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HOLOMORPHIC SUPERPOSITION OPERATORS BETWEEN BANACH FUNCTION SPACES

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Abstract

We prove that for a large class of Banach function spaces continuity and holomorphy of superposition operators are equivalent and that bounded superposition operators are continuous. We also use techniques from infinite dimensional holomorphy to establish the boundedness of certain superposition operators. Finally, we apply our results to the study of superposition operators on weighted spaces of holomorphic functions and the $F(p,\alpha,\beta)$ spaces of Zhao. Some independent properties on these spaces are also obtained.

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1. Introduction

One of the most fundamental operations which arises when working with functions is that of composition. Even the most elementary of mathematical courses, however, emphasises that the operation of composition is, in general, not commutative. This lack of commutativity is reflected when studying composition in function spaces and we are forced to consider two distinct operations. The operation obtained from composition on the right gives rise to the study of composition operators. Composition operators are linear operators. They have been studied by many authors, initially on function spaces over finite dimensional domains but more recently over infinite dimensional spaces (see [4, 5, 10, 21–23, 26]). Fundamental problems studied in the theory of composition operators are characterisations of continuity, compactness and nuclearity of the composition operator in terms of the symbol, the topological structure of the set of composition operators and the spectra of composition operators. The operators obtained from composition on the left are called superposition operators. In contrast to composition operators, superposition operators are, in general, highly nonlinear. This

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nonlinearity is one of the fundamental differences between superposition operators and composition operators. The lack of linearity means that continuity and boundedness of the operator may differ and have to be studied separately.

In this paper we will concentrate on the more theoretic aspect of superposition operators. In Section 3 we will show that for a large class of function spaces boundedness of a superposition operator implies continuity. Moreover, using techniques from infinite dimensional holomorphy we will prove that for such spaces a superposition operator is locally bounded if and only if it is holomorphic and hence, if and only if it is continuous. In Section 4 we specialize our study to two classes of function spaces: weighted spaces of holomorphic functions and $F(p, \alpha, \beta)$ spaces. We show that in many cases bounded or compact superposition operators can only be obtained from affine linear or constant mappings.

2. Background and notation

Given an open subset U of $\mathbb{C}^n$ let X, Y be complex Banach spaces of scalar-valued functions on U. We say that an entire function ϕ induces a superposition operator from X into Y if for each f in X the composition $\phi \circ f$ belongs to Y. When it exists we use S_ϕ to denote the superposition operator induced by ϕ, that is $S_\phi(f) = \phi \circ f$.

Given an entire function ϕ and $r > 0$ we let $M_\phi(r)$ denote the maximum of $|\phi(z)|$ over the closed disc with centre 0 and radius r, that is $M_\phi(r) = \sup_{|z|\le r} |\phi(z)| = \sup_{|z|=r} |\phi(z)|$. We say that ϕ has order ρ if

$$\limsup_{r\to\infty} \frac{\log\log M_\phi(r)}{\log r} = \rho.$$

If ϕ has order ρ we shall say that it is of type τ if

$$\limsup_{r\to\infty} \frac{\log M_\phi(r)}{r^\rho} = \tau.$$

A number of authors have investigated superposition operators between spaces of analytic functions on the ball. Cámera and Giménez, [20], show that ϕ induces a superposition operator from the Bergman space of order p, A^p, into the Hardy space of order q, H^q if and only if ϕ is constant, while it maps H^p into A^q if and only if it is a polynomial of degree at most $[2p/q]$. Necessary and sufficient conditions for ϕ to map A^p to A^q are also obtained while [19] considers the case of superposition operators between Hardy spaces. A number of results for superposition operators on Dirichlet type spaces have been obtained in [17]. Álvarez *et al.* [2] show that S_ϕ maps the Bloch space into Bergman space if and only if it is of order less than one or of order one and type zero. Recently, Buckley and Vukotić, [18], have shown that if we wish to replace the Bloch space with the little Bloch space ϕ must have order less than one or have order one and finite type. In the same paper they give a necessary and sufficient condition for ϕ to induce a superposition operator from the Besov space, B^p, into the Bergman space A^q.

Section 4 is concerned mainly with weighted spaces of holomorphic functions and $F(p,\alpha,\beta)$ spaces. Let us say a little more on each of these classes.

