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Abstract

Sen attached to each $p$-adic Galois representation of a $p$-adic field a multiset of numbers called generalized Hodge–Tate weights. In this paper, we discuss a rigidity of these numbers in a geometric family. More precisely, we consider a $p$-adic local system on a rigid analytic variety over a $p$-adic field and show that the multiset of generalized Hodge–Tate weights of the local system is constant. The proof uses the $p$-adic Riemann–Hilbert correspondence by Liu and Zhu, a Sen–Fontaine decompletion theory in the relative setting, and the theory of formal connections. We also discuss basic properties of Hodge–Tate sheaves on a rigid analytic variety.

## 1. Introduction

In the celebrated paper [Tat67], Tate studied the Galois cohomology of $p$-adic fields and obtained the so-called Hodge–Tate decomposition of the Tate module of a $p$-divisible group with good reduction. The paper has been influential in the developments of $p$-adic Hodge theory, and one of the earliest progresses was done by Sen. In [Sen81], he attached to each $p$-adic Galois representation of a $p$-adic field $k$ a multiset of numbers that are algebraic over $k$. These numbers are called generalized Hodge–Tate weights, and they serve as one of the basic invariants in $p$-adic Hodge theory, especially for the study of Galois representations that may not be Hodge–Tate (e.g. Galois representations attached to finite slope overconvergent modular forms).

In this paper, we study how generalized Hodge–Tate weights vary in a geometric family. To be precise, we consider an étale $\mathbb{Q}_p$-local system on a rigid analytic varieties over $k$ and regard it as a family of Galois representations of residue fields of its classical points. Here is one of the main theorems of this paper.

Theorem 1.1 (Corollary 4.9). *Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X$. Then the generalized Hodge–Tate weights of the $p$-adic Galois representations $\mathbb{L}_{\overline{x}}$ of $k(x)$ are constant on the set of classical points $x$ of $X$.*

The theorem gives one instance of the rigidity of a geometric family of Galois representations. It is worth noting that arithmetic families of Galois representations do not have such rigidity; consider a representation of the absolute Galois group of $k$ with coefficients in some $\mathbb{Q}_p$-affinoid algebra. One can associate to each maximal ideal a Galois representation of $k$. In such a situation, the generalized Hodge–Tate weights vary over the maximal ideals.

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To explain ideas of the proof of Theorem 1.1 as well as other results of this paper, let us recall the work of Sen mentioned above. For each $p$-adic Galois representation $V$ of $k$, we set

$$\mathcal{H}(V) := (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\mathrm{Gal}(\overline{k}/k_\infty)},$$

where $\mathbb{C}_p$ is the $p$-adic completion of $\overline{k}$ and $k_\infty := k(\mu_{p^\infty})$ is the cyclotomic extension of $k$. This is a vector space over the $p$-adic completion $K$ of $k_\infty$ equipped with a continuous semilinear action of $\mathrm{Gal}(k_\infty/k)$ and satisfies $\dim_K \mathcal{H}(V) = \dim_{\mathbb{Q}_p} V$. Sen developed a theory of decompletion; he found a natural $k_\infty$-vector subspace $\mathcal{H}(V)_{\mathrm{fin}} \subset \mathcal{H}(V)$ that is stable under $\mathrm{Gal}(k_\infty/k)$-action and satisfies $\mathcal{H}(V)_{\mathrm{fin}} \otimes_{k_\infty} K = \mathcal{H}(V)$. He then defined a $k_\infty$-endomorphism $\phi_V$ on $\mathcal{H}(V)_{\mathrm{fin}}$, called the Sen endomorphism of $V$, by considering the infinitesimal action of $\mathrm{Gal}(k_\infty/k)$. The generalized Hodge–Tate weights are defined to be eigenvalues of $\phi_V$.

Therefore, the first step toward Theorem 1.1 is to define generalizations of $\mathcal{H}(V)$ and $\phi_V$ for each $\mathbb{Q}_p$-local system. For this, we use the $p$-adic Simpson correspondence by Liu and Zhu [LZ17]; based on recent developments in relative $p$-adic Hodge theory by Kedlaya–Liu and Scholze, Liu and Zhu associated to each $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X$ a vector bundle $\mathcal{H}(\mathbb{L})$ of the same rank on $X_K$ equipped with a $\mathrm{Gal}(k_\infty/k)$-action and a Higgs field, where $X_K$ is the base change of $X$ to $K$. When $X$ is a point and $\mathbb{L}$ corresponds to $V$, this agrees with $\mathcal{H}(V)$ as the notation suggests. Following Sen, we will define the arithmetic Sen endomorphism $\phi_{\mathbb{L}}$ of $\mathbb{L}$ by decompleting $\mathcal{H}(\mathbb{L})$ and considering the infinitesimal action of $\mathrm{Gal}(k_\infty/k)$. Then Theorem 1.1 is reduced to the following.

Theorem 1.2 (Theorem 4.8). *The eigenvalues of $\phi_{\mathbb{L},x}$ for $x \in X_K$ are algebraic over $k$ and constant on $X_K$.*

Before discussing ideas of the proof, let us mention consequences of Theorem 1.2. Sen proved that a $p$-adic Galois representation $V$ is Hodge–Tate if and only if $\phi_V$ is semisimple with integer eigenvalues. In the same way, we use $\phi_{\mathbb{L}}$ to study Hodge–Tate sheaves. We define a sheaf $D_{\mathrm{HT}}(\mathbb{L})$ on the étale site $X_{\mathrm{\acute{e}t}}$ by

$$D_{\mathrm{HT}}(\mathbb{L}) := \nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbb{B}_{\mathrm{HT}}),$$

where $\mathcal{O}\mathbb{B}_{\mathrm{HT}}$ is the Hodge–Tate period sheaf on the pro-étale site $X_{\mathrm{pro\acute{e}t}}$ and $\nu : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$ is the projection (see § 5). A $\mathbb{Q}_p$-local system $\mathbb{L}$ is called *Hodge–Tate* if $D_{\mathrm{HT}}(\mathbb{L})$ is a vector bundle on $X$ of rank equal to $\operatorname{rank} \mathbb{L}$.

Theorem 1.3 (Theorem 5.5). *The following conditions are equivalent for a $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X$:*

(i) *$\mathbb{L}$ is Hodge–Tate;*

(ii) *$\phi_{\mathbb{L}}$ is semisimple with integer eigenvalues.*

The study of the Sen endomorphism for a geometric family was initiated by Brinon as a generalization of Sen's theory to the case of non-perfect residue fields [Bri03]. Tsuji obtained Theorem 1.3 in the case of schemes with semistable reduction [Tsu11].

Using this characterization, we prove the following basic property of Hodge–Tate sheaves.

Theorem 1.4 (Theorem 5.10). *Let $f : X \to Y$ be a smooth proper morphism between smooth rigid analytic varieties over $k$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\mathrm{\acute{e}t}}$. Then if $\mathbb{L}$ is a Hodge–Tate sheaf on $X_{\mathrm{\acute{e}t}}$, $R^i f_* \mathbb{L}$ is a Hodge–Tate sheaf on $Y_{\mathrm{\acute{e}t}}$.*

Hyodo introduced the notion of Hodge–Tate sheaves and proved Theorem 1.4 in the case of schemes [Hyo86]. Links between Hodge–Tate sheaves and the $p$-adic Simpson correspondence can be seen in his work and were also studied by Abbes–Gros–Tsuji [AGT16] and Tsuji [Tsu18]. In fact, they undertook a systematic development of the $p$-adic Simpson correspondence started by Faltings [Fal05] and their focus is much broader than ours. Andreatta and Brinon also studied Higgs modules and Sen endomorphisms in a different setting [AB10]. In these works, one is restricted to working with schemes or log schemes, whereas we work with rigid analytic varieties.

