\end{aligned}$$

where $0 \leq i \leq m - 1$ and $m \geq 2$. The infimum of this last expression, as i and m change over the above ranges, is strictly positive. This follows easily from the observations that the infinite product

$$\prod \left(1 + \frac{\beta}{n}\right) \exp(-\beta/n)$$

is convergent for $\beta > -1$, and that the sequence

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

is bounded.

2. *B*-Sequences.

NOTATION 2.1. All the intervals we are going to consider are bounded intervals in $\mathbb{R}$ and have nonempty interiors, either by construction or assumption. If I is an interval with the end points p and q, and a and r are real numbers, then $a + rI$ denotes the interval with the end points $a + rp$ and $a + rq$. We will always assume that $r > 0$. In this case $\ell(a + rI) = r\ell(I)$.

B-SEQUENCES OF INTERVALS 2.2. Let $\{I_k\}$ be a sequence of intervals. Then we will say that $\{I_k\}$ is a B-sequence if there is a constant B such that, for all intervals I,

$$\ell(\{t \mid \exists k \in \mathbb{N}, t + I_k \subset I\}) \leq B\ell(I).$$

REMARK. If $\{I_k\}$ is a B-sequence of intervals then

$$\ell(\{t \mid \exists k \in \mathbb{N}, t + I_k \subset E\}) \leq B\ell(E)$$

for any Borel set E.

LEMMA 2.3. *Let $\{(a_k, r_k)\}$ be a sequence in $\mathbb{R} \times \mathbb{R}$ and assume that $r_k > 0$ for all k. Let I and J be two intervals. Then $I_k = a_k + r_k I$ is a B-sequence if and only if $J_k = a_k + r_k J$ is a B-sequence.*

PROOF. Let $V = (-R, R)$ be a fixed symmetric interval about zero that contains both I and J. Find $\alpha \geq 0$ such that $\ell(V) = (1 + \alpha)\ell(J)$. Given any interval W, let $\tilde{W}$ be the interval with the same center as W and satisfying $\ell(\tilde{W}) = (1 + 2\alpha)\ell(W)$. If $t + J_k = t + a_k + r_k J \subset W$ then we see that $t + a_k + r_k V \subset \tilde{W}$ and, consequently, $t + a_k + r_k I \subset \tilde{W}$. Hence, if $\{I_k\}$ is a B-sequence with a constant B, then $\{J_k\}$ is also a B-sequence with a constant $(1 + 2\alpha)B$.

DEFINITION 2.4. Let $\{(a_k, r_k)\}$ be a sequence in $\mathbb{R} \times \mathbb{R}$ such that $r_k > 0$ for all k. We say that $\{(a_k, r_k)\}$ is a B-sequence if $a_k + r_k I$ is a B-sequence of intervals for an interval I. The previous lemma shows that this definition is independent of the choice of the interval I.

REMARK. The defining condition for being a B-sequence is equivalent to the "Cone Condition", formulated and used by Nagel, Stein, Sueiro, Bellow, Jones, and Rosenblatt in [13, 15, 1] (see also [14]). This equivalence is obtained below, although we are not going to use the Cone Condition, formulated as (C) below.

LEMMA 2.5. *Let $\{(a_k, r_k)\}$ be a sequence in $\mathbb{R} \times \mathbb{R}$ such that $r_k > 0$ for all k. Then the following two conditions on this sequence are equivalent.*

(B) There is a constant B, such that for all intervals I,

$$\ell\Big(\{t \mid \exists k, (t + a_k, t + a_k + r_k) \subset I\}\Big) \leq B\ell(I).$$

(C) There is a constant C, such that for all positive numbers s,

$$\ell\Big(\{t \mid \exists k, |t - a_k| \leq (s - r_k)\}\Big) \leq Cs.$$

PROOF. Assume (B). Given $s > 0$, let $I = (-s, s)$ and define

$$E = \{t \mid \exists k, |t - a_k| \leq (s - r_k)\},$$

and

$$E' = \{t \mid \exists k, (t + a_k, t + a_k + r_k) \subset I\}.$$

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press

Then we verify that $E \subset -E'$. Hence,

$$\ell(E) \leq \ell(-E') = \ell(E') \leq B\ell(I) = 2Bs,$$

which means that (C) is satisfied with $C = 2B$.

Now assume (C). Let $I = (u, v)$ be an interval. Define

$$E' = \{t \mid \exists k, (t + a_k, t + a_k + r_k) \subset I\},$$

and, for $s > 0$,

$$E(s) = \{t \mid \exists k, |t - a_k| \leq (s - r_k)\}.$$

We verify that if $t \in E'$ then $-t + u \in E(v - u)$. Hence

$$\ell(E') \leq \ell\big(u - E(v - u)\big) = \ell\big(E(v - u)\big) \leq C(v - u),$$

which is (B) with $B = C$.

3. **Divergence of averages.** In this section we prove Theorem A. This result shows that the moving ergodic averages over non B-sequences, with respect to any weight function, always diverge for some bounded functions.

LEMMA 3.1. *Assume that $\{(a_k, r_k)\}$ is not a B-sequence. Then for each $R > 0$, K, and $\varepsilon > 0$ we can find two sets C and D in X such that*

$$0 < R\mu(C) < \mu(D),$$

and such that

$$\sup_{k \geq K} \mathcal{A}_k^{\varphi} \chi_C(x) \geq 1 - \varepsilon$$

for all $x \in D$.

PROOF. Since the flow is aperiodic, given any $S > 0$ we can find a mapping

$$\Gamma\colon (-S, S) \longrightarrow \mathcal{F}$$

and a number $\gamma > 0$ such that for any Borel subset E of $(-S, S)$ and for any numbers s and t, where s, t, and $s + t$ are all in $(-S, S)$,

(a) $\Gamma(E) \in \mathcal{F}$,

(b) $\mu\big(\Gamma(E)\big) = \gamma\ell(E)$,

(c) $\Gamma(s + t) = T_t\Gamma(s)$,

where $\Gamma(E) = \bigcup_{t \in E} \Gamma(t)$. This follows easily by considering the given flow as the standard flow under a ceiling function and by taking a sufficiently tall Rohlin tower for the base transformation. In fact we can also make $\mu\big(\Gamma((-S, S))\big)$ arbitrarily close to $\mu(X)$ but this fact is not needed.