A weight v on an open subset U of $\mathbb{C}^n$ is a continuous bounded function $v\colon U \to \mathbb{R}$ with $v(z) > 0$ for z in U. We use $\mathcal{H}_v(U)$ to denote the space of all holomorphic functions f on U such that $\|f\|_v := \sup_{z\in U} v(z)|f(z)| < \infty$. Endowed with the norm $\|\cdot\|_v$, $\mathcal{H}_v(U)$ becomes a Banach space. We let $\mathcal{H}_{v_o}(U)$ denote the subspace of $\mathcal{H}_v(U)$ of all f in $\mathcal{H}_v(U)$ with the property that $|f(z)|v(z)$ converges to 0 as z converges to the boundary of U. By this we mean that given $\epsilon > 0$ there is a compact subset K of U such that $v(z)|f(z)| < \epsilon$ for all z in $U \setminus K$. Thus $\mathcal{H}_v(U)$ may be regarded as all holomorphic functions on U which satisfy a growth rate of order $O(1/v(z))$ while $\mathcal{H}_{v_o}(U)$ are those holomorphic functions with a growth rate of order $o(1/v(z))$.

We denote by $\mathcal{B}$ the Bloch space of all holomorphic functions f on the open unit disc, Δ, such that $\|f\|_{\mathcal{B}} := |f(0)| + \sup_{z\in\Delta}(1-|z|^2)|f'(z)| < \infty$. The little Bloch space, $\mathcal{B}_o$, is the set of all f in $\mathcal{B}$ with $\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0$.

An integrable function, f, on the unit circle $\mathcal{T}$ is said to be of bounded mean oscillation (BMO) if

$$\|f\|_{\mathrm{BMO}} := \sup_{I\subset\mathcal{T}}\bigg(\frac{1}{|I|}\int_I |f(\zeta) - f_I|^2\,d\zeta\bigg)^{1/2} < \infty,$$

where $|I|$ denotes the arclength of I and $f_I = \int_I f(\zeta)\,d\zeta$. Given an interval I in $\mathcal{T}$ let $R(I) = \{re^{i\theta} : 1 - |I|/2\pi < r < 1, e^{i\theta} \in I\}$. A positive measure μ on Δ is said to be a Carleson measure if

$$\nu(\mu) := \sup_{I\subset\mathcal{T}} \frac{\mu(R(I))}{|I|}$$

is finite. Given an integrable function f on $\mathcal{T}$ define $d\mu_f$ by $d\mu_f(z) = (1/2)|(\nabla f)(z)|^2(1-|z|)\,dA(z)$. Fefferman shows that f belongs to BMO if and only if μ_f is a Carleson measure. A function f in BMO is said to have vanishing mean oscillation (VMO) if

$$\lim_{|I|\to 0} \frac{\mu_f(R(I))}{|I|} = 0.$$

The space of all analytic functions which are the Poisson integral of functions in BMO (respectively VMO) is denoted by BMOA (respectively VMOA).

For $1 < p < \infty$, $-2 < \alpha < \infty$ and $0 \le \beta < \infty$ the space $F(p,\alpha,\beta)$, introduced by Zhao in [31], consists of all holomorphic functions g in the unit disc such that

$$\|g\|_{F(p,\alpha,\beta)} := \sup_{a\in\Delta}\bigg(\int_\Delta |g'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta\,dA(z)\bigg)^{1/p} < \infty,$$

where σ_a is the Möbius Transformation of the disc $\sigma_a(z) = (a-z)/(1-\bar{a}z)$. The norm on $F(p,\alpha,\beta)$ is given by $|g(0)| + \|g\|_{F(p,\alpha,\beta)}$. The space $F_0(p,\alpha,\beta)$ is the subspace of $F(p,\alpha,\beta)$ of all functions g such that

$$\lim_{|a|\to 1^-}\bigg(\int_\Delta |g'(z)|^p(1-|z|^2)^\alpha(1-|\sigma_a(z)|^2)^\beta\,dA(z)\bigg) = 0.$$

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Special cases of $F(p, \alpha, \beta)$ spaces are known. For example $F(2, 0, \beta)$ and $F_0(2, 0, \beta)$ are the Q_β and $Q_{\beta,0}$ spaces respectively, introduced by Aulaskari *et al.* [1] (see also [30]), $F(p, \alpha p - 2, \beta)$ and $F_0(p, \alpha p - 2, \beta)$, $\beta > 1$ are the Bloch-type spaces $\mathcal{B}^\alpha$ and $\mathcal{B}^\alpha_0$ ($\alpha = 0$ gives the Bloch $\mathcal{B}$ and little Bloch space $\mathcal{B}_0$), $F(p, \alpha p - 2, 1)$ and $F_0(p, \alpha p - 2, 1)$ are the spaces BMOA^α_p and VMOA^α_p. When $\alpha = 1$ and $p = 2$ we obtain BMOA and VMOA. In [28] Lindström and Palmberg develop the duality theory of $F(p, \alpha, \beta)$ spaces. They show that when $\alpha \geq 0$ and $p > 1 + \alpha + \beta$, $F(p, \alpha, \beta)$ is isometrically isomorphic to the bidual of $F_0(p, \alpha, \beta)$. Furthermore, [28, Corollary 4.3] and the comments following [28, Theorem 4.1] imply that $\{\delta_z : z \in \Delta\}$, $\delta_z(f) = f(z)$, spans a dense linear subspace of $F_0(p, \alpha, \beta)'$.