We now turn to the proof of Theorem 1.2. The key idea to obtain such constancy is to describe $\phi_{\mathbb{L}}$ as the residue of a certain formal integrable connection. Such an idea occurs in the work [AB10] of Andreatta and Brinon. Roughly speaking, they associated to $\mathbb{L}$ a formal connection over some pro-étale cover of $X_K$ when $X$ is an affine scheme admitting invertible coordinates. In our case, we want to work over $X_K$, and thus we use the geometric $p$-adic Riemann–Hilbert correspondence by Liu and Zhu [LZ17] and Fontaine's decompletion theory for the de Rham period ring $B_{\mathrm{dR}}(K)$ in the relative setting.

Liu and Zhu associated to each $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X$ a locally free $\mathcal{O}_X \mathbin{\hat{\otimes}} B_{\mathrm{dR}}(K)$-module $\mathcal{RH}(\mathbb{L})$ equipped with a filtration, an integrable connection

$$\nabla : \mathcal{RH}(\mathbb{L}) \to \mathcal{RH}(\mathbb{L}) \otimes \Omega^1_X,$$

and a $\mathrm{Gal}(k_\infty/k)$-action (see § 4.1 for the notation). To regard $\phi_{\mathbb{L}}$ as a residue, we also need a connection in the arithmetic direction $B_{\mathrm{dR}}(K)$. For this we use Fontaine's decompletion theory [Fon04]; recall the natural inclusion $k_\infty((t)) \subset B_{\mathrm{dR}}(K)$ where $t$ is the $p$-adic analogue of the complex period $2\pi i$. Fontaine extended the work of Sen and developed a decompletion theory for $B_{\mathrm{dR}}(K)$-representations of $\mathrm{Gal}(k_\infty/k)$. We generalize Fontaine's decompletion theory to the relative setting, i.e. that for $\mathcal{O}_X \mathbin{\hat{\otimes}} B_{\mathrm{dR}}(K)$-modules (Theorem 2.5 and Proposition 2.24), which yields an endomorphism $\phi_{\mathrm{dR},\mathbb{L}}$ on $\mathcal{RH}(\mathbb{L})_{\mathrm{fin}}$ satisfying

$$\phi_{\mathrm{dR},\mathbb{L}}(t^n v) = n t^n v + t^n \phi_{\mathrm{dR},\mathbb{L}}(v)$$

and $\mathrm{gr}^0 \phi_{\mathrm{dR},\mathbb{L}} = \phi_{\mathbb{L}}$. Informally, this means that we have an integrable connection

$$\nabla + \frac{\phi_{\mathrm{dR},\mathbb{L}}}{t} \otimes dt : \mathcal{RH}(\mathbb{L}) \to \mathcal{RH}(\mathbb{L}) \otimes ((\mathcal{O}_X \mathbin{\hat{\otimes}} B_{\mathrm{dR}}(K)) \otimes \Omega^1_X + (\mathcal{O}_X \mathbin{\hat{\otimes}} B_{\mathrm{dR}}(K)) \otimes dt)$$

over $X \mathbin{\hat{\otimes}} B_{\mathrm{dR}}(K)$ whose residue along $t = 0$ coincides with the arithmetic Sen endomorphism $\phi_{\mathbb{L}}$. We develop a theory of formal connections to analyze our connection and prove Theorem 1.2.

Finally, let us mention two more results in this paper. The first result is a rigidity of Hodge–Tate local systems of rank at most two.

Theorem 1.5 (Theorem 5.12). *Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. Assume that* $\operatorname{rank} \mathbb{L}$ *is at most two. If $\mathbb{L}_{\overline{x}}$ is a Hodge–Tate representation at a classical point $x \in X$, then $\mathbb{L}$ is a Hodge–Tate sheaf. In particular, $\mathbb{L}_{\overline{y}}$ is a Hodge–Tate representation at every classical point $y \in X$.*

Liu and Zhu proved such a rigidity for de Rham local systems [LZ17, Theorem 1.3]. We do not know whether a similar statement holds for Hodge–Tate local systems of higher rank.

The second result concerns the relative $p$-adic monodromy conjecture for de Rham local systems; the conjecture states that a de Rham local system on $X$ becomes semistable at every classical point after a finite étale extension of $X$ (cf. [KL15, § 0.8], [LZ17, Remark 1.4]). This is a relative version of the $p$-adic monodromy theorem proved by Berger [Ber02], and it is a major open problem in relative $p$-adic Hodge theory. We work on the case of de Rham local systems with a single Hodge–Tate weight, in which case the result follows from a theorem of Sen (Theorem 5.13).

Theorem 1.6 (Theorem 5.15). *Let $X$ be a smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\acute{e}t}$. Assume that $\mathbb{L}$ is a Hodge–Tate sheaf with a single Hodge–Tate weight. Then there exists a finite étale cover $f : Y \to X$ such that $(f^*\mathbb{L})_{\overline{y}}$ is semistable at every classical point $y$ of $Y$.*

This is the simplest case of the relative $p$-adic monodromy conjecture. In [Col08], Colmez gave a proof of the $p$-adic monodromy theorem for de Rham Galois representations using Sen's theorem mentioned above. It is an interesting question whether one can adapt Colmez's strategy to the relative setting using Theorem 1.6.

The organization of the paper is as follows: § 2 presents Sen–Fontaine's decompletion theory in the relative setting. In § 3, we review the $p$-adic Simpson correspondence by Liu and Zhu, and define the arithmetic Sen endomorphism $\phi_{\mathbb{L}}$. Section 4 discusses a Fontaine-type decompletion for the geometric $p$-adic Riemann–Hilbert correspondence by Liu and Zhu, and develops a theory of formal connections. Combining them together we prove Theorem 1.1. Section 5 presents applications of the study of the arithmetic Sen endomorphism including basic properties of Hodge–Tate sheaves, a rigidity of Hodge–Tate sheaves, and the relative $p$-adic monodromy conjecture.

*Conventions.* We will use Huber's adic spaces as our language for non-Archimedean analytic geometry. In particular, a rigid analytic variety over $\mathbb{Q}_p$ will refer to a quasi-separated adic space that is locally of finite type over $\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$. See [Hub94, § 4], [Hub96, 1.11.1].

We will use Scholze's theory of perfectoid spaces and pro-étale site. For the pro-étale site, we will use the one introduced in [Sch13, Sch16].

## 2. Sen–Fontaine's decompletion theory for an arithmetic family

### 2.1 Set-up

Let $k$ be a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p$. We set $k_m := k(\mu_{p^m})$ and $k_\infty := \varinjlim_m k_m$. Let $K$ denote the $p$-adic completion of $k_\infty$. We set $\Gamma_k := \mathrm{Gal}(k_\infty/k)$. Then $\Gamma_k$ is identified with an open subgroup of $\mathbb{Z}_p^\times$ via the cyclotomic character $\chi : \Gamma_k \to \mathbb{Z}_p^\times$ and it acts continuously on $K$.

Let $L_{\mathrm{dR}}^+$ (respectively $L_{\mathrm{dR}}$) denote the de Rham period ring $B_{\mathrm{dR}}^+(K)$ (respectively $B_{\mathrm{dR}}(K)$) introduced by Fontaine. We fix a compatible sequence of $p$-power roots of unity $(\zeta_{p^n})$ and set $t := \log[\varepsilon]$ where $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{K^\flat}$. Then $\Gamma_k$ acts on $t$ via the cyclotomic character and the $\mathbb{Z}_p$-submodule $\mathbb{Z}_p t \subset L_{\mathrm{dR}}^+$ does not depend on the choice of $(\zeta_{p^n})$. Note that $L_{\mathrm{dR}}$ is a discrete valuation ring with residue field $K$, fraction field $L_{\mathrm{dR}}$, and uniformizer $t$, and that $k_\infty[[t]]$ is embedded into $L_{\mathrm{dR}}^+$.

We now recall the Sen–Fontaine's decompletion theory [Sen81, Theorem 3], [Fon04, Théorème 3.6].