Find an interval J such that the integral of the given weight function φ on J is greater than $1 - \varepsilon$. Then the same is also true for the integral of ψ_k on $J_k = a_k + r_kJ$, for any

k Since $\{J_k\}$ is not a B-sequence, its tail $\{J_k\}_{k>K}$ is not a B-sequence either, for any K Hence we can find an interval I such that

$$\ell(\{t \mid \exists k \geq K, t + J_k \subset I\}) > R\ell(I) > 0$$

Then there is an integer L such that if

$$F = \{t \mid \exists k,\ K \leq k \leq L,\ t + J_k \subset I\}$$

then $\ell(F) > R\ell(I)$ We now find a number S such that the interval $(-S, S)$ contains I, F, and all the intervals of the form $t + J_k$, where $t = 0$ or $t \in F$ and $K \leq k \leq L$ If $\Gamma\ (-S, S) \longrightarrow X$ is a mapping as described at the beginning of this proof, we then let $C = \Gamma(I)$ and $D = \Gamma(F)$

We will verify that the requirements of the lemma are satisfied Since

$$0 < R\ell(I) < \ell(F),$$

we see that

$$0 < R\mu(C) < \mu(D),$$

by (b) Let $x \in D$ Then there is a $t \in F$ such that $x \in \Gamma(t)$ Hence there is a $k \geq K$ such that $t + J_k \in I$, which implies that $T_s x \in C$, for all $s \in J_k$ Then, with these choices,

$$\begin{aligned} \mathcal{A}_k^{\varphi} \chi_C(x) &= \int_{\mathbb{R}} \chi_C(T_s x)\, \psi_k(s)\, ds \\ &\geq \int_{J_k} \chi_C(T_s x)\, \psi_k(s)\, ds \\ &= \int_{J_k} \psi_k(s)\, ds > 1 - \varepsilon \end{aligned}$$

PROOF OF THEOREM A 3 2 Given two numbers $\varepsilon > 0$ and $\eta > 0$, we use the previous lemma to find two sequences of sets $\{C_n\}$ and $\{D_n\}$ in X and a sequence of numbers $\{K_n\}$ such that

(a) K_n converges to infinity,
(b) $\sum \mu(C_n) < \eta$,
(c) $\sum \mu(D_n) = \infty$,
(d) $\sup_{k>K_n} \mathcal{A}_k^{\varphi} \chi_C\ (x) \geq 1 - \varepsilon$ for all $x \in D_n$

We can do this easily in stages, at each stage choosing a finite segment of our sequences Suppose that at the m-th stage we have the sets C and D satisfying the conditions of Lemma 3 1 with $R = 2^m$ and $K = m$ Hence $\mu(C) \leq 2^{-m}\mu(X)$ Then the m-th segments of the sequences $\{C_n\}$, $\{D_n\}$ and $\{K_n\}$ will have u_m terms, obtained by repeating C, D, and m u_m-times, respectively, where $u_m \geq 1$ is the smallest integer such that $u_m\mu(C) > 2^{-m-1}\mu(X)$ Note that the sum of the terms of $\{\mu(C_n)\}$ over this segment, which is just $u_m\mu(C)$, is still less than $2^{-m}\mu(X)$ The sum of the terms $\{\mu(D_n)\}$ over the same segment is $u_m\mu(D)$, which is greater than $2^m u_m \mu(C) > 2^{-1}\mu(X)$ Hence we see that $\sum \mu(C_n) < \infty$ and $\sum \mu(D_n) = \infty$ Then by cutting off an initial segment of these sequences we obtain the desired sequences

Now we note that if $\{C_n\}$, and $\{D_n\}$ are two sequences satisfying the conditions above, then, for any choice of the numbers t_n, the sequences $\{T_{t_n}C_n\}$, and $\{T_{t_n}D_n\}$ will also satisfy the same conditions. Since $\sum \mu(D_n) = \infty$, we claim that there is a choice for the sequence $\{t_n\}$ such that

$$\mu\Big(\bigcup_{n\geq N} T_{t_n}D_n\Big) = \mu(X)$$

for all N. This follows easily from the fact that, given any two sets A and B in X and any number $M > 1$, there is a t such that $\mu(A \cap T_t B) < M\mu(A)\mu(B)$. This fact, in turn, is a direct consequence of the mean ergodic theorem (see also the related Lemma 1.24 in Chapter XIII of Zygmund [16]). We now let f be the characteristic function of $\cup_n T_{t_n} C_n$, with this choice of $\{t_n\}$. Then it is clear that $\int_X f\,d\mu < \eta$ but

$$\limsup_k \mathcal{A}_k^{\varphi} f > 1 - \varepsilon$$

a.e. on X. Hence the averages for this function f can not be convergent.

4. **Rearrangements.**

REARRANGEMENTS 4.1. The definitions we are going to give apply to nonnegative functions f on a measure space $(X, \mathcal{F}, \mu)$ having supports

$$S_f = \{x \mid f(x) > 0\}$$

of finite measure. The distribution of such a function is the finite measure D_f on $(\mathbb{R}, \mathcal{B})$ defined by

$$D_f(B) = \mu(S_f \cap f^{-1}B), \quad B \in \mathcal{B}.$$

Note that this measure is contained in $(0, \infty)$. We will call any finite measure on the Borel subsets of $(0,\infty)$ a *distribution*. A nonnegative function on the measure space $(\mathbb{R}, \mathcal{B}, \ell)$ will be called a *(decreasing) rearrangement function* if it has compact support, is zero on $(-\infty, 0]$, and decreasing on $(0,\infty)$. It is easy to see that given any distribution there is a rearrangement function whose distribution is the given one. Also, any two rearrangement functions with the same distribution differ only on a set of Lebesgue measure zero. The rearrangement of f is the rearrangement function f^* which has the same distribution as f. We see that there is a measure preserving map $\tau\colon S_f \longrightarrow S_{f^*}$, which is not necessarily invertible, such that $f(x) = f^*(\tau x)$ for almost all $x \in S_f$. This map can be extended in an arbitrary way to the outside of these supports, if convenient.