Given complex Banach spaces E and F we say that a function $f : E \to F$ is holomorphic if it is continuous and for each ϕ in F', and each a, b in E, the function $\lambda \mapsto \phi(f(a + \lambda b))$ is holomorphic in a neighbourhood of O. A holomorphic function is said to be of bounded type if it maps bounded sets to bounded sets. The space of all holomorphic functions of bounded type from X into Y is denoted by $\mathcal{H}_b(X, Y)$. There is, in general, no Hahn–Banach theorem for spaces of homogeneous polynomials and hence for holomorphic functions. In 1978, however, Aron and Berner, [3], showed that each holomorphic function of bounded type f on a Banach space E has an extension to a holomorphic function of bounded type on E''. We shall denote this extension by $AB(f)$ and refer to it as the Aron–Berner extension of f. For further reading on holomorphic functions we refer the reader to [24].

3. Continuity and boundedness of superposition operators

Let U be an open subset of $\mathbb{C}^n$. Given a complex Banach space X of scalar-valued functions on U we use X' to denote the dual of X. Given $z \in U$, we denote by δ^X_z the evaluation map given by $\delta^X_z(g) = g(z)$ for $g \in X$. A subset H of X' is said to be separating if $\eta(g) = 0$ for all η in H implies $g = 0$. A function f between Banach spaces X and Y is locally bounded (compact, weakly compact respectively) if for each x in X there is a neighbourhood V_x of x such that $f(V_x)$ is bounded (relatively compact, relatively weakly compact respectively). We denote the compact-open topology of uniform convergence on compact subsets of U by τ_0.

Much of the theory of superposition operators to date has been devoted to establishing the continuity or boundedness of superposition operators between given function spaces. Our first result shows that for a wide class of function spaces, which includes weighted spaces of holomorphic functions and spaces of Zhao, continuity, local boundedness and holomorphy of superposition operators are equivalent.

THEOREM 3.1. *Let U be an open subset of $\mathbb{C}^n$ and let X and Y be complex Banach spaces of scalar-valued functions on U such that δ^Y_z belongs to Y' for all z in U. Let ϕ be an entire function which induces a superposition operator S_ϕ from X into Y. Then the following are equivalent:*

(a) *S_ϕ is holomorphic;*
(b) *S_ϕ is continuous;*
(c) *S_ϕ is locally bounded.*

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Proof. Clearly (a) implies (b) and (b) implies (c). Let us prove (c) implies (a). Since $\{\delta_z^Y : z \in U\}$ is a separating subset of Y', by [27, Theorem 1] it is sufficient to show that for every $z \in U$ and every f_o, f in X the function $\lambda \mapsto \delta_z^Y \circ S_\phi(f_o + \lambda f)$ is holomorphic on some neighbourhood of 0. For every $z \in U$, f_o, f in X we have that $\delta_z^Y \circ S_\phi(f_o + \lambda f) = \phi(f_o(z) + \lambda f(z))$ and the result follows. □

The hypothesis that δ_z^Y belongs to Y' for all z in U is satisfied by many spaces of scalar functions. For instance, it is well known that given any weight v defined on an open set $U \subset \mathbb{C}^n$ the evaluation maps $\delta_z : \mathcal{H}_v(U) \to \mathbb{C}$ (and so $\delta_z : \mathcal{H}_{v_0}(U) \to \mathbb{C}$) are continuous for any $z \in U$. So, all weighted spaces of holomorphic functions satisfy the condition. For $\alpha \geq 0, \beta > 0$ and $p \geq 1 + \alpha + \beta$, the evaluation maps $\delta_z : F(p, \alpha, \beta) \to \mathbb{C}$ (and so $\delta_z : F_0(p, \alpha, \beta) \to \mathbb{C}$) are continuous for any $z \in U$ (see [28]). These examples include the spaces Q_β and $Q_{\beta,0}$ for $0 < \beta \leq 1$, the Bloch-type spaces $\mathcal{B}_0^\alpha$ for $\beta > 1$ and $(1 - \alpha)p \geq \beta - 1$, in particular the Bloch space $\mathcal{B}$ and little Bloch space $\mathcal{B}_0$, and the spaces BMOA_p^α and VMOA_p^α for $0 < \alpha \leq 1$ and $\beta = 1$, in particular BMOA and VMOA.