Theorem 2.1. (i) *(Sen) Let $V$ be a $K$-representation of $\Gamma_k$. Denote by $V_{\mathrm{fin}}$ the union of finite-dimensional $k$-vector subspaces of $V$ that are stable under the action of $\Gamma_k$. Then the natural map*

$$V_{\mathrm{fin}} \otimes_{k_\infty} K \to V$$

*is an isomorphism.*



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(ii) *(Fontaine) Let $V$ be an $L^+_{dR}$-representation of $\Gamma_k$ and set*

$$V_{\mathrm{fin}} := \varprojlim_n (V/t^nV)_{\mathrm{fin}},$$

*where $(V/t^nV)_{\mathrm{fin}}$ is defined to be the union of finite-dimensional $k$-vector subspaces of $V/t^nV$ that are stable under the action of $\Gamma_k$. Then the natural map*

$$V_{\mathrm{fin}} \otimes_{k_\infty[[t]]} L^+_{\mathrm{dR}}$$

*is an isomorphism.*

Using this theorem, Sen defined the so-called Sen endomorphism $\phi_V$ on $V_\infty$ for a $K$-representation $V$ of $\Gamma_k$ (cf. [Sen81, Theorem 4]), and Fontaine defined a formal connection on $V_{\mathrm{fin}}$ for an $L^+_{\mathrm{dR}}$-representation $V$ of $\Gamma_k$ (cf. [Fon04, Proposition 3.7]).

We now turn to the relative setting. Let $A$ be a Tate $k$-algebra that is reduced and topologically of finite type over $k$. It is equipped with the supremum norm and we use this norm when we regard $A$ as a Banach $k$-algebra. We further assume that $(A, A^\circ)$ is smooth over $(k, \mathcal{O}_k)$. We set

$$A_{k_m} := A \hat{\otimes}_k k_m, \quad A_\infty := \varinjlim_m A_{k_m} \quad \text{and} \quad A_K := A \hat{\otimes}_k K.$$

Here we use a slightly heavy notation $A_{k_m}$ to reserve $A_m$ for a different ring in a later section. Since $A$, $k_m$, and $K$ are all complete Tate $k$-algebras, the completed tensor product is well-defined (or one can use Banach $k$-algebra structures). Note that $A_{k_m}$ (respectively $A_K$) is a complete Tate $k_m$-algebra (respectively $K$-algebra), that $A_\infty$ is a Tate $k_\infty$-algebra and that $A_K$ is the completion of $A_\infty$.

We introduce the relative versions of $k_\infty[[t]]$, $L^+_{\mathrm{dR}}$, and $L_{\mathrm{dR}}$ over $A$. We set

$$A_\infty[[t]] := \varprojlim_n A_\infty[t]/(t^n),$$

and equip $A_\infty[[t]]$ with the inverse limit topology of Tate $k_\infty$-algebras $A_\infty[t]/(t^n)$. We also set

$$A \hat{\otimes} L^+_{\mathrm{dR}} := \varprojlim_n A \hat{\otimes}_k L^+_{\mathrm{dR}}/(t^n),$$

and equip $A \hat{\otimes} L^+_{\mathrm{dR}}$ with the inverse limit topology. We finally set

$$A \hat{\otimes} L_{\mathrm{dR}} = (A \hat{\otimes} L^+_{\mathrm{dR}})[t^{-1}]$$

and equip $A \hat{\otimes} L_{\mathrm{dR}}$ with the inductive limit topology. Note that $\Gamma_k$ acts continuously on these rings (cf. [Bel15, Appendix]).

Definition 2.2. In this paper, an $A \hat{\otimes} L^+_{\mathrm{dR}}$-*representation* of $\Gamma_k$ is an $A \hat{\otimes} L^+_{\mathrm{dR}}$-module $V$ that is isomorphic to either $(A \hat{\otimes} L^+_{\mathrm{dR}})^r$ or $(A \hat{\otimes} L^+_{\mathrm{dR}}/(t^n))^r$ for some $r$ and $n$, equipped with a continuous $A \hat{\otimes} L^+_{\mathrm{dR}}$-semilinear action of $\Gamma_k$. We denote the category of $A \hat{\otimes} L^+_{\mathrm{dR}}$-representations of $\Gamma_k$ by $\mathrm{Rep}_{\Gamma_k}(A \hat{\otimes} L^+_{\mathrm{dR}})$. An $A \hat{\otimes} L^+_{\mathrm{dR}}$-representation of $\Gamma_k$ that is annihilated by $t$ is also called an $A_K$-*representation of* $\Gamma_k$.



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If $V$ is isomorphic to either $(A \hat{\otimes} L_{\mathrm{dR}}^+)^r$ or $(A \hat{\otimes} L_{\mathrm{dR}}^+/(t^n))^r$ then $V$ admits a topology by taking a basis and the topology is independent of the choice of the basis. Thus the continuity condition of the action of $\Gamma_k$ makes sense. Note that if $V$ is an $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation of $\Gamma_k$, then so are $t^nV$ and $V/t^nV$.

We are going to discuss the relative version of Sen–Fontaine's theory. Namely, we will work on $A_K$-representations of $\Gamma_k$ and $A \hat{\otimes} L_{\mathrm{dR}}^+$-representations of $\Gamma_k$. Note that Sen's theory in the relative setting is established by Sen himself [Sen88, Sen93] and that Fontaine's decompletion theory in the relative setting is established by Berger–Colmez and Bellovin for representations which come from $A$-representations of $\mathrm{Gal}(\overline{k}/k)$ via the theory of $(\varphi, \Gamma)$-modules [BC08, Bel15]. Since we need a Fontaine-type decompletion theory for arbitrary $A \hat{\otimes} L_{\mathrm{dR}}^+$-representations of $\Gamma_k$, we give detailed arguments; we will discuss the decompletion theory in the next subsection, and define Sen's endomorphism and Fontaine's connection in § 2.3.

We end this subsection with establishing basic properties of the rings we have introduced.

Proposition 2.3.

(i) *For each $n \geqslant 1$, $A \hat{\otimes}_k L_{\mathrm{dR}}^+/(t^n)$ is Noetherian and faithfully flat over $A_\infty[t]/(t^n)$.*

(ii) *$A \hat{\otimes} L_{\mathrm{dR}}^+$ is a $t$-adically complete flat $L_{\mathrm{dR}}^+$-algebra with $(A \hat{\otimes} L_{\mathrm{dR}}^+)/(t^n) = A \hat{\otimes}_k L_{\mathrm{dR}}^+/(t^n)$.*

*Proof.* For (i), the first assertion is proved in [BMS18, Lemma 13.4]. We prove that $A \hat{\otimes}_k L_{\mathrm{dR}}^+/(t^n)$ is faithfully flat over $A_\infty[t]/(t^n)$.

First we deal with the case $n = 1$, i.e. faithful flatness of $A_K$ over $A_\infty$. The proof is similar to that of [AB10, Lemme 5.9]. Recall $A_\infty = \varinjlim_n A_{k_m}$. Since $k_m$ and $K$ are both complete valuation fields, $A_K = A_{k_m} \hat{\otimes}_{k_m} K$ is faithfully flat over $A_{k_m}$ (e.g. use [BGR84, Proposition 2.1.7/8 and Theorem 2.8.2/2]).

We prove that $A_K$ is flat over $A_\infty$. For this it suffices to show that for any finitely generated ideal $I \subset A_\infty$, the map $I \otimes_{A_\infty} A_K \to A_K$ is injective. Take such an ideal $I$. As $I$ is finitely generated, there exist a positive integer $m$ and a finitely generated ideal $I_m \subset A_{k_m}$ such that $I = \mathrm{Im}(I_m \otimes_{A_{k_m}} A_\infty \to A_\infty)$. Since $A_K$ is flat over $A_{k_m}$, the map $I_m \otimes_{A_{k_m}} A_K \to A_K$ is injective. On the other hand, this map factors as $I_m \otimes_{A_{k_m}} A_K \to I \otimes_{A_\infty} A_K \to A_K$ and the first map is surjective by the choice of $I_m$. Hence the second map $I \otimes_{A_\infty} A_K \to A_K$ is injective.