LEMMA 4.2. *Let f be a nonnegative function on $\mathbb{R}$ with a support of finite measure. Given a rearrangement function ξ on $\mathbb{R}$, there is a function g such that $g^* = \xi$ and such that*

$$\int_{\mathbb{R}} fg\,d\ell = \int_{\mathbb{R}} f^*g^*\,d\ell = \int_{\mathbb{R}} f^*\xi\,d\ell.$$

Furthermore, if the support of ξ is contained in the support of f^*, then the support of g is contained in the support of f.

The proof follows easily from the observation made above, by letting $g(t) = \xi(\tau t)$. Another property of rearrangements is given in the following theorem, which we state without proof. For a further discussion see [10].

THEOREM 4.3. *Let f and g be two nonnegative functions on a measure space $(X, \mathcal{F}, \mu)$. Let $A \in \mathcal{F}$ and let $\alpha = \mu(A)$. Then*

$$\begin{aligned}\int_A fg\,d\mu &\leq \int_{\mathbb{R}} (f\chi_A)^*(g\chi_A)^*\,d\ell \\ &\leq \int_{\mathbb{R}} (f\chi_A)^* g^*\,d\ell \\ &\leq \int_0^\alpha f^*(t)g^*(t)\,dt.\end{aligned}$$

We now consider a sequence of nonnegative functions $\{f_\iota\}$ on X with pairwise disjoint supports. Let f be the sum of these functions. The distribution of f is the sum of the distributions of f_ι's. In general there may not be an easy way to express the rearrangement of f in terms of the rearrangements of f_ι's. As an important special case, however, assume that the rearrangement of each f_ι is of the form $f_\iota^*(t) = \xi(t/k_\iota)$, where ξ is a fixed rearrangement function and k_ι's are constants, $k_\iota > 0$. In this case the rearrangement function of f is given by $f^*(t) = \xi(t/K)$, where $K = \sum_\iota k_\iota$, assumed to be finite. In fact, $D_{f_\iota} = k_\iota D_\xi$, and, consequently, $D_f = KD_\xi$. A continuous version of this result is also true, which will be stated at the end of the next paragraph.

Let $(Y, \mathcal{G}, \nu)$ be another measure space and consider the Cartesian product space $X \times Y$ with the product measure $\mu \times \nu$. For a nonnegative function f on $X \times Y$ let $\tilde{f}$ denote the function on $X \times \mathbb{R}$ such that, for any $x \in X$, the function $\tilde{f}(x, \cdot)$ on $\mathbb{R}$ is the rearrangement of the function $f(x, \cdot)$ on Y. It is easy to see that there is a measurable function $\tilde{f}$ with these properties. Note that the function f on $X \times Y$ has the same distribution as the function $\tilde{f}$ on $X \times \mathbb{R}$. Hence f^* and $\tilde{f}^*$ are the same functions on $\mathbb{R}$. Then, for two non-negative functions f and g on $X \times Y$,

$$\begin{aligned}\int_X \int_Y f(x,y)\,g(x,y)\,\nu(dy)\,\mu(dx) &\leq \int_X \int_{\mathbb{R}} \tilde{f}(x,s)\tilde{g}(x,s)\,ds\,\mu(dx) \\ &\leq \int_{\mathbb{R}} f^*(t)\,g^*(t)\,dt,\end{aligned}$$

by applying the theorem above twice. In particular, if there is a single rearrangement function ξ and a strictly positive integrable function k on X such that $\tilde{g}(x,t) = \xi\big(t/k(x)\big)$, then we see easily that $g^*(t) = \xi(t/K)$, where $K = \int_X k(x)\mu(dx)$. We collect these observations in the following lemma.

LEMMA 4.4. *Let f and g be two nonnegative functions on the Cartesian product $X \times Y$ of two measure spaces. Assume that there is a fixed rearrangement function ξ such that, for each fixed $x \in X$, the rearrangement of $g(x, \cdot)\colon Y \longrightarrow \mathbb{R}$ is given by $\xi\big(t/k(x)\big)$, where k is a strictly positive and integrable function on X. Then*

$$\int_X \int_Y fg\,d\nu\,d\mu \leq \int_{\mathbb{R}} f^*(t)\,\xi(t/K)\,dt,$$

where $K = \int_X k(x)\mu(dx)$.

Finally, we mention the following simple result which will be used on several occasions. It is proved by an obvious change of variables.

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press

LEMMA 4.5. *Let ξ and η be two rearrangement functions and let $a > 0$. Then*

$$\int_{\mathbb{R}} \xi(t)\,\eta(t/a)\,dt \leq \max(1,a)\int_{\mathbb{R}} \xi(t)\,\eta(t)\,dt.$$

5. **Maximal theorems.** Our purpose in this section is to prove the following maximal inequality.

THEOREM 5.1. *Consider a weight function φ with a compact support, and a B-sequence $\{(a_k, r_k)\}$. Let $\{T_t\}$ be a measure preserving flow on a finite measure space $(X, \mathcal{F}, \mu)$. We let*

$$\mathcal{A}_k^{\varphi} f(x) = \int_{\mathbb{R}} f(T_t x)\,\psi_k(t)\,dt,$$

where

$$\psi_k(t) = (1/r_k)\varphi\big((t-a_k)/r_k\big),$$

and f is a function on X. Then there is a constant C such that

$$\mu\big(\{x \mid \exists k, \mathcal{A}_k^{\varphi} f(x) > \lambda\}\big) \leq \frac{C}{\lambda}\int_{\mathbb{R}} f^* \varphi^*\,d\ell$$

for all nonnegative functions f on X, and for all $\lambda > 0$. If J is an interval containing the support of the weight function φ then the constant C can be taken as

$$C = 4B\big(\ell(J) + \mu(X)\big)$$

where B is a constant corresponding to the B-sequence of intervals

$$J_k = a_k + r_k J,$$

as defined in 2.2, i.e. a constant such that

$$\ell(\{t \mid \exists k, t + J_k \subset I\}) \leq B\ell(I)$$

for any interval I.