We recall that a function defined between metric spaces is bounded if it maps bounded sets into bounded sets. Bounded holomorphic functions between Banach spaces are usually called holomorphic functions of bounded type.

It is shown in [18] that there are continuous superposition operators which are not bounded. Since every bounded (alternatively compact or weakly compact) function between Banach spaces is locally bounded we obtain the following corollary.

Corollary 3.2. *Let U be an open subset of $\mathbb{C}^n$ and let X and Y be complex Banach spaces of scalar-valued functions on U such that δ_z^Y belongs to Y' for all z in U. Let ϕ be an entire function which induces a bounded (alternatively compact or weakly compact) superposition operator S_ϕ from X into Y. Then S_ϕ is holomorphic and hence continuous.*

The following proposition shows that superposition operators are always continuous for the compact-open topology τ_o.

Proposition 3.3. *Let U be an open subset of $\mathbb{C}^n$ and X, Y be spaces of scalar-valued functions on U. Let ϕ be an entire function which induces a superposition operator S_ϕ from X into Y. Then $S_\phi : (X, \tau_o) \to (Y, \tau_o)$ is continuous.*

Proof. Let $(f_i)_i$ be a net in X that converges uniformly on compact subsets of U to $f \in X$. Let K be a compact subset of U and let $\epsilon > 0$. As K is compact, choose $r > 0$ so that $\sup_{z \in K} |f(z)| < r$. Since ϕ is uniformly continuous on the compact set $2r\overline{\Delta}$, there is $\delta > 0$, $\delta < r$, so that

$$|\phi(w) - \phi(w')| < \epsilon \tag{3.1}$$

for every $w, w' \in 2r\overline{\Delta}$ with $|w - w'| < \delta$. Choose i_0 such that $|f_i(z) - f(z)| < \delta$ for all $z \in K$ and all $i \geq i_0$. As

$$|f_i(z)| \leq |f_i(z) - f(z)| + |f(z)| < \delta + r < 2r,$$

from (3.1) it follows that $|\phi(f_i(z)) - \phi(f(z))| < \epsilon$ for all $z \in K$ and all $i \geq i_0$. This shows that $\phi \circ f_i$ converges to $\phi \circ f$ uniformly on K. □

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We have seen in Corollary 3.2 that bounded superposition operators are continuous under the mild condition that each evaluation map is continuous on the range space. The next result shows that, under the same condition, entire functions that induce a bounded superposition operator between complex Banach spaces of scalar-valued functions actually induce a bounded superposition operator between their biduals.

THEOREM 3.4. *Let U be an open subset of $\mathbb{C}^n$ and X, Y be complex Banach spaces of scalar-valued functions on U such that X'' and Y'' are also Banach spaces of scalar-valued functions on U. Suppose that δ_z^Y belongs to Y' for every z in U. If ϕ is an entire function which induces a bounded superposition operator from X into Y then ϕ induces a bounded superposition operator from X'' into Y''.*

PROOF. To avoid confusion we use $S_\phi^{X,Y}$ to denote the superposition operator from X to Y induced by ϕ. It follows from Corollary 3.2 that $S_\phi^{X,Y}$ is a holomorphic function of bounded type. Then its Aron–Berner extension $AB(S_\phi^{X,Y})$ is a holomorphic function of bounded type from X'' into Y''. As ϕ is an entire function we can write it as $\phi(z) = \sum_{m=0}^\infty a_m z^m$ for $z \in \mathbb{C}$. Then $S_\phi^{X,Y}(f) = \sum_{m=0}^\infty a_m f^m$ is the Taylor series expansion of $S_\phi^{X,Y}$ about the origin. Thus

$$AB(S_\phi^{X,Y})(f) = \sum_{m=0}^{\infty} AB(g \mapsto a_m g^m)(f) = \sum_{m=0}^{\infty} a_m f^m = \phi \circ f$$

for all $f \in X''$, proving that $S_\phi^{X'',Y''} = AB(S_\phi^{X,Y})$ is a bounded holomorphic function from X'' into Y''. □

THEOREM 3.5. *Let X and Y be Banach spaces of scalar-valued functions defined on some open subset U of $\mathbb{C}^n$. Suppose that ϕ is an entire function that induces a holomorphic superposition operator S_ϕ from X into Y. Then S_ϕ is bounded if and only if:*

(a) *for each n in $\mathbb{N}$ with $\phi^{(n)}(0) \neq 0$ the set $\{f^n : f \in X, \|f\|_X \leq 1\}$ is bounded in Y; and*
(b) *if $M_n = \sup\{\|f^n\|_Y : \|f\|_X \leq 1\}$ then $\lim_{n\to\infty} |\phi^{(n)}(0)/n!|^{1/n} M_n^{1/n} = 0$.*