For faithful flatness, it remains to prove that the map $\mathrm{Spec}\, A_K \to \mathrm{Spec}\, A_\infty$ is surjective. Assume the contrary and take a prime ideal $\mathfrak{P} \in \mathrm{Spec}\, A_\infty$ that is not in the image of the map. Set $\mathfrak{p} = \mathfrak{P} \cap A \in \mathrm{Spec}\, A$. Note that the prime ideals of $A_\infty$ above $\mathfrak{p}$ are conjugate to each other by the action of $\Gamma_k$. From this we see that no prime ideal of $A_\infty$ above $\mathfrak{p}$ is in the image of $\mathrm{Spec}\, A_K \to \mathrm{Spec}\, A_\infty$. Hence $\mathfrak{p}$ does not lie in the image of $\mathrm{Spec}\, A_K \to \mathrm{Spec}\, A$, which contradicts that $A_K$ is faithfully flat over $A$.

Next we deal with the general $n$. By the local flatness criterion [Mat89, Theorem 22.3] applied to the nilpotent ideal $(t) \subset A_\infty[t]/(t^n)$, the flatness follows from the case $n = 1$. Moreover, since $\mathrm{Spec}\, A_K \to \mathrm{Spec}\, A_\infty$ is surjective, so is $\mathrm{Spec}\, A \hat{\otimes}_k L_{\mathrm{dR}}^+/(t^n) \to \mathrm{Spec}\, A_\infty[t]/(t^n)$. Hence $A \hat{\otimes}_k L_{\mathrm{dR}}^+/(t^n)$ is faithfully flat over $A_\infty[t]/(t^n)$.

Assertion (ii) is proved in [BMS18, Lemma 13.4]. Note that the proof of [BMS18, Lemma 13.4] works in our setting since we assume the smoothness of $A$. □

## 2.2 Sen–Fontaine's decompletion theory in the relative setting

Definition 2.4. For an $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation $V$ of $\Gamma_k$, we define the subspace $V_{\mathrm{fin}}$ as follows.

– If $V$ is annihilated by $t^n$ for some $n \geqslant 1$, then $V_{\mathrm{fin}}$ is defined to be the union of finitely generated $A$-submodules of $V$ that are stable under the action of $\Gamma_k$.



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– In general, define

$$V_{\mathrm{fin}} := \varprojlim_n (V/t^n V)_{\mathrm{fin}}.$$

If $V$ is killed by $t^n$, then $V_{\mathrm{fin}}$ is an $A_\infty[t]/(t^n)$-module. In general, $V_{\mathrm{fin}}$ is an $A_\infty[[t]]$-module equipped with a semilinear action of $\Gamma_k$.

The following theorem is the main goal of this subsection.

Theorem 2.5. *For an $A \mathbin{\hat{\otimes}} L^+_{\mathrm{dR}}$-representation $V$ of $\Gamma_k$ that is finite free of rank $r$ over $A \mathbin{\hat{\otimes}} L^+_{\mathrm{dR}}$, the $A_\infty[[t]]$-module $V_{\mathrm{fin}}$ is finite free of rank $r$. Moreover, the natural map*

$$V_{\mathrm{fin}} \otimes_{A_\infty[[t]]} (A \mathbin{\hat{\otimes}} L^+_{\mathrm{dR}}) \to V$$

*is an isomorphism, and $V_{\mathrm{fin}}/t^n V_{\mathrm{fin}}$ is isomorphic to $(V/t^n V)_{\mathrm{fin}}$ for each $n \geqslant 1$.*

The key tool in the proof is the Sen method, which is axiomatized in [BC08, § 3]. We review parts of the Tate–Sen conditions that are used in our proofs. For a thorough treatment, we refer the reader to [BC08, § 3].

Consider Tate's normalized trace map

$$R_{k,m} = R_m : K \to k_m.$$

On $k_{m+m'} \subset K$, this map is defined as

$$[k_{m+m'} : k_m]^{-1} \operatorname{tr}_{k_{m+m'}/k_m} : k_{m+m'} \to k_m,$$

and it extends continuously to $R_{k,m} : K \to k_m$. We denote the kernel $\operatorname{Ker} R_{k,m}$ by $X_m$. The map $R_{k,m}$ extends $A$-linearly to the map $R_{A,m} : A_K \to A_{k_m}$. Fix a real number $c_3 > 1$. By work of Tate and Sen [BC08, Propositions 3.1.4 and 4.1.1], $G_0 = \Gamma_k$, $\tilde{\Lambda} = A_K$, $R_m$, and the valuation val on $A_K$ satisfy the Tate–Sen axioms in [BC08, § 3] for any fixed positive numbers $c_1$ and $c_2$.

In particular, $X_{A,m} := A \mathbin{\hat{\otimes}}_k X_m$ is the kernel of $R_{A,m}$, and we have topological splitting $A_K = A_{k_m} \oplus X_{A,m}$. For $\gamma \in \Gamma_k$, let $m(\gamma) \in \mathbb{Z}$ be the valuation of $\chi(\gamma) - 1 \in \mathbb{Z}_p$. Then there exists a positive integer $m(k)$ such that for each $m \geqslant m(k)$ and $\gamma \in \Gamma_k$ with $m(\gamma) \leqslant m$, $\gamma - 1$ is invertible on $X_{A,m}$ and

$$\operatorname{val}((\gamma - 1)^{-1} a) \geqslant \operatorname{val}(a) - c_3$$

for each $a \in A_K$.

Finally, for each matrix $U = (a_{ij}) \in M_r(A_K)$, we set $\operatorname{val} U := \min_{i,j} \operatorname{val} a_{ij}$.

Proposition 2.6. *Each finitely generated $A$-submodule of $A_K$ that is stable under the action of an open subgroup of $\Gamma_k$ is contained in $A_\infty$.*

*Proof.* We follow the proof of [Sen81, Proposition 3]. By [BC08, Corollaire 2.1.4], there exist complete discrete valuation fields $E_1, \ldots, E_s$ and an isometric embedding $A \hookrightarrow \prod_{i=1}^s E_i$. Then extending the scalar yields an isometric embedding

$$A_K = A_{k_m} \oplus X_{A,m} \hookrightarrow \prod_{i=1}^s E_i \mathbin{\hat{\otimes}}_k K = \prod_{i=1}^s (E_i \mathbin{\hat{\otimes}}_k k_m \oplus E_i \mathbin{\hat{\otimes}}_k X_m)$$

preserving the topological splittings.



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Let $\Gamma'_k$ be an open subgroup of $\Gamma_k$ and $W$ a finitely generated $A$-submodule of $A_K$ that is stable under the action of $\Gamma'_k$. Let $W_i$ be the finite-dimensional $E_i$-vector subspace of $E_i \hat{\otimes}_k K$ generated by the image of $W$ under the map $A_K \to \prod_{i=1}^s E_i \hat{\otimes}_k K \to E_i \hat{\otimes}_k K$. To prove that $W$ is contained in $A_\infty = \bigcup_m A_{k_m}$, it suffices to prove that for each $i$, there exists a large integer $m$ such that $W_i$ is contained in $E_i \hat{\otimes}_k k_m$.

Replacing $\Gamma'_k$ by a smaller open subgroup if necessary, we may assume that there exists a topological generator $\gamma$ of $\Gamma'_k$. Replacing $E_i$ by a finite field extension, we may also assume that all the eigenvalues of the $E_i$-endomorphism $\gamma$ on $W_i$ lie in $E_i$.

Let $w \in W_i$ be an eigenvector for $\gamma$ and let $\lambda_i \in E$ be its eigenvalue. Note that $\Gamma'_k$ acts continuously on $W_i$. When $j$ goes to infinity, $\gamma^{p^j}$ approaches 1 and thus $\lambda^{p^j}$ approaches 1. This implies that $\lambda$ is a principal unit, i.e. $|\lambda - 1|_{E_i} < 1$.

Lemma 2.7. *The eigenvalue $\lambda$ is a $p$-power root of unity.*

*Proof.* We follow the proof of [Tat67, Proposition 7(c)]. Assume the contrary. We will prove that $\gamma - \lambda : E_i \hat{\otimes}_k K \to E_i \hat{\otimes}_k K$ is bijective, which would contradict that the non-zero element $w \in W_i \subset E_i \hat{\otimes}_k K$ satisfies $(\gamma - \lambda)w = 0$.