The proof will consist of several lemmas. Theorem 5.5 is the corresponding maximal inequality on $\mathbb{R}$. The main inequality on X is then obtained by an application of Calderón's Transfer Principle.

NOTATION 5.2. Let φ be a weight function which has a compact support contained in an interval J. Let F be another nonnegative function on $\mathbb{R}$ with compact support. Starting with φ and F we define, for each Borel set E with nonzero finite measure,

$$\rho(E) = \frac{\ell(J)}{\ell(E)}\int_{\mathbb{R}} (F\chi_E)^*(t)\varphi^*\left(t\frac{\ell(J)}{\ell(E)}\right)dt.$$

Then we see that for each Borel set E there is a function ψ_E with support in E such that

$$\psi_E^*(t) = \varphi^*\left(t\frac{\ell(J)}{\ell(E)}\right)$$

and such that

$$\rho(E) = \frac{\ell(J)}{\ell(E)}\int_{\mathbb{R}} F\psi_E\,d\ell.$$

The reason for defining ρ is given by the following lemma.

LEMMA 5.3. *Let $a \in \mathbb{R}$ and $r > 0$. If*

$$(1/r)\int_{\mathbb{R}} F(t)\,\varphi\big((t-a)/r\big)\,dt > \lambda$$

then

$$\rho(a + rJ) > \lambda.$$

PROOF. Let $\psi(t) = \varphi\big((t-a)/r\big)$. Since the support of φ is contained in J, the support of ψ is contained in $a + rJ$. Also, the rearrangement function of ψ is given by $\psi^*(t) = \varphi^*(t/r)$. Hence we see that

$$\begin{aligned}\lambda &< (1/r)\int_{\mathbb{R}} F(t)\,\varphi\big((t-a)/r\big)\,dt \\ &= (1/r)\int_{a+rJ} F(t)\,\varphi\big((t-a)/r\big)\,dt \\ &\leq (1/r)\int_{\mathbb{R}} (F\chi_{a+rJ})^*(t)\,\varphi^*(t/r)\,dt \\ &= \frac{\ell(J)}{\ell(a+rJ)}\int_{\mathbb{R}} (F\chi_{a+rJ})^*(t)\,\varphi^*\Big(t\frac{\ell(J)}{\ell(a+rJ)}\Big)\,dt \\ &= \rho(a+rJ).\end{aligned}$$

LEMMA 5.4. *Let $\{E_\iota\}$ be a sequence of pairwise disjoint Borel sets and let $E = \cup_\iota E_\iota$. Then*

$$\sum_\iota \ell(E_\iota)\rho(E_\iota) \leq \ell(E)\rho(E) \leq \ell(E)\sum_\iota \rho(E_\iota).$$

PROOF. For the first inequality, with the notations above,

$$\begin{aligned}\sum_\iota \frac{\ell(E_\iota)}{\ell(J)}\rho(E_\iota) &= \sum_\iota \int_{\mathbb{R}} F\psi_{E_\iota}\,d\ell \\ &= \int_{\mathbb{R}} F\sum_\iota \psi_{E_\iota}\,d\ell \\ &= \int_{\mathbb{R}} (F\chi_E)\sum_\iota \psi_{E_\iota}\,d\ell \\ &\leq \int_{\mathbb{R}} (F\chi_E)^*\Big(\sum_\iota \psi E_\iota\Big)^*\,d\ell \\ &= \int_{\mathbb{R}} (F\chi_E)^*(t)\varphi^*\Big(t\frac{\ell(J)}{\ell(E)}\Big)\,dt \\ &= \frac{\ell(E)}{\ell(J)}\rho(E).\end{aligned}$$

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press

For the second inequality,

$$
\begin{aligned}
\frac{\ell(E)}{\ell(J)}\rho(E) &= \int_{\mathbb{R}} F\psi_E \, d\ell \\
&= \int_E F\psi_E \, d\ell \\
&= \sum_i \int_{E_i} F\psi_E \, d\ell \\
&\leq \sum_i \int_{\mathbb{R}} (F\chi_{E_i})^* \psi_E^* \, d\ell \\
&= \sum_i \int_{\mathbb{R}} (F\chi_{E_i})^*(t)\varphi^*\left(t\frac{\ell(J)}{\ell(E)}\right) dt \\
&\leq \sum_i \frac{\ell(E)}{\ell(E_i)} \int_{\mathbb{R}} (F\chi_{E_i})^*(t)\, \varphi^*\left(t\frac{\ell(J)}{\ell(E_i)}\right) dt \\
&= \frac{\ell(E)}{\ell(J)} \sum_i \rho(E_i),
\end{aligned}
$$

where the last inequality follows from Lemma 4.5.