PROOF. We observe that S_ϕ is bounded if and only if it belongs to $\mathcal{H}_b(X, Y)$. If we write $\phi(z)$ as $\phi(z) = \sum_{n=0}^\infty (\phi^{(n)}(0)/n!) z^n$, $z \in \mathbb{C}$, then $S_\phi(f) = \sum_{n=0}^\infty (\phi^{(n)}(0)/n!) f^n$ is the Taylor series expansion of S_ϕ. Thus $(\hat{d}^n S_\phi(0)/n!)(f) = (\phi^{(n)}(0)/n!) f^n$ for any $f \in X$. The result now follows from [24, page 165]. □

THEOREM 3.6. *Let X and Y be Banach spaces of scalar-valued functions defined on some open subset U of $\mathbb{C}^n$. Suppose that ϕ is an entire function that induces a holomorphic superposition operator S_ϕ from X into Y. Then S_ϕ is compact (respectively weakly compact) if and only if for each n in $\mathbb{N}$ with $\phi^{(n)}(0) \neq 0$ the set $\{f^n : f \in X, \|f\|_X \leq 1\}$ is relatively compact (respectively relatively weakly compact) in Y.*

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Proof. By [6, Proposition 3.4] S_ϕ is compact if and only if $\hat{d}^n S_\phi(0)/n!$ is compact for each positive integer n. The result now follows from the fact that $(\hat{d}^n S_\phi(0)/n!)(f) = (\phi^{(n)}(0)/n!) f^n$.

Replacing [6, Proposition 3.4] with [29, Proposition 3.2] we get the weakly compact case. □

4. Weighted spaces of holomorphic functions and spaces of Zhao

In this section we study superposition operators between weighted spaces of holomorphic functions and spaces of Zhao. Some of the results are applications of the general ones obtained in the previous section. Others are independent as they use techniques inherent to the specific spaces.

Given a weight $v\colon U \to \mathbb{R}$ we define $w\colon U \to \mathbb{R}$ by $w(z) = 1/v(z)$. The closed unit ball of $\mathcal{H}_{v_o}(U)$ is $\{f \in \mathcal{H}_{v_o}(U) : |f(z)| \leq w(z),\ \text{for all } z \in U\}$ whereas the closed unit ball of $\mathcal{H}_v(U)$ is equal to $\{f \in \mathcal{H}_v(U) : |f(z)| \leq w(z),\ \text{for all } z \in U\}$. We define $\tilde{w}_o\colon U \to \mathbb{R}$ by

$$\tilde{w}_o(z) = \sup\{|f(z)| : f \in B_{\mathcal{H}_{v_o}(U)}\}$$

and $\tilde{w}\colon U \to \mathbb{R}$ by

$$\tilde{w}(z) = \sup\{|f(z)| : f \in B_{\mathcal{H}_v(U)}\}.$$

Let $\tilde{v}_o(z) = 1/\tilde{w}_o(z)$ and $\tilde{v}(z) = 1/\tilde{w}(z)$. Then $\tilde{v}_o$ and $\tilde{v}$ are continuous strictly positive weights which satisfy $0 < v \leq \tilde{v} \leq \tilde{v}_o$. The associated weights $\tilde{v}$ and $\tilde{v}_o$ are studied in [7].

Theorem 4.1. *Let m be a non-negative integer. Let U be an open subset of $\mathbb{C}^n$ and v, w be weights on U. Then every polynomial of degree m induces a continuous superposition operator from $\mathcal{H}_v(U)$ into $\mathcal{H}_w(U)$ if and only if for every integer k, $k \leq m$, we have $\sup_{z\in U}(w(z)/\tilde{v}(z)^k) < \infty$.*

Proof. Let $\phi(z) = \sum_{k=0}^m a_k z^k$ be a polynomial of degree m. Then ϕ induces a continuous superposition operator from $\mathcal{H}_v(U)$ into $\mathcal{H}_w(U)$ if and only if the mapping $f \to a_k f^k$ is a continuous k-homogeneous polynomial for each $k \leq m$. This happens if and only if $\{f^k : \|f\|_v \leq 1\}$ is bounded in $\mathcal{H}_w(U)$ for $k \leq m$. Since

$$\begin{aligned}\sup_{\|f\|_v\leq 1} \|f^k\|_w &= \sup_{\|f\|_v\leq 1} \sup_{z\in U} w(z)|f^k(z)| \\ &= \sup_{z\in U} w(z) \sup_{\|f\|_v\leq 1} |f(z)|^k \\ &= \sup_{z\in U} w(z)/\tilde{v}(z)^k\end{aligned}$$

the result follows. □

We note that the above result is also valid for v and w replaced with v_o and w_o.

