BULL. AUSTRAL. MATH. SOC. 49D25
VOL. 38 (1988) [365-371]

A DISCRETE CALCULUS OF VARIATIONS ALGORITHM

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An algorithm which has been developed to solve the problem of determining an optimal path of the hand of a robot is applied to various classical problems in the calculus of variations.

1. INTRODUCTION

In a recent paper [**6**] an algorithm has been described which gives a general and efficient solution to the problem of determining an optimal path of the hand or end effector of a robot manipulator subject to a variety of geometric, kinematic and dynamic constraints. The algorithm is based on a discrete form of the Lagrange equations of motion for the manipulator [**1**, **3**, **4**].

To test the algorithm it has been applied to several classical calculus of variations problems for which the solutions are known. It is the purpose of this paper to report the success of the algorithm when applied to these and to illustrate its potential use for a wide class of optimisation problems.

2. OPTICAL REFLECTION

The optical problem of reflection of light from a surface will be used to introduce notation and the general features of the discrete algorithm. Figure 1 illustrates the situation. A ray of light from $A(a_1, a_2)$ to $B(b_1, b_2)$ in a uniform medium in the upper half of the q_1, q_2 plane is reflected at $R(r, 0)$ on the boundary $q_2 = 0$ of a denser medium. According to optical theory, the actual path of the light ray is that which minimises the total passage time from A to B, and from this the reflection law — straight line rays with angle of incidence equal to angle of reflection — can be readily deduced.

Suppose a possible path, in general allowed to be curved, is incident at point R on the surface. Discretise the path by introducing $L+1$ knots, $k = 0, \ldots, L$, on the segment AR and a further M knots, $k = L+1, \ldots, L+M$, on the segment RB. For each of the L intervals on the segment AR, the passage time is assumed to be Δt, and for each of the M intervals on the segment RB, Δu, so that the total passage time, which has to be minimised, is

$$Z = L\Delta t + M\Delta u. \tag{2.1}$$

Received 4th February, 1988

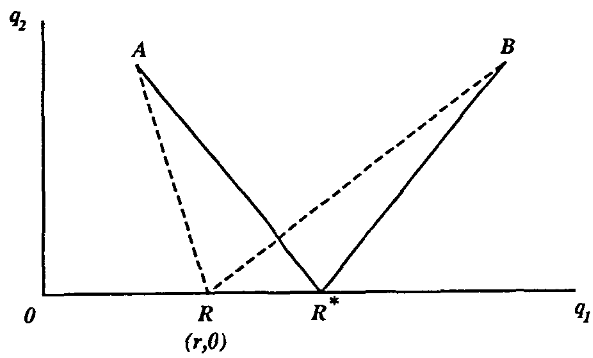

Figure 1 : A ray of light from $A(a_1, a_2)$ to $B(b_1, b_2)$ in a uniform medium is reflected at R on the boundary $q_2 = 0$. The optimal solution gives straight lines AR^* and R^*B with the angle of incidence equal to the angle of reflection. The dashed lines indicate the initial solution taken for the discrete algorithm.

Suppose the coordinates of the path at each knot k are denoted by $q_p(k)$, $k = 0, \ldots, L+M$, with $p = 1, 2$. The displacements are denoted by the forward differences

$$\Delta q_p(k) = q_p(k+1) - q_p(k) \qquad k = 0, \ldots, L+M-1. \tag{2.2}$$

Basic to the discrete path planner algorithm is the use of the approximate trapezoidal or smoothing formulae used extensively by Greenspan [**2**]
for $k = 1, \ldots, L-1$

$$\begin{aligned} \dot{q}_p(k) &= -\dot{q}_p(k-1) + 2[\Delta t]^{-1} \Delta q_p(k-1) \\ &= (-1)^k \dot{q}_p(0) - 2(-1)^k [\Delta t]^{-1} \sum_{i=0}^{k-1} (-1)^i \Delta q_p(i), \end{aligned} \tag{2.3}$$

and for $k = L+1, \ldots, L+M$

$$\begin{aligned} \dot{q}_p(k) &= -\dot{q}_p(k-1) + 2[\Delta u]^{-1} \Delta q_p(k-1) \\ &= (-1)^{k-L} \dot{q}_p(L) - 2(-1)^k [\Delta u]^{-1} \sum_{i=L}^{k-1} (-1)^i \Delta q_p(i). \end{aligned} \tag{2.4}$$

The speed of light throughout the uniform medium above the reflecting surface is constant so that

$$\dot{q}_1(k)^2 + \dot{q}_2(k)^2 = C \qquad k = 0, \ldots, L+M, \; k \neq L. \tag{2.5}$$