THEOREM 5.5. *Let $\lambda > 0$ and $M > 0$ be fixed. Let G be the union of all open intervals $I \subset (-M, M)$ for which $\rho(I) > \lambda$. Then*

$$
\ell(G) \leq \frac{4\ell(J)}{\lambda} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t\frac{\ell(J)}{2M}\right) dt.
$$

PROOF. Let

$$
L = \left\{ t \mid \exists s, -M < t < s < M, \rho\big((t,s)\big) > \lambda/2 \right\}.
$$

It is easy to see that L is an open subset of $(-M, M)$, and that, for each $\varepsilon > 0$, there are finitely many pairwise disjoint intervals $I_i \subset (-M, M)$ such that $\rho(I_i) > \lambda/2$ for each i and such that $\ell(L) - \varepsilon \leq \ell(E) \leq 2M$, where E is the union of these intervals. Hence $\rho(E) > \lambda/2$ by the previous lemma. This means that

$$
\begin{aligned}
\lambda/2 &< \frac{\ell(J)}{\ell(E)} \int_{\mathbb{R}} (F\chi_E)^*(t)\, \varphi^*\left(t\frac{\ell(J)}{2M}\right) dt \\
&\leq \frac{\ell(J)}{\ell(E)} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t\frac{\ell(J)}{2M}\right).
\end{aligned}
$$

Then we conclude that

$$
\ell(L) - \varepsilon \leq \ell(E) \leq \frac{2\ell(J)}{\lambda} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t\frac{\ell(J)}{2M}\right) dt.
$$

Hence

$$
\ell(L) \leq \frac{2\ell(J)}{\lambda} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t\frac{\ell(J)}{2M}\right) dt.
$$

Similarly we see that if

$$L' = \{t \mid \exists s, -M < s < t < M, \rho\big((s,t)\big) > \lambda/2\},$$

then

$$\ell(L') \leq \frac{2\ell(J)}{\lambda} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t \frac{\ell(J)}{2M}\right) dt.$$

We claim that $G \subset L \cup L'$. To see this let $t \in G$. Then there is an interval $I \subset (-M, M)$ that contains t such that $\rho(I) > \lambda$. If t is an end point of I then our claim is clear. Otherwise I is divided into two intervals I_1 and I_2 by t. The previous lemma shows that $\lambda < \rho(I) \leq \rho(I_1) + \rho(I_2)$. Hence either $\rho(I_1) > \lambda/2$ or $\rho(I_2) > \lambda/2$. This means that t belongs to at least one of L or L'. Hence our claim has been proved. Then

$$\ell(G) \leq \ell(L) + \ell(L') \leq \frac{4\ell(J)}{\lambda} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t \frac{\ell(J)}{2M}\right) dt.$$

NOTATION 5.6. Let $M > 0$ be a fixed number and $I = (-M, M)$. Instead of a single function $F\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ with compact support, we will now deal with a (measurable) function $F\colon X \times \mathbb{R} \longrightarrow \mathbb{R}^+$ whose support is contained in $X \times I$, where $(X, \mathcal{F}, \mu)$ is finite measure space. We fix, as before, a single weight function φ whose support is contained in an interval J with $\ell(J) \geq 1$. With a given $\lambda > 0$ and with the same $M > 0$ as above we define ρ and G for each $x \in X$ separately, using the function $F(x, \cdot)\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ in each case. To denote the dependence on x we will now use the notations $\rho(E, x)$ and $G(x)$ instead of their corresponding previous versions. We also let

$$H = \{(x,t) \mid t \in G(x)\}.$$

It is easy to see that H is a measurable subset of $X \times \mathbb{R}$. In fact, in the definition of $G(x)$ it is enough, for example, to consider only intervals with rational end points.

LEMMA 5.7. *We have*

$$(\mu \times \ell)(H) \leq \frac{4\ell(J)}{\lambda} \int_{\mathbb{R}} F^*(t)\, \varphi^*\left(t \frac{\ell(J)}{2M\mu(X)}\right) dt$$

where $F^*\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ *is the rearrangement function of* $F\colon X \times \mathbb{R} \longrightarrow \mathbb{R}^+$ *and* $\varphi^*\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ *is the rearrangement function of* $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}^+$. *In particular, if* $F(\cdot, t)\colon X \longrightarrow \mathbb{R}^+$ *has the same rearrangement function* ξ *for each* $t \in I$ *and is zero for* $t \notin I$, *then*

$$(\mu \times \ell)(H) \leq \frac{4\ell(J)}{\lambda} \int_{\mathbb{R}} \xi\big(t/(2M)\big)\, \varphi^*\left(t \frac{\ell(J)}{2M\mu(X)}\right) dt.$$

PROOF. Let the (measurable) function $\tilde{F}\colon X \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be defined by the condition that $\tilde{F}(x, \cdot)$ is the rearrangement of $F(x, \cdot)$, for each fixed $x \in X$. Note that F and $\tilde{F}$ have the same rearrangement function. By Fubini's theorem and by Theorem 5.5 and Lemma 4.4,

$$\begin{aligned}(\mu \times \ell)(H) &= \int_X \ell\big(G(x)\big)\mu(dx)\\ &\leq \frac{4\ell(J)}{\lambda}\int_X\int_R \tilde{F}(x,t)\,\varphi^*\left(t\frac{\ell(J)}{2M}\right)dt\,\mu(dx)\\ &\leq \frac{4\ell(J)}{\lambda}\int_R F^*(t)\,\varphi^*\left(t\frac{\ell(J)}{2M\mu(X)}\right)dt.\end{aligned}$$

The second statement follows again from Lemma 4.4.

NOTATION 5.8. Let $\{(a_k, r_k)\}$ be a B-sequence and define $J_k = a_k + r_k J$, where, as before, J is an interval containing the support of the weight function φ and $\ell(J) \geq 1$. We fix an integer $K \geq 1$ and find a number $R > 0$ such that the interval $(-R, R)$ contains J_k for $k \leq K$. We then fix an arbitrary number $S > 0$ and let $M = R + S$ and $I = (-M, M)$. We then consider a nonnegative function f on X and define F on $X \times \mathbb{R}$ by $F(x,t) = f(T_t x)$ if $|t| < M$ and $F(x,t) = 0$ otherwise, where $\{T_t\}_{t\in\mathbb{R}}$ is a measure preserving flow acting on the finite measure space $(X, \mathcal{F}, \mu)$. The previous notations will now apply to this particular choice of F. Note that F satisfies the hypothesis stated in the second part of the Lemma above. In fact, since the flow is measure preserving, the distribution of $F(\cdot, t) = f(T_t\cdot)$ is equal to the distribution of f for each $t \in I$ and zero otherwise. Hence we see that $F^*(t) = f^*\big(t/(2M)\big)$. We define $G(x)$ as before with these choices of F and M and with a fixed $\lambda > 0$. Recall that

$$\mathcal{A}_k^{\varphi} f(x) = \int_{\mathbb{R}} f(T_t x)\,\psi_k(t)\,dt,$$

where

$$\psi_k(t) = (1/r_k)\varphi\big((t-a_k)/r_k\big).$$

LEMMA 5.9. *Let $k \leq K$ and $|t| < S$. If $\mathcal{A}_k^{\varphi} f(T_t x) > \lambda$ then $t + J_k \subset G(x)$.*