The next result shows that whenever u and v coincide and converge to 0 on the boundary of U then the only superposition operators are those induced by affine linear maps.

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Proposition 4.2. *Let U be an open subset of $\mathbb{C}^n$. Let v be a weight on U such that $v(z)$ converges to 0 as z tends to the boundary of U. Then ϕ induces a continuous superposition operator from $\mathcal{H}_v(U)$ to $\mathcal{H}_v(U)$ if and only if ϕ is affine linear.*

Proof. If ϕ is affine linear then S_ϕ is clearly a continuous and bounded superposition operator. Conversely if ϕ induces a continuous superposition operator from $\mathcal{H}_v(U)$ to $\mathcal{H}_v(U)$ then $(\hat{d}^k S_\phi(0)/k!)(f)$ is a continuous k-homogeneous polynomial from $\mathcal{H}_v(U)$ into $\mathcal{H}_v(U)$ for every k. Writing $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{C}$, we have that $(\hat{d}^k S_\phi(0)/k!)(f) = a_k f^k$. It follows from the proof of [12, Corollary 2] that $\mathcal{H}_{v_o}(U)$ contains a copy of c_o. Hence ℓ_1 is a quotient of $\mathcal{H}_{v_o}(U)$ and therefore by [8, Remark 1] a quotient of $\mathcal{G}_v(U)$, the canonical predual of $\mathcal{H}_v(U)$ constructed in [8]. It follows that ℓ_∞ is a subspace of $\mathcal{H}_v(U)$. Since $\mathcal{H}_{v_o}(U)$ is separable we have $\mathcal{H}_{v_o}(U) \neq \mathcal{H}_v(U)$. Choosing f in $\mathcal{H}_v(U) \setminus \mathcal{H}_{v_o}(U)$ and using the fact that $v(z)$ converges to 0 as z converges to the boundary of U we have that $\sup_{z\in U} |f(z)| = \infty$. Since $g \mapsto a_k g^k$ is a continuous k-homogeneous polynomial from $\mathcal{H}_v(U)$ into $\mathcal{H}_v(U)$ we have that $a_k = 0$ for $k \geq 2$ and thus ϕ is affine linear. □

Note that the above proposition is false if we drop the assumption that $v(z)$ converges to 0 as z tends to the boundary of U. To see this consider the weight $v(z) \equiv 1$ on the unit disc Δ. In this case $\mathcal{H}_v(\Delta) = \mathcal{H}^\infty(\Delta)$ while $\mathcal{H}_{v_o}(\Delta) = \{0\}$. Since $\mathcal{H}^\infty(\Delta)$ is a Banach algebra we see that for each polynomial $p\colon \mathbb{C} \to \mathbb{C}$ the function $f \mapsto p(f)$ is continuous and hence, by Theorem 3.1, holomorphic.

Let U be an open subset of $\mathbb{C}^n$ and $v\colon U \to \mathbb{R}$ be a weight on U. We shall say that there is a positive solution to the biduality problem for v if $\mathcal{H}_v(U)$ is canonically isometrically isomorphic to $\mathcal{H}_{v_o}(U)''$. In [8] Bierstedt and Summers showed that a necessary and sufficient condition for a positive solution to the biduality problem is that the closed unit ball of $\mathcal{H}_{v_o}(U)$ is dense in the closed unit ball of $\mathcal{H}_v(U)$ for the compact-open topology. In [15] the authors prove that $\mathcal{H}_v(U)$ is canonically the bidual of $\mathcal{H}_{v_o}(U)$ if and only if $\mathcal{H}_{v_o}(U)$ is an M-ideal in $\mathcal{H}_v(U)$ and we have equality of the associated weights $\tilde{v}_o$ and $\tilde{v}$. For discussions on the biduality problem we refer the reader to [8, 15]. From Theorem 3.4 we have the following corollaries.

Corollary 4.3. *Let U be an open subset of $\mathbb{C}^n$. Let v, w be weights on U with a positive solution to the biduality problem. If ϕ is an entire function that induces a bounded superposition operator S_ϕ from $\mathcal{H}_{v_o}(U)$ into $\mathcal{H}_{w_o}(U)$ then ϕ induces a bounded superposition operator from $\mathcal{H}_v(U)$ into $\mathcal{H}_w(U)$.*

Corollary 4.4. *For $i = 1, 2$ let $\alpha_i \geq 0$, $\beta_i > 0$ and $p_i \geq 1 + \alpha_i + \beta_i$. Let ϕ be an entire function which induces a bounded superposition operator from $F_0(p_1, \alpha_1, \beta_1)$ into $F_0(p_2, \alpha_2, \beta_2)$. Then ϕ induces a bounded superposition operator from $F(p_1, \alpha_1, \beta_1)$ into $F(p_2, \alpha_2, \beta_2)$.*

Example 4.5. In particular, Corollary 4.4 gives us the following cases.