The problem of determining the optimum ray path from A to B *via* a point on the reflecting surface is now expressed as a nonlinear program. The $7+2(L+M)$ program variables are $\Delta t, \Delta u, r, \dot{q}_p(0), \dot{q}_p(L), \Delta q_p(k)$ for $k = 0,\ldots,L+M-1$, $p = 1,2$. The initial and final points are given with $q_p(0) = a_p$, $q_p(L+M) = b_p$. The linear objective function to be minimised is given by (2.1). The constraints to be satisfied are the linear relations

$$\sum_{k=0}^{L-1} \Delta q_1(k) = r - a_1, \tag{2.6}$$

$$\sum_{k=L}^{L+M-1} \Delta q_1(k) = b_1 - r, \tag{2.7}$$

$$\sum_{k=0}^{L-1} \Delta q_2(k) = -a_2, \tag{2.8}$$

$$\sum_{k=L}^{L+M-1} \Delta q_2(k) = b_2, \tag{2.9}$$

and the $L+M+1$ nonlinear relations (2.5), the nonlinearity arising from the necessity to introduce (2.3) and (2.4).

The NLP can be solved using sequential linear programming (SLP), sequential quadratic programming (SQP) or the augmented Lagrangian method and in particular the NPSOL and MINOS packages [5] were used. These techniques require the Jacobian matrix formed by the partial derivatives of the nonlinear expressions, in this case only (2.5), with respect to the program variables. These derivatives are simple to evaluate. The NLP algorithms require an initial solution, if possible feasible, and a simple such choice is to assume straight line paths AR and RB.

A particular test of the algorithm was run with the following data :

$$C = 1, \quad a_1 = -0.5, \; a_2 = 0.5, \; b_1 = b_2 = 0.5.$$

From a cold start with $L = M = 5$, the initial solution was chosen with $\Delta t = \Delta u = 0.1, r = -0.3$, and straight line segments from A to R and R to B with

$$\Delta q_1(k) = (r - a_1)/L, \quad \Delta q_2(k) = -a_2/L, \quad k = 0,\ldots,L-1$$

and

$$\Delta q_1(k) = (b_1 - r)/M, \quad \Delta q_2(k) = b_2/M, \qquad k = L,\ldots,L+M-1.$$

The initial values of the velocity variables were chosen to be

$$\dot{q}_1(0) = 0.3714, \; \dot{q}_2(0) = -0.9285, \; \dot{q}_1(L) = 0.8480, \; \dot{q}_2(L) = 0.5300.$$

With the use of the NPSOL package, the expected theoretical result of straight line paths AR^* and R^*B with $R^* = (0,0)$ was obtained with about 6 figure accuracy and with a CPU time of 10 seconds on a VAX-11/785 computer.

3. Optical Refraction

A similar test of the optimal path planner was performed for the problem of optical refraction. The reflection problem was modified by choosing below the boundary $q_2 = 0$ a uniform medium of different density and taking B within this medium so that $b_2 < 0$. The modifications to the formulation are straightforward with the right hand side of the velocity constraints set to the appropriate values.

A particular test of the algorithm was run with

$$a_1 = -0.5,\ a_2 = 0.5,\ b_1 = \sqrt{3}/6,\ b_2 = -0.5$$

and with the refractive index of the medium above the interface $q_2 = 0$ equal to 1 and that below equal to $\sqrt{2}$. For a test run with $L = M = 5$, the initial values taken were :

$$\Delta t = \Delta u = 0.1, r = -0.3, \Delta q_1(k) = (r - a_1)/L, \Delta q_2(k) = -a_2/L, k = 0, \dots, L-1$$

and

$$\Delta q_1(k) = (b_1 - r)/M,\ \Delta q_2(k) = b_2/M,\ k = L, \dots, L+M-1.$$

The initial values of the velocity variables were taken as :

$$\dot{q}_1(0) = 0.3714,\ \dot{q}_2(0) = -0.9285,\ \dot{q}_1(L) = 0.5389,\ \dot{q}_2(L) = -0.4578.$$

With NPSOL, the expected result of straight line paths from A to R^* and R^* to B with $R^* = (0,0)$ was obtained with 6 figure accuracy and a CPU time of 17 seconds.

4. Brachistochrone Problem

As a third test of the algorithm, the classic brachistochrone problem was considered — determine the path OB in the vertical q_1, q_2 plane (see Figure 2) giving minimum travel time for a frictionless particle starting from rest and acted upon by gravity. To be specific, if $B = (\pi, -2)$, the path is the cycloid

$$q_1(t) = t - \sin t, \qquad q_2(t) = -1 + \cos t, \tag{4.1}$$

with travel time equal to π. The effect of gravity forces the nonlinear constraint

$$\dot{q}_1^2 + \dot{q}_2^2 + 2q_2 = 0. \tag{4.2}$$

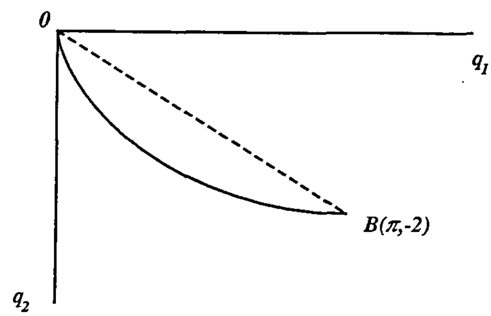

Figure 2: In the brachistochrone problem a frictionless particle falls under the action of gravity from O at rest to a given point B. The dashed line is the straight path taken as the initial solution for the discrete algorithm; the curved path is the optimal solution, portion of a cycloid.