PROOF. We have

$$\lambda < \mathcal{A}_k^{\varphi} f(T_t x) = \int_{\mathbb{R}} f(T_{t+s}x)\,\varphi\big((s-a_k)/r_k\big)(1/r_k)\,ds.$$

Since the support of φ is contained in J, the integral above can be taken over J_k only, instead of $\mathbb{R}$. By a change of variables we obtain

$$\lambda < \mathcal{A}_k^{\varphi} f(T_t x) = \int_{t+J_k} f(T_s x)\,\varphi\Big(\big(s-(t+a_k)\big)/r_k\Big)(1/r_k)\,ds.$$

Since $k \leq K$ and $|t| < S$ we see that $t + J_k \subset I$. Hence we can also replace $f(T_s x)$ by $F(x,s)$ in the integral above. Then Lemma 5.3 shows that this integral is dominated by $\rho(t+J_k, x)$. Therefore $t + J_k \subset G(x)$.

LEMMA 5.10. *Let*

$$V = \{x \mid \exists k \leq K, \mathcal{A}_k^{\varphi} f(x) > \lambda\}.$$

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press

Then

$$\mu(V) \leq \frac{4BM\ell(J)}{\lambda S} \int_{\mathbb{R}} f^*(t)\, \varphi^*\left(t \frac{\ell(J)}{\mu(X)}\right) dt,$$

where B is the constant for the B-sequence of intervals

$$J_k = a_k + r_k J,$$

as defined in 2.2.

PROOF. Let

$$W = \{(x,t) \mid |t| < S, T_t x \in V\}.$$

We calculate $(\mu \times \ell)(W)$ in two different ways, by Fubini's Theorem. First,

$$\begin{aligned}(\mu \times \ell)(W) &= \int_{\mathbb{R}} \int_X \chi_W(x,t)\, \mu(dx)\, d\ell \\ &= \int_{-S}^{S} \int_X \chi_{T_t^{-1}V}(x)\, \mu(dx)\, dt \\ &= 2S\mu(V).\end{aligned}$$

For the second order we integrate on $\mathbb{R}$ first, for a fixed $x \in X$. We see that $(x,t) \in W$ if $|t| < S$ and if there is a $k \leq K$ such that

$$\mathcal{A}_k^{\varphi} f(T_t x) > \lambda.$$

Then the previous lemma shows that

$$W(x) = \{t \mid (x,t) \in W\} \subset \big\{t \mid \exists k, t + J_k \subset G(x)\big\}.$$

Hence $\ell\big(W(x)\big) \leq B\ell\big(G(x)\big)$. Therefore,

$$(\mu \times \ell)(W) = \int_X \ell\big(W(x)\big)\mu(dx) \leq B \int_X \ell\big(G(x)\big)\mu(dx).$$

The last integral gives $(\mu \times \ell)(H)$, with the notations of Lemma 5.7. Hence, by applying this lemma and observing that the rearrangement function ξ is now f^*,

$$\begin{aligned}(\mu \times \ell)(W) &\leq \frac{4B\ell(J)}{\lambda} \int_{\mathbb{R}} f^*\big(t/(2M)\big)\, \varphi^*\left(t \frac{\ell(J)}{2M\mu(X)}\right) dt \\ &= \frac{8BM\ell(J)}{\lambda} \int_{\mathbb{R}} f^*(t)\, \varphi^*\left(t \frac{\ell(J)}{\mu(X)}\right) dt.\end{aligned}$$

Combining these two expressions for $(\mu \times \ell)(W)$ we conclude the proof.

PROOF OF THE MAIN THEOREM 5.1. We recall that $M = R + S$, where $R > 0$ was a fixed number determined by $K \in \mathbb{N}$ and $S > 0$ was arbitrary. By letting S go to infinity we see that, for any $K \in \mathbb{N}$,

$$\mu\big(\{x \mid \exists k \leq K, \mathcal{A}_k^{\varphi} f(x) > \lambda\}\big) \leq \frac{4B\ell(J)}{\lambda} \int_{\mathbb{R}} f^*(t)\, \varphi^*\left(t \frac{\ell(J)}{\mu(X)}\right) dt.$$

Hence we conclude that

$$\mu\big(\{x \mid \exists k, \mathcal{A}_k^\varphi f(x) > \lambda\}\big) \leq \frac{4B\ell(J)}{\lambda} \int_{\mathbb{R}} f^*(t)\, \varphi^*\left(t\frac{\ell(J)}{\mu(X)}\right) dt.$$

Finally, by Lemma 4.5, the last integral is dominated by

$$\left(1 + \frac{\mu(X)}{\ell(J)}\right) \int_{\mathbb{R}} f^*(t)\, \varphi^*(t)\, dt.$$

Hence the constant

$$C = 4B\ell(J)\left(1 + \frac{\mu(X)}{\ell(J)}\right) = 4B\big(\ell(J) + \mu(X)\big)$$

satisfies the requirements of the theorem.