(a) Let $0 \leq \beta, \beta' < 1$ and ϕ be an entire function which induces a bounded superposition operator from $Q_{\beta,0}$ into $Q_{\beta',0}$. Then it induces a bounded superposition operator from Q_β into $Q_{\beta'}$.
(b) Let ϕ be an entire function which induces a bounded superposition operator from VMOA into VMOA. Then it induces a bounded superposition operator from BMOA into BMOA.
(c) Let $1 < p < \infty$, $0 < \alpha < 1$ and $\beta < 1$. Suppose ϕ is an entire function which induces a bounded superposition operator from VMOA_p^α into $Q_{\beta,0}$. Then ϕ induces a bounded superposition operator from BMOA_p^α into Q_β.

Theorem 4.6. *Let U be an open subset of $\mathbb{C}^n$. Let v be a weight on U with a positive solution to the biduality problem which converges to 0 on the boundary of U. Then the only weakly compact superposition operators from $\mathcal{H}_v(U)$ into $\mathcal{H}_v(U)$ are the constant mappings.*

Proof. Let S_ϕ be a weakly compact superposition operator from $\mathcal{H}_v(U)$ into $\mathcal{H}_v(U)$. For each z in U, δ_z belongs to $\mathcal{H}_v(U)'$. Then by Corollary 3.2, S_ϕ is continuous and even holomorphic. It follows from Proposition 4.2 that ϕ must be affine linear. Therefore we write $\phi(z) = az + b$, $z \in \mathbb{C}$. If $a \neq 0$ then by Theorem 3.6 the unit ball of $\mathcal{H}_v(U)$ is weakly compact. This means that $\mathcal{H}_v(U)$, and hence $\mathcal{H}_{v_o}(U)$, is reflexive, which contradicts [12, Corollary 2]. □

If in addition we assume that S_ϕ is bounded then the above theorem remains valid for $\mathcal{H}_{v_o}(U)$. The only point in the proof that needs to be checked is that, under the assumptions, the only bounded superposition operators from $\mathcal{H}_{v_o}(U)$ into $\mathcal{H}_{v_o}(U)$ are induced by affine linear symbols. This is an easy consequence of Proposition 4.2 and Corollary 4.3.

Corollary 4.7. *Let U be an open subset of $\mathbb{C}^n$. Let v be a weight on U with a positive solution to the biduality problem which converges to 0 on the boundary of U. Then ϕ induces a bounded superposition operator from $\mathcal{H}_{v_o}(U)$ to $\mathcal{H}_{v_o}(U)$ if and only if ϕ is affine linear.*

In the previous section we gave a mild condition under which boundedness of superposition operators implies continuity and continuous and holomorphic superposition operators coincide. This is the case for weighted spaces of holomorphic functions and spaces of Zhao.

Proposition 4.8. *(a) Let U be an open subset of $\mathbb{C}^n$ and v, w be weights on U. Every bounded superposition operator from $\mathcal{H}_v(U)$ into $\mathcal{H}_w(U)$ is holomorphic and hence continuous.*

(b) *For $i = 1, 2$ let $\alpha_i \geq 0$, $\beta > 0$ and $p_i \geq 1 + \alpha_i + \beta_i$. Every bounded superposition operator from $F(p_1, \alpha_1, \beta_1)$ into $F(p_2, \alpha_2, \beta_2)$ is holomorphic and hence continuous.*

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Proof. (a) For each z in U, δ_z belongs to $\mathcal{H}_v(U)'$. We can therefore apply Theorem 3.1 and Corollary 3.2 with Y equal to $\mathcal{H}_v(U)'$. Part (b) follows in an similar way. □

The analogous result holds for $F_0(p, \alpha, \beta)$ and $\mathcal{H}_{v_0}(U)$.

When we consider spaces of weighted holomorphic functions, Theorem 3.5 yields the following result.

Corollary 4.9. *Let U be an open subset of $\mathbb{C}^n$ and $v, w\colon U \to \mathbb{R}$ be weights on U. Let ϕ be an entire function which induces a continuous superposition operator S_ϕ from $\mathcal{H}_{v_o}(U)$ into $\mathcal{H}_{w_o}(U)$. Then S_ϕ is bounded if and only if:*

(a) *for each n in $\mathbb{N}$ with $\phi^{(n)}(0) \neq 0$ the set $\{f^n : f \in \mathcal{H}_{v_o}(U), \|f\|_v \leq 1\}$ is bounded in $\mathcal{H}_{w_o}(U)$;*
(b) $\lim_{n\to\infty} |\phi^{(n)}(0)/n!|^{1/n} \sup_{z\in U} w(z)^{1/n}/\tilde{v}_o(z) = 0$.