The discrete path planner takes the following form of an NLP with $2L+1$ variables Δt and $\Delta q_p(k)$ for $k = 0, \ldots, L-1$, $p = 1, 2$:

(4.3) $$\text{minimise} \qquad Z = \Delta t,$$

(4.4) $$\text{subject to} \qquad \sum_0^{L-1} \Delta q_1(k) = \pi,$$

(4.5) $$\sum_0^{L-1} \Delta q_2(k) = -2,$$

(4.6) $$\dot{q}_1(k)^2 + \dot{q}_2(k)^2 + 2q_2(k) = 0, \qquad k = 1, \ldots, L.$$

The values of $\dot{q}_p(k)$ are given by (2.3), with $\dot{q}_p(0) = 0$, and the values of $q_2(k)$ by

(4.7) $$q_2(k) = \sum_{i=0}^{k-1} \Delta q_2(i) \qquad k = 1, \ldots, L.$$

Again a convenient initial solution for the NLP algorithm is the straight line path OB with

(4.8) $$\Delta q_1(k) = \pi/L, \qquad \Delta q_2(k) = -2/L, \qquad k = 0, \ldots, L-1.$$

The result for $L = 5$, obtained with a CPU time of 7.5 seconds, is shown in Figure 2. The travel time is 3.188 compared to the theoretical value π. For $L = 10$, the graph of the solution, obtained with a CPU time of 26.9 seconds, is indistinguishable from the cycloid, and the travel time is now 3.153.

5. Minimum Surface of Revolution Area

As a final example consider the problem of determining the minimum surface of revolution area which in a simple case is formulated as the determination of the function $q_2 = q_2(q_1)$ such that

$$I = \int_0^1 q_2(q_1)[1 + q_2'(q_1)^2]^{\frac{1}{2}}\, dq_1 \tag{5.1}$$

is a minimum subject to

$$q_2(0) = 1, \qquad q_2(1) = \cosh 1. \tag{5.2}$$

The analytic solution is

$$q_2^* = \cosh q_1, \qquad I^* = \frac{1}{2} + \frac{1}{4}\sinh 2. \tag{5.3}$$

In the usual way, the path is discretised into L intervals by taking $\Delta q_1 = 1/L$. The $L+1$ variables for the nonlinear program are $\Delta q_2(k), k = 0, \ldots, L-1$ and $q_2'(0)$. The remaining derivatives are given by the smoothing formulae

$$q_2'(k+1) = -q_2'(k) + 2L\Delta q_2(k), \qquad k = 0, \ldots, L-1. \tag{5.4}$$

The objective function to be minimised is taken as a discrete approximation to the integral I given by (5.1). For example, the trapezoidal rule gives, apart from a factor $1/(2L)$,

$$Z = q_2(0)[1 + q_2'(0)^2]^{\frac{1}{2}} + 2\sum_1^{L-1} q_2(k)[1 + q_2'(k)^2]^{\frac{1}{2}} + q_2(L)[1 + q_2'(L)^2]^{\frac{1}{2}}. \tag{5.5}$$

The constraints are the linear relation

$$\sum_0^{L-1} \Delta q_2(k) = -1 + \cosh 1, \tag{5.6}$$

and the non-negativity relations

$$q_2(k) \geqslant 0, \qquad k = 1, \ldots, L-1. \tag{5.7}$$

The initial solution was taken as

$$\Delta q_2(k) = [-1 + \cosh 1]/L, \quad k = 0, \ldots, L-1, \qquad q_2'(0) = -1 + \cosh 1.$$

For a test run using NPSOL and with $L = 10$, about 3 figure accuracy was obtained with a CPU time of 7 seconds. For $L = 20$ about 4 figure accuracy was obtained with a CPU time of 34 seconds.

6. Discussion

The numerical solution of problems in calculus of variations can be difficult because a potentially large function space has to be searched in order to find a good approximation to the function optimising the objective function. Practical problems in robotics involve functions of six variables corresponding to a robot's six degrees of freedom, and the 'curse of dimensions' has frustrated some attempts at solving optimal path planning. An algorithm which has been used successfully in robotics [**6**] has been applied in this paper to solve several classical calculus of variations problems. The algorithm has been proved to be applicable to general problems with various objectives and constraints.

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https://doi.org/10.1017/S0004972700027726 Published online by Cambridge University Press