6. **Convergence of averages.**

NOTATION 6.1. Let $\{(a_k, r_k)\}$ be a sequence in $\mathbb{R} \times \mathbb{R}$ such that $r_k > 0$ for all k. We recall that

$$\mathcal{A}_k^\varphi f(x) = \int_{\mathbb{R}} f(T_t x)\, \psi_k(t)\, dt,$$

where φ is a weight function and

$$\psi_k(t) = \varphi\big((t - a_k)/r_k\big)(1/r_k).$$

We also let

$$\mathcal{M}^\varphi f(x) = \sup_k |\mathcal{A}_k^\varphi f(x)|,$$

and observe that

$$\mu\big(\{x \mid \mathcal{M}^\varphi f(x) > \lambda\}\big) \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f|^* \varphi^*\, d\ell,$$

by the maximal theorem. Note that if the support of φ is contained in an interval J then the support of ψ_k is contained in the interval $J_k = a_k + r_k J$. The following simple lemma is useful for the investigation of $\mathcal{A}_k^\varphi f$ for bounded functions f.

LEMMA 6.2. *Let φ and φ' be two weight functions. Then*

$$|\mathcal{A}_k^\varphi f(x) = \mathcal{A}_k^{\varphi'} f(x)| \leq \|\varphi - \varphi'\|_1 \|f\|_\infty.$$

Hence, for a given bounded function f, if $\mathcal{A}_k^\varphi f$ converges a.e. for a class of weight functions that approximate any weight function in the $L_1(\mathbb{R})$-norm, then $\mathcal{A}_k^\varphi f$ converges a.e. for any weight function.

PROOF. Follows from the fact that

$$\int_{\mathbb{R}} \left|\varphi\big((t - a_k)/r_k\big) - \varphi'\big((t - a_k)/r_k\big)\right| dt = \|\varphi - \varphi'\|_1.$$

LEMMA 6 3 *Let φ be a weight function and let $\{(a_k, r_k)\}$ be a sequence such that both a_k and r_k converge to zero Then $\mathcal{A}_k^{\varphi} f$ converges a e for any function f of the form $f(x) = \int_0^a g(T_s x)\, ds$, where g is a bounded function*

PROOF Without loss of generality we can assume that the weight function φ has compact support, because of the previous lemma The function $f(T_t x) = \int_t^{t+a} g(T_s x)\, ds$ is a bounded and continuous function of t for a a x To complete the proof it is enough to show that if F $\mathbb{R} \longrightarrow \mathbb{R}$ is a bounded and continuous function then

$$\int_{\mathbb{R}} F(t_0 + t)\, \psi_k(t)\, dt$$

converges to $F(t_0)$, for each $t_0 \in \mathbb{R}$ To see this assume that $F(t_0) = 0$, without loss of generality Let J be an interval containing the support of φ Find a number $\delta > 0$ such that $|F(t)| < \varepsilon$ whenever $|t - t_0| < \delta$ Then find a $k_0 \in \mathbb{N}$ such that $J_k \subset (-\delta, \delta)$ for all $k \geq k_0$ Then, for $k \geq k_0$,

$$\begin{aligned} \left| \int_{\mathbb{R}} F(t_0 + t)\, \psi_k(t)\, dt \right| &= \left| \int_{J_k} F(t_0 + t)\, \psi_k(t)\, dt \right| \\ &\leq \int_{J_k} |F(t_0 + t)|\, \psi_k(t)\, dt \leq \varepsilon \end{aligned}$$

THEOREM 6 4 *Let $\{(a_k, r_k)\}$ be a B-sequence such that both a_k and r_k converge to zero Let φ be a weight function Then $\mathcal{A}_k^{\varphi} f$ converges a e for each bounded function f*

PROOF Again we will assume that the weight function φ has compact support, without loss of generality Let f be a bounded function with $M = \|f\|_\infty$ Then the family of functions $f_a(x) = (1/a) \int_0^a f(T_t x)\, dt$ converge to f in $L_1(X)$, as a approaches to 0^+ Also, this family of functions is uniformly bounded by the same bound M If $\mathcal{A}_k^{\varphi} f$ does not converge on a e , then there is a set $E \subset X$ of μ-measure $m > 0$ and a number $\lambda > 0$ such that liminf and limsup of $\mathcal{A}_k^{\varphi} f$ differ by more than λ on E Since $\mathcal{A}_k^{\varphi} f_a$ is convergent, we see that liminf and limsup of $\mathcal{A}_k^{\varphi}(f - f_a)$ also differ by more than λ on E, for each $a > 0$ This means that

$$E \subset \{x \mid \mathcal{M}^{\varphi} |f - f_a|(x) > \lambda/2\},$$

for any $a > 0$ This, however, violates the Maximal Inequality 5 1, which says that the measure of the last set is dominated by

$$\frac{2C}{\lambda} \int_{\mathbb{R}} |f - f_a|^* \varphi^*\, d\ell$$

In fact, this integral converges to zero as $a \longrightarrow 0^+$, since the integrand converges to zero ℓ-a e on $\mathbb{R}$ and is dominated by an L_1 function

THEOREM 6 5 *Let $\{(a_k, r_k)\}$ be a B-sequence such that r_k converges to infinity Let φ be a weight function The $\mathcal{A}_k^{\varphi} f$ converges a e for each bounded function f*

PROOF Let u and v be two numbers, $u < v$, and assume that

$$\varphi = \bigl(1/(v - u)\bigr) \chi_{(u\ v)}$$

In this case the a.e. convergence of $\mathcal{A}_k^\varphi f$ is obtained easily if f is of the form

$$f(x) = h(x) + g(T_1 x) - g(x),$$

where g and h are two bounded functions and h satisfies $h(T_1 x) = h(x)$. Any bounded function can be approximated by functions of this form in the $L_1(X)$-norm, as, for example, an application of the mean ergodic theorem for T_1 in $L_2(X)$ shows. Then we see that $\mathcal{A}_k^\varphi f$ converges a.e. for every bounded function, applying the maximal inequality as in the proof of the previous theorem. If φ is a linear combination of weight functions of this special form, that is, if φ is a step function, then it is clear that $\mathcal{A}_k^\varphi f$ also converges for each bounded function f. Since any weight function can be approximated by step functions in the L_1-norm, the proof is completed by an application of Lemma 6.2.