Let $m$ be the integer such that $k_m$ is the fixed subfield of $k_\infty$ by $\gamma$. Consider the map $\gamma - 1 : E_i \hat{\otimes}_k K \to E_i \hat{\otimes}_k K$. This map preserves the decomposition $E_i \hat{\otimes}_k K = E_i \hat{\otimes}_k k_m \oplus E_i \hat{\otimes}_k X_m$. Moreover, it is zero on $E_i \hat{\otimes}_k k_m$ and bijective on $E_i \hat{\otimes}_k X_m$ with continuous inverse. Denote the inverse by $\rho$. Then $\rho$ is a bounded $E_i \hat{\otimes}_k k_m$-linear operator with operator norm at most $p^{c_3}$. Since $\lambda \in E_i$ and $\lambda \neq 1$, the map $\gamma - \lambda$ is bijective on $E_i \hat{\otimes}_k k_m$. So it suffices to prove that $\gamma - \lambda$ is bijective on $E_i \hat{\otimes}_k X_m$.

As operators on $E_i \hat{\otimes}_k X_m$, we have

$$(\gamma - \lambda)\rho = ((\gamma - 1) - (\lambda - 1))\rho = 1 - (\lambda - 1)\rho.$$

Thus if $|\lambda - 1|_{E_i} p^{c_3} < 1$, then $1 - (\lambda - 1)\rho$ has an inverse on $E_i \hat{\otimes}_k X_m$ given by a geometric series, and hence $\gamma - \lambda$ admits a continuous inverse on $E_i \hat{\otimes}_k X_m$. If $|\lambda - 1|_{E_i} p^{c_3} \geqslant 1$, first take a large integer $j$ with $|\lambda^{p^j} - 1|_{E_i} p^{c_3} < 1$. Then we can prove that $\gamma^{p^j} - \lambda^{p^j}$ has a bounded inverse on $E_i \hat{\otimes}_k X_m$. Hence so does $\gamma - \lambda$. □

We continue the proof of the proposition. Since each eigenvalue of $\gamma$ on $W_i$ is a $p$-power root of unity, we replace $\gamma$ by a higher $p$-power and may assume that $\gamma$ acts on $W_i$ unipotently. Thus $\gamma - 1$ acts on $W_i$ nilpotently.

Let $m$ be the integer such that $k_m$ is the fixed subfield of $k_\infty$ by $\gamma$. Then the map $\gamma - 1 : E_i \hat{\otimes}_k K \to E_i \hat{\otimes}_k K$ is zero on $E_i \hat{\otimes}_k k_m$ and bijective on $E_i \hat{\otimes}_k X_m$. This implies that the nilpotent endomorphism $\gamma - 1$ on $W_i$ is actually zero and thus $W_i$ is contained in $E_i \hat{\otimes}_k k_m$. □

*Example* 2.8. For the trivial $A_K$-representation $V = A_K$ of $\Gamma_k$, we have $V_{\mathrm{fin}} = A_\infty$ by Proposition 2.6.

The following theorem describes $V_{\mathrm{fin}}$ for a general $A_K$-representation $V$ of $\Gamma_k$, and it was first proved by Sen [Sen88, Sen93].

Theorem 2.9. *For an $A_K$-representation $V$ of $\Gamma_k$, the $A_\infty$-module $V_{\mathrm{fin}}$ is finite free. Moreover, the natural map*

$$V_{\mathrm{fin}} \otimes_{A_\infty} A_K \to V$$

*is an isomorphism.*

*Proof.* First we prove the following lemma.

Lemma 2.10. *There exist an $A_K$-basis $v_1, \dots, v_r \in V$ and a large positive integer $m$ such that the transformation matrix of $\gamma$ with respect to this basis has entries in $A_{k_m}$ for each $\gamma \in \Gamma_k$.*

*Proof.* This follows from the Tate–Sen method for $\Gamma_k$-representations in the relative setting. By [Che09, Lemme 3.18], $V$ has a $\Gamma_k$-stable $A_K^{\circ}$-lattice. Note that [Che09, Lemme 3.18] only concerns reduced affinoid algebras over a finite extension of $\mathbb{Q}_p$ but the same proof works for $A_K$ since one can apply Raynaud's theory to $A_K$.

By [BC08, Corollaire 3.2.4], there exist an $A_K$-basis $v_1, \dots, v_r \in V$, a large positive integer $m$, and an open subgroup $\Gamma'_k$ of $\Gamma_k$ such that the transformation matrix of $\gamma$ with respect to this basis has entries in $A_{k_m}$ for each $\gamma \in \Gamma'_k$. By shrinking $\Gamma'_k$ if necessary, we may also assume that $\Gamma'_k$ acts trivially on $A_{k_m}$.

For each $\gamma \in \Gamma_k$, we denote by $U_\gamma \in \mathrm{GL}_r(A_K)$ the transformation matrix of $\gamma$ with respect to $v_1, \dots, v_r$. Note that $U_{\gamma\gamma'} = U_\gamma \gamma(U_{\gamma'})$ for $\gamma, \gamma' \in \Gamma_k$.

Take a set $\{\gamma_1, \dots, \gamma_s\}$ of coset representatives of $\Gamma_k/\Gamma'_k$ and let $W$ be the finitely generated $A_{k_m}$-submodule of $A_K$ generated by the entries of $U_{\gamma_1}, \dots, U_{\gamma_s}$. Since $U_{\gamma_i\gamma'} = U_{\gamma_i}\gamma_i(U_{\gamma'})$ and $\gamma_i(U_{\gamma'})$ has entries in $A_{k_m}$ for $\gamma' \in \Gamma'_k$ by our construction, it follows that $W$ is independent of the choice of the representatives $\gamma_1, \dots, \gamma_s$. Moreover, we have $\gamma'(U_{\gamma_i}) = U_{\gamma'}^{-1} U_{\gamma'\gamma_i}$ for $\gamma' \in \Gamma'_k$. From this we see that $W$ is stable under the action of $\Gamma'_k$.

Proposition 2.6 implies that $W \subset A_\infty$, namely, $U_{\gamma_1}, \dots, U_{\gamma_s} \in \mathrm{GL}_r(A_\infty)$. Thus if we increase $m$ so that $U_{\gamma_1}, \dots, U_{\gamma_s} \in \mathrm{GL}_r(A_{k_m})$, then $U_\gamma \in \mathrm{GL}_r(A_{k_m})$ for any $\gamma \in \Gamma_k$. □

We keep the notation in the proof of the lemma. From the lemma, we see that $\bigoplus_{i=1}^r A_\infty v_i \subset V_{\mathrm{fin}}$. So it suffices to prove that this is an equality.

Take any $v \in V_{\mathrm{fin}}$. Let $W_v$ be the $A_{k_m}$-submodule of $A_K$ generated by the coordinates of $\gamma v$ with respect to the basis $v_1, \dots, v_r$ where $\gamma$ runs over all elements of $\Gamma_k$. Since $v \in V_{\mathrm{fin}}$, this is a finitely generated $A_{k_m}$-module.

Write $v = \sum_{i=1}^r a_i v_i$ with $a_i \in A_K$ and denote the column vector of the $a_i$ by $\vec{a}$. Then it is easy to see that $W_v$ is generated by the entries of $U_\gamma \gamma(\vec{a})$ ($\gamma \in \Gamma_k$). Since $U_{\gamma'\gamma} = U_{\gamma'}\gamma'(U_\gamma)$ for $\gamma, \gamma' \in \Gamma_k$, we compute

$$\gamma'(U_\gamma \gamma(\vec{a})) = U_{\gamma'}^{-1} U_{\gamma'\gamma}(\gamma'\gamma)(\vec{a}).$$

From this we see that $W_v$ is stable under the action of $\Gamma_k$.