Proof. For each n in $\mathbb{N}$ with $\phi^{(n)}(0) \neq 0$ we have

$$M_n = \sup_{\|f\|_v \leq 1} \|f^n\|_w = \sup_{z\in U} w(z)/\tilde{v}_o(z)^n.$$

The result now follows from Theorem 3.5. □

Example 4.10. Consider the weights $v(z) = 1 - |z|$ and $w(z) = e^{-2/(1-|z|)}$ on Δ. By [16, Theorem 5] any entire function ϕ of exponential type 0 (of order one and type 0) induces a continuous superposition operator S_ϕ from $\mathcal{H}_{v_o}(\Delta)$ into $\mathcal{H}_{w_o}(\Delta)$. It follows from Theorem 3.1 that S_ϕ is holomorphic. On the other hand, we see that $\sup_{z\in\Delta} w(z)^{1/n}/\tilde{v}_o(z) = n/2e$. Thus S_ϕ is a holomorphic function of bounded type if $\phi^{(n)}(0)/n! \sim o(1/n)$. Hence if ϕ is of exponential type 0 (of order one and type 0) it follows from [9, Theorem 2.2.10] that $\lim_{n\to\infty} |\phi^{(n)}(0)/n!|^{1/n} \sup_{z\in U}(w(z)^{1/n}/\tilde{v}_o(z)) = 0$ and S_ϕ will be of bounded type.

Necessary and sufficient conditions for the existence, boundedness and continuity of superposition operators between weighted spaces of holomorphic functions for weights of the form $v(z) = (1 - |z|)^p$, $w(z) = \exp(-1/(1 - |z|)^p)$ and $v(z) = (1 - \log(1 - |z|))^{-1}$ have also been obtained by Bonet and Vukotić [11].

Since $\mathcal{H}_{v_o}(U)$ is isometrically isomorphic to a subspace of a $C(K)$−space (see the proof of [14, Proposition 2.1]) a result of Bourgain and Talagrand (see [13, Theorem 1]), yields the following result.

Theorem 4.11. *Let U be an open subset of $\mathbb{C}^n$ and v, w be weights on U. Let ϕ be an entire function which induces a bounded superposition operator, S_ϕ, from $\mathcal{H}_{v_o}(U)$ into $\mathcal{H}_{w_o}(U)$. Then S_ϕ is weakly compact if and only if for each n in $\mathbb{N}$ with $\phi^{(n)}(0) \neq 0$ the set $\{f^n : f \in X, \|f\|_v \leq 1\}$ is relatively compact for the topology of pointwise convergence on U.*

Given $0 < q < \infty$ the Bergman space, A^q, is defined as those holomorphic f on the disc with the property that

$$\|f\|_{A^q} := \left(\int_\Delta |f(z)|^q \, d\lambda \right)^{1/q} < \infty,$$

where λ denotes Lebesgue measure on the disc. Let us conclude the paper showing that the Aron–Berner extension together with [2, Theorem 3] allows us to give an alternative proof of [18, Theorem 2].

Theorem 4.12 (Buckley, Vukotić). *Let* $0 < q < \infty$. *Then* S_ϕ *is bounded from* $\mathcal{B}_0$ *into the Bergman space* A^q *if and only if* ϕ *is of order less than one or of order one and type* 0.

Proof. If ϕ is of order less than one or of order one and type 0 then [2, Theorem 3] implies that S_ϕ is a bounded superposition operator from $\mathcal{B}$ into A^q. Its restriction to $\mathcal{B}_0$ will map $\mathcal{B}_0$ into A^q. Conversely, suppose that ϕ induces a bounded superposition operator from $\mathcal{B}_0$ into A^q. [25, Theorem 1.1] implies that each δ_z is continuous on A^q. Hence the span of $\{\delta_z : z \in \Delta\}$ is equal to $(A^q)'$ and therefore Corollary 3.2 implies that S_ϕ is a holomorphic function of bounded type from $\mathcal{B}_0$ into A^q. The Aron–Berner extension of $AB(S_\phi)$ is a bounded holomorphic function from $\mathcal{B}$ into A^q. However, it follows as in Theorem 3.4 that $AB(S_\phi) = S_\phi$. An application of [2, Theorem 3] proves that ϕ is of order less than one or of order one and type 0. □

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