THEOREM 6.6. *Assume that $\{(a_k, r_k)\}$ is a B-sequence such that either both a_k and r_k converge to zero or r_k converges to infinity. Let φ be a weight function with a compact support. Then $\mathcal{A}_k^\varphi f$ converges a.e. for all functions f on X such that*

$$\int_{\mathbb{R}} |f|^* \varphi^* \, d\ell < \infty.$$

PROOF. It is enough to restrict the attention to nonnegative functions. We know that $\mathcal{A}_k^\varphi f$ converges for bounded functions. The proof is completed by an application of the maximal inequality, as in the proof of Theorem 6.4. We give some of the details. Let f be a nonnegative function such that

$$\int_{\mathbb{R}} f^* \varphi^* \, d\ell < \infty.$$

Let f_n be the minimum of f and the constant function n. Then each f_n is a bounded function and

$$\int_{\mathbb{R}} (f - f_n)^* \varphi^* \, d\ell$$

converges to zero as n approaches infinity. If $\mathcal{A}_k^\varphi f$ diverges on a set of positive measure, there is a constant $\lambda > 0$ and a set E of positive measure such that limsup and liminf of $\mathcal{A}_k^\varphi f$ differ by more than λ on E. Then this is also true for $\mathcal{A}_k^\varphi (f - f_n)$, for any n. Hence

$$E \subset \{x \mid \mathcal{M}^\varphi (f - f_n)(x) > \lambda\},$$

for all n. The maximal inequality is then again violated, as the measure of the last set must be less than

$$\frac{C}{\lambda} \int_{\mathbb{R}} (f - f_n)^* \varphi^* \, d\ell,$$

which is arbitrarily small if n is sufficiently large.

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press

THEOREM 6.7. *Let $\{\varphi_k\}$ be a sequence of weight functions. Assume that this sequence is dominated by an $L_1(\mathbb{R}) = L_1(\mathbb{R}, \mathcal{B}, \ell)$ function Φ of compact support and converges ℓ-a.e. to a function φ, which is necessarily another weight function. Let*

$$\mathcal{A}_k f(x) = \int_X f(T_t x)\, \theta_k(t)\, dt$$

where

$$\theta_k(t) = (1/r_k)\varphi_k\big((t - a_k)/r_k\big).$$

If the function f satisfies

$$\int_{\mathbb{R}} |f|^* \Phi^*\, d\ell < \infty,$$

then

$$F_k = |\mathcal{A}_k^{\varphi} f - \mathcal{A}_k f|$$

converges a.e. to zero.

PROOF. Let

$$\Phi_n(t) = \sup_{m \geq n} |\varphi(t) - \varphi_m(t)|$$

and

$$\vartheta_{n,k}(t) = (1/r_k)\Phi_n\big((t - a_k)/r_k\big).$$

Then we see that Φ_n^* is dominated by Φ^* and converges ℓ-a.e. to zero. Hence, if

$$\int_{\mathbb{R}} |f|^* \Phi^*\, d\ell < \infty,$$

then

$$\int_{\mathbb{R}} |f|^* \Phi_n^*\, d\ell$$

converges to zero. Also, if $k \geq n$ then

$$\begin{aligned} F_k &= |\mathcal{A}_k^{\varphi} f - \mathcal{A}_k f| \\ &\leq \int_{\mathbb{R}} |f|\, |\psi_k - \theta_k|\, d\ell \\ &\leq \int_{\mathbb{R}} |f| \vartheta_{n,k}\, d\ell. \end{aligned}$$

Hence, by the maximal inequality, for any $\lambda > 0$,

$$\mu\Big(\Big\{x \mid \sup_{k \geq n} F_k(x) \geq \lambda\Big\}\Big) \leq \frac{C}{\lambda} \int_{\mathbb{R}} |f|^* \Phi_n^*\, d\ell.$$

This completes the proof, as the last integral converges to zero as n approaches infinity.

REFERENCES

1. A Bellow, R Jones and J Rosenblatt, *Convergence for moving averages*, J Erg Th Dyn Sys **10**(1990), 43–62

2. ———, *Almost everywhere convergence of weighted averages*, preprint

3. ———, *Almost everywhere convergence of powers*, preprint

4. A Bellow and V Losert, *The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences*, Trans Amer Math Soc **288**(1985), 349–353

5. M Broise, Y Deniel and Y Derriennic, *Rearrangement inegalites maximales et theoremes ergodiques fractionnaires*, Ann Inst Fourier, Grenoble **39**(1989), 689–714

6. ———, *Maximal inequalities and ergodic theorems for Cesaro α or weighted averages*, preprint

7. A P Calderon, *Ergodic theory and translation invariant operators*, Proc Nat Acad Sci **59**(1968), 349–353

8. Y Deniel, *On the a e Cesaro α convergence for stationary or orthogonal random variables*, J Theoretical Probability **2**(1989), 475–485

9. Y Deniel and Y Derriennic, *Sur la convergence presque sûre au sens de Cesaro d ordre α $0 < \alpha < 1$ de v a i i d*, Prob Th **79**(1988), 629–639

10. G H Hardy, J E Littlewood and G Polya, *Inequalities*, Cambridge Univ Press, 1934

11. R Jones and J Olsen, *Subsequence ergodic theorems for operators*, preprint

12. R Jones, J Olsen and M Wierdl, *Subsequence ergodic theorems for L^p contractions*, preprint

13. A Nagel and E M Stein, *On certain maximal functions and approach regions*, Adv Math **54**(1984), 83–106

14. J Rosenblatt and M Wierdl, *A new maximal inequality and applications*, preprint

15. J Sueiro, *A note on maximal operators of Hardy Littlewood type*, Math Proc Camb Phil Soc **102**(1987), 131–134

16. A Zygmund, *Trigonometric Series*, Cambridge Univ Press, 1968

Department of Mathematics
University of Toronto
Toronto Ontario
M5S 1A1

Departement de Mathematiques et Informatique
Universite de Bretagne occidentale
6 avenue Victor le Gorgeu
29287 Brest
France

https://doi.org/10.4153/CJM-1993-023-x Published online by Cambridge University Press