By Proposition 2.6, we have $W_v \subset A_\infty$. In particular, $a_1, \dots, a_r \in A_\infty$ and thus $v \in \bigoplus_{i=1}^r A_\infty v_i$. □

Proposition 2.11. *Let $V$ be an $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation of $\Gamma_k$. If $V$ is finite free of rank $r$ over $A \hat{\otimes} L_{\mathrm{dR}}^+/(t^n)$, then $V_{\mathrm{fin}}$ is finite free of rank $r$ over $A_\infty[t]/(t^n)$. Moreover, the natural map*

$$V_{\mathrm{fin}} \otimes_{A_\infty[[t]]} (A \hat{\otimes} L_{\mathrm{dR}}^+) \to V$$

*is an isomorphism.*

*Proof.* We prove this proposition by induction on $n$. When $n = 1$, this is Theorem 2.9. So we assume $n > 1$.

Set $V' := t^{n-1}V$ and $V'' := V/V'$. They are $A \hat{\otimes} L_{\mathrm{dR}}^+$-representations of $\Gamma_k$ and $V''$ is finite free of rank $r$ over $A \hat{\otimes} L_{\mathrm{dR}}^+/(t^{n-1})$. By induction hypothesis, $V''_{\mathrm{fin}}$ is finite free of rank $r$ over $A_\infty[t]/(t^{n-1})$ and $V''_{\mathrm{fin}} \otimes_{A_\infty[t]/(t^{n-1})} A \hat{\otimes} L_{\mathrm{dR}}^+/(t^{n-1}) \cong V''$.



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Take lifts $v_1, \dots, v_r$ of a basis of $V''_{\mathrm{fin}}$ to $V$. Then $v_1, \dots, v_r$ form an $A_\infty[t]/(t^n)$-basis of $V$. We will prove that after a suitable modification of $v_1, \dots, v_r$ the transformation matrix of $\gamma$ on $V$ with respect to the new basis has entries in $A_\infty[t]/(t^n)$ for every $\gamma \in \Gamma_k$.

Suppose that we are given an element $\gamma$ of $\Gamma_k$. For each $1 \leqslant j \leqslant r$, write $\gamma v_j = \sum_{i=1}^r a_{ij} v_i$ with $a_{ij} \in A \hat{\otimes} L^+_{\mathrm{dR}}/(t^n)$. Then the $r \times r$ matrix $T := (a_{ij})$ is invertible since it is invertible modulo $t^{n-1}$. By the property of $V''_{\mathrm{fin}}$, we can write

$$a_{ij} = a^0_{ij} + t^{n-1} a^1_{ij}, \quad a^0_{ij} \in A_\infty[t]/(t^n), \quad a^1_{ij} \in A_K = A \hat{\otimes} L^+_{\mathrm{dR}}/(t).$$

Set $U := (a^0_{ij} \bmod t) \in M_r(A_\infty)$. This is invertible. In fact, $U$ is the transformation matrix of $\gamma$ acting on $V/tV$ with respect to the basis $(v_i \bmod t)$.

Since $\Gamma_k$ acts continuously on $V/tV$, $\mathrm{val}(U-1) > c_3$ and $m(\gamma) > \max\{c_3, m(k)\}$ for some $\gamma \neq 1$ close to 1. From now on, we fix such $\gamma$.

CLAIM 2.12. *There exists an element in* $\mathrm{GL}_r(A \hat{\otimes} L^+_{\mathrm{dR}}/(t^n))$ *of the form* $1 + t^{n-1}M$ *with* $M \in M_r(A_K)$ *such that the* $r \times r$ *matrix*

$$(1 + t^{n-1}M)^{-1} T \gamma (1 + t^{n-1}M)$$

*lies in* $\mathrm{GL}_r(A_\infty[t]/(t^n))$.

*Proof.* Noting that every element in $A \hat{\otimes} L^+_{\mathrm{dR}}/(t^n)$ is annihilated by $t^n$, we compute

$$\begin{aligned}(1 + t^{n-1}M)^{-1} T \gamma (1 + t^{n-1}M) &= (1 - t^{n-1}M) T (1 + \chi(\gamma)^{n-1} t^{n-1} \gamma(M)) \\ &= T - t^{n-1}(MT - \chi(\gamma)^{n-1} T \gamma(M)) \\ &\quad - t^{2(n-1)} \chi(\gamma)^{n-1} M T \gamma(M) \\ &= T - t^{n-1}(MU - \chi(\gamma)^{n-1} U \gamma(M)).\end{aligned}$$

Since $T = (a^0_{ij}) + t^{n-1}(a^1_{ij})$ with $(a^0_{ij}) \in \mathrm{GL}_r(A_\infty[t]/(t^n))$, it suffices to find $M \in M_r(A_K)$ such that

$$(a^1_{ij}) - (MU - \chi(\gamma)^{n-1} U \gamma(M)) \in M_r(A_\infty).$$

We will apply Lemma 2.13 below to $U$, $U' = U^{-1}$ and $s = n-1$. Take $m \geqslant m(\gamma)$ large enough so that $U$ and $U^{-1}$ lie in $\mathrm{GL}_r(A_{k_m})$. Recall the normalized trace map $R_{A,m} : A_K \to A_{k_m}$ with kernel $X_{A,m}$. Since $R_{A,m}$ is $A_{k_m}$-linear, we see that $((1 - R_{A,m})(a^1_{ij}))U^{-1} \in M_r(X_{A,m})$. Therefore, by Lemma 2.13, there exists $M_0 \in M_r(X_{A,m})$ such that

$$((1 - R_{A,m})(a^1_{ij}))U^{-1} = M_0 - \chi(\gamma)^{n-1} U \gamma(M_0) U^{-1}.$$

From this we have

$$(a^1_{ij}) - (M_0 U - \chi(\gamma)^{n-1} U \gamma(M_0)) = R_{A,m}(a^1_{ij}) \in M_r(A_{k_m}),$$

and the matrix $1 + t^{n-1}M_0$ satisfies the condition of the lemma. □

We continue the proof of the proposition. We replace the basis $v_1, \dots, v_r$ by the one corresponding to the matrix $1 + t^{n-1}M$ in the lemma. Then the transformation matrix of our fixed $\gamma$ with respect to the new $v_1, \dots, v_r$ has entries in $A_{k_m}[t]/(t^n)$. Thus for each $1 \leqslant i \leqslant r$, the $\gamma^{\mathbb{Z}_p}$-orbit of $v_i$ is contained in a finitely generated $A_{k_m}[t]/(t^n)$-submodule of $V$ that is stable under $\gamma^{\mathbb{Z}_p}$. Since $\gamma^{\mathbb{Z}_p}$ is of finite index in $\Gamma_k$, the $\Gamma_k$-orbit of $v_i$ is also contained in a finitely generated $A_{k_m}[t]/(t^n)$-submodule of $V$ that is stable under $\Gamma_k$. This means that $v_1,\ldots,v_r \in V_{\mathrm{fin}}$. Hence $\bigoplus_{i=1}^r A_\infty[t]/(t^n)v_i \subset V_{\mathrm{fin}}$.

It remains to prove that $\bigoplus_{i=1}^r A_\infty[t]/(t^n)v_i = V_{\mathrm{fin}}$. Since $A_\infty[t]/(t^n) \to A\,\hat{\otimes}\,L_{\mathrm{dR}}^+/(t^n)$ is faithfully flat and $V = \bigoplus_{i=1}^r A\,\hat{\otimes}\,L_{\mathrm{dR}}^+/(t^n)v_i$, it is enough to show that the natural map $V_{\mathrm{fin}}\otimes_{A_\infty[t]/(t^n)} A\,\hat{\otimes}\,L_{\mathrm{dR}}^+/(t^n) \to V$ is injective. Note that $V_{\mathrm{fin}}\otimes_{A_\infty[t]/(t^n)} A\,\hat{\otimes}\,L_{\mathrm{dR}}^+/(t^n) = V_{\mathrm{fin}}\otimes_{A_\infty[[t]]} A\,\hat{\otimes}\,L_{\mathrm{dR}}^+$.

Recall the exact sequence $0 \to V' \to V \to V'' \to 0$. From this we have an exact sequence $0 \to V'_{\mathrm{fin}} \to V_{\mathrm{fin}} \to V''_{\mathrm{fin}}$, and it yields the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & V'_{\mathrm{fin}} \otimes (A\,\hat{\otimes}\,L_{\mathrm{dR}}^+) \arrow[r] \arrow[d] & V_{\mathrm{fin}} \otimes (A\,\hat{\otimes}\,L_{\mathrm{dR}}^+) \arrow[r] \arrow[d] & V''_{\mathrm{fin}} \otimes (A\,\hat{\otimes}\,L_{\mathrm{dR}}^+) \arrow[d] & \\
0 \arrow[r] & V' \arrow[r] & V \arrow[r] & V'' \arrow[r] & 0
\end{tikzcd}$$

where the tensor products in the first row are taken over $A_\infty[[t]]$. By induction hypothesis, the first and the third vertical maps are isomorphisms. Hence the second vertical map is injective and this completes the proof. □

The following lemma is used in the proof of Proposition 2.11.

Lemma 2.13. *Let $s$ be a positive integer. Let $U, U'$ be elements in $M_r(A_\infty)$ satisfying* $\mathrm{val}(U-1) > c_3$ *and* $\mathrm{val}(U'-1) > c_3$. *Take a positive integer $m$ such that $m > \max\{m(k), c_3\}$ and $U$, $U' \in M_r(A_{k_m})$. Then for any $\gamma \in \Gamma_k$ with $c_3 < m(\gamma) \leqslant m$, the map*

$$f : M_r(A_K) \to M_r(A_K), \quad M \mapsto M - \chi(\gamma)^s U\gamma(M)U'$$

*is bijective on the subset $M_r(X_{A,m})$ consisting of the $r\times r$ matrices with entries in the kernel $X_{A,m}$ of $R_{A,m} : A_K \to A_{k_m}$.*

*Proof.* The proof of [BC09, Lemma 15.3.9] works in our setting. For the convenience of the reader, we reproduce their proof here.

We first check that $f$ restricts to an endomorphism on $M_r(X_{A,m})$. This follows from the fact that the map $R_{A,m}$ is $A_{k_m}$-linear and $\Gamma_k$-equivariant and thus $X_{A,m}$ is an $A_{k_m}$-module stable under the action of $\Gamma_k$.

We define a map $h : M_r(A_K) \to M_r(A_K)$ by

$$\begin{aligned} h(N) &:= N - \chi(\gamma)^s UNU' \\ &= (N - \chi(\gamma)^s N) + \chi(\gamma)^s((N-UN) + UN(1-U')). \end{aligned}$$

Then the same argument as above shows that $h$ restricts to an endomorphism on $M_r(X_{A,m})$. We also have $f(M) = (1-\gamma)M + h(\gamma M)$.

Recall that the map $1-\gamma : M_r(X_{A,m}) \to M_r(X_{A,m})$ admits a continuous inverse with operator norm at most $p^{c_3}$. We denote this inverse by $\rho$. Since $(f\circ\rho - \mathrm{id})M = h(\gamma\rho(M))$, it suffices to prove that the operator norm of $h$ is less than $p^{-c_3}$; this will imply that the operator norm of $h\circ\gamma\circ\rho$ is less than 1. Thus $f\circ\rho$ admits a continuous inverse given by a geometric series and hence $f$ is bijective on $M_r(X_{A,m})$.

By the second expression of $h$, we have

$$\begin{aligned} \mathrm{val}(h(N)) &\geqslant \min\{\mathrm{val}((1-\chi(\gamma)^s)N), \mathrm{val}((U-1)N), \mathrm{val}(UN(1-U'))\} \\ &\geqslant \min\{\mathrm{val}((1-\chi(\gamma))N), \mathrm{val}((U-1)N), \mathrm{val}(N(1-U'))\}. \end{aligned}$$

From this we have

$$\mathrm{val}(h(N)) \geqslant \mathrm{val}(N) + \delta,$$

where $\delta := \min\{m(\gamma), \mathrm{val}(U-1), \mathrm{val}(U'-1)\}$. Thus the operator norm of $h$ is at most $p^{-\delta}$. Since $\delta > c_3$ by assumption, this completes the proof. □

*Proof of Theorem 2.5.* For each $n \geqslant 1$, put $V_n := V/t^nV$. This is an $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation of $\Gamma_k$ that is finite free of rank $r$ over $A \hat{\otimes} L_{\mathrm{dR}}^+/(t^n)$. Thus by Proposition 2.11, $(V_n)_{\mathrm{fin}}$ is finite free of rank $r$ over $A_\infty[t]/(t^n)$, and $(V_n)_{\mathrm{fin}} \otimes_{A_\infty[[t]]} (A \hat{\otimes} L_{\mathrm{dR}}^+) \to V_n$ is an isomorphism.

By definition, we have $V_{\mathrm{fin}} = \varprojlim_n (V_n)_{\mathrm{fin}}$. Since the natural map $V_{n+1} \to V_n$ is surjective, so is the map $(V_{n+1})_{\mathrm{fin}} \to (V_n)_{\mathrm{fin}}$ by the faithfully flatness of $A_\infty[t]/(t^{n+1}) \to A \hat{\otimes} L_{\mathrm{dR}}^+/(t^{n+1})$. Thus lifting a basis of $(V_n)_{\mathrm{fin}}$ gives a basis of $V_{\mathrm{fin}}$ and we see that $V_{\mathrm{fin}}$ is finite free of rank $r$ over $A_\infty[[t]]$. The remaining assertions also follow from this. □

Proposition 2.14. *For an $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation $V$ of $\Gamma_k$ that is finite free of rank $r$ over $A \hat{\otimes} L_{\mathrm{dR}}^+$, the $A_\infty[[t]]$-module $V_{\mathrm{fin}}$ is the union of finitely generated $A_\infty[[t]]$-submodules of $V$ that are stable under the action of $\Gamma_k$. In particular, the natural inclusion*

$$(V_{\mathrm{fin}})^{\Gamma_k} \hookrightarrow V^{\Gamma_k}$$

*is an isomorphism.*

*Proof.* Let $V'_{\mathrm{fin}}$ denote the union of finitely generated $A_\infty[[t]]$-submodules of $V$ that are stable under the action of $\Gamma_k$. Then $V_{\mathrm{fin}} \subset V'_{\mathrm{fin}}$ by Theorem 2.5. So it remains to prove the opposite inclusion. For this it suffices to prove $V'_{\mathrm{fin}}/t^nV'_{\mathrm{fin}} \subset V_{\mathrm{fin}}/t^nV_{\mathrm{fin}}$ for each $n \geqslant 1$. Since $V_{\mathrm{fin}}/t^nV_{\mathrm{fin}} = (V/t^nV)_{\mathrm{fin}}$ by Theorem 2.5, the desired inclusion follows from the definition of $(V/t^nV)_{\mathrm{fin}}$ noting $A_\infty[t]/(t^n) = \bigcup_m A_{k_m}[t]/(t^n)$. The second assertion follows from the first. □

*Example* 2.15. For the trivial $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation $V = A \hat{\otimes} L_{\mathrm{dR}}^+$ of $\Gamma_k$, we have $V_{\mathrm{fin}} = A_\infty[[t]]$.

Finally, we discuss topologies on $V_{\mathrm{fin}}$ and the continuity of the action of $\Gamma_k$.

Lemma 2.16. *Let $W$ be a finite free $A_\infty[[t]]/(t^n)$-module equipped with an action of $\Gamma_k$. Then $\Gamma_k$-action is continuous with respect to the topology on $W$ induced from the product topology on $A_\infty[[t]]/(t^n) \cong A_\infty^n$ if and only if it is continuous with respect to the topology on $W$ induced from the subspace topology on $A_\infty[[t]]/(t^n) \subset A \hat{\otimes} L_{\mathrm{dR}}^+/(t^n)$.*

*Proof.* For each of the two topologies on $W$, the continuity of $\Gamma_k$ implies that there exist an $A_\infty[[t]]/(t^n)$-basis $w_1, \ldots, w_r$ of $W$ and a large positive integer $m$ such that $W_m := \bigoplus_{i=1}^r A_{k_m}[[t]]/(t^n)w_i$ is stable under $\Gamma_k$ and its action on $W_m$ is continuous with respect to the induced topology $W_m \subset W$. Conversely, if the $\Gamma_k$-action on $W_m$ is continuous with respect to the induced topology $W_m \subset W$ for such $\Gamma_k$-stable $A_{k_m}[[t]]/(t^n)$-submodule $W_m$ with $W_m \otimes_{A_{k_m}[[t]]/(t^n)} A_\infty[[t]]/(t^n) = W$, the $\Gamma_k$-action on $W$ is continuous.

The subspace topology on $A_{k_m}[[t]]/(t^n)$ from $A \hat{\otimes} L_{\mathrm{dR}}^+/(t^n)$ coincides with the product topology on $A_{k_m}[[t]]/(t^n) \cong A_{k_m}^n$. From this we find that the continuity conditions on the action of $\Gamma_k$ on $W_m$ with respect to the two topologies coincide. Hence the two continuity properties of the action of $\Gamma_k$ on $W$ are equivalent. □



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Definition 2.17. Let $V$ be an $A \hat{\otimes} L_{\mathrm{dR}}^{+}$-representation of $\Gamma_k$.

- If $V$ is finite free over $A \hat{\otimes} L_{\mathrm{dR}}^{+}/(t^n)$ for some $n \geqslant 1$, we equip $V_{\mathrm{fin}}$ with the topology acquired from topologizing $A_\infty[[t]]/(t^n)$ with the product topology of the $p$-adic topology on $A_\infty$. Then $\Gamma_k$ acts continuously on $V_{\mathrm{fin}}$ by Lemma 2.16.
- If $V$ is finite free over $A \hat{\otimes} L_{\mathrm{dR}}^{+}$, we equip $V_{\mathrm{fin}}$ with the inverse limit topology via $V_{\mathrm{fin}} = \varprojlim_n (V/t^n V)_{\mathrm{fin}}$. Then $\Gamma_k$ acts continuously on $V_{\mathrm{fin}}$.

Definition 2.18. An $A_\infty[[t]]$-*representation* of $\Gamma_k$ is an $A_\infty[[t]]$-module $W$ that is isomorphic to either $(A_\infty[[t]])^r$ or $(A_\infty[[t]]/(t^n))^r$ for some $r$ and $n$, equipped with a continuous $A_\infty[[t]]$-semilinear action of $\Gamma_k$ (here the topology on $W$ is acquired from the $p$-adic topology on $A_\infty$ by considering the product topology and the inverse limit topology as before). We denote the category of $A_\infty[[t]]$-representations of $\Gamma_k$ by $\mathrm{Rep}_{\Gamma_k}(A_\infty[[t]])$. An $A_\infty[[t]]$-representation of $\Gamma_k$ that is annihilated by $t$ is also called an $A_\infty$-*representation* of $\Gamma_k$.

Theorem 2.19. *The decompletion functor*

$$\mathrm{Rep}_{\Gamma_k}(A \hat{\otimes} L_{\mathrm{dR}}^{+}) \to \mathrm{Rep}_{\Gamma_k}(A_\infty[[t]]), \quad V \mapsto V_{\mathrm{fin}}$$

*is an equivalence of categories. A quasi-inverse is given by* $W \mapsto W \otimes_{A_\infty[[t]]} (A \hat{\otimes} L_{\mathrm{dR}}^{+})$.

*Proof.* By Theorem 2.5, Proposition 2.11, and Lemma 2.16, the functor is well-defined and essentially surjective. The full faithfulness follows from Proposition 2.14. □

### 2.3 Sen's endomorphism and Fontaine's connection in the relative setting

Proposition 2.20. *Let $W$ be an $A_\infty$-representation of $\Gamma_k$. Then there exists a unique $A_\infty$-linear map $\phi_W : W \to W$ satisfying the following property: for any $w \in W$, there exists an open subgroup $\Gamma_{k,w}$ of $\Gamma_k$ such that*

$$\gamma w = \exp(\log(\chi(\gamma))\phi_W)(w)$$

*for $\gamma \in \Gamma_{k,w}$. Here* $\log$ *(respectively* $\exp$*) is the $p$-adic logarithm (respectively exponential). Moreover, $\phi_W$ is $\Gamma_k$-equivariant and functorial with respect to $W$.*

*Remark* 2.21. The proposition says that the endomorphism $\phi_W$ is computed as

$$\phi_W(w) = \lim_{\gamma \to 1} \frac{\gamma w - w}{\log \chi(\gamma)}$$

for $w \in W$.

*Proof.* This is standard; arguments in [Sen81, Theorem 4] also work in our setting. See also [Sen93, § 2], [Sen88, Proposition 4], [Fon04, Proposition 2.5], and [BC09, § 15.1]. □

The following lemma is also proved by standard arguments.

Lemma 2.22. *Let $W_1$ and $W_2$ be $A_\infty$-representations of $\Gamma_k$. Then we have the following equalities:*

- $\phi_{W_1 \oplus W_2} = \phi_{W_1} \oplus \phi_{W_2}$ *on* $W_1 \oplus W_2$;
- $\phi_{W_1 \otimes W_2} = \phi_{W_1} \otimes \mathrm{id}_{W_2} + \mathrm{id}_{W_1} \otimes \phi_{W_2}$ *on* $W_1 \otimes W_2$;
- $\phi_{\mathrm{Hom}(W_1,W_2)}(f) = \phi_{W_2} \circ f - f \circ \phi_{W_1}$ *for* $f \in \mathrm{Hom}(W_1, W_2)$.



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Definition 2.23. Let $V$ be an $A_K$-representation of $\Gamma_k$. We denote by $\phi_V$ the $A_K$-linear endomorphism $\phi_{V_{\mathrm{fin}}} \otimes \mathrm{id}_{A_K}$ on $V = V_{\mathrm{fin}} \otimes_{A_\infty} A_K$.

Proposition 2.24. *Let $W$ be an $A_\infty[[t]]$-representation of $\Gamma_k$. Then there exists a unique $A_\infty$-linear map $\phi_{\mathrm{dR},W} : W \to W$ satisfying the following property: for each $n \in \mathbb{N}$ and $w \in W$, there exists an open subgroup $\Gamma_{k,n,w}$ of $\Gamma_k$ such that*

$$\gamma w \equiv \exp(\log(\chi(\gamma))\phi_{\mathrm{dR},W})(w) \pmod{t^n W}$$

*for $\gamma \in \Gamma_{k,n,w}$.*

*Proof.* Note that $A_\infty[[t]]/(t^n)$ is a finite free $A_\infty$-module of rank $n$ and thus $W/t^nW$ can be regarded an $A_\infty$-representation of $\Gamma_k$. So the proposition follows from Proposition 2.20. □

Definition 2.25. Set $A_\infty((t)) := A_\infty[[t]][t^{-1}]$. We denote by $\partial_t$ the $A_\infty$-linear endomorphism

$$A_\infty((t)) \to A_\infty((t)), \quad \textstyle\sum_{j \gg -\infty} a_j t^j \mapsto \sum_{j \gg -\infty} j a_j t^{j-1}.$$

The restriction of $\partial_t$ to $A_\infty[[t]]$ is also denoted by $\partial_t$.

Proposition 2.26. *For an $A_\infty[[t]]$-representation $W$ of $\Gamma_k$, the endomorphism $\phi_{\mathrm{dR},W} : W \to W$ satisfies*

$$\phi_{\mathrm{dR},W}(\alpha w) = t\partial_t(\alpha)w + \alpha\phi_{\mathrm{dR},W}(w)$$

*for every $\alpha \in A_\infty[[t]]$ and $w \in W$.*

*Proof.* By the characterizing property of $\phi_{\mathrm{dR},W}$, we may assume that $W$ is annihilated by some power of $t$. In this case, it is enough to check the equality for $\alpha = t^j$ by $A_\infty$-linearity of $\phi_{\mathrm{dR},W}$. By induction on $j$, we may further assume that $\alpha = t$.

So we need to show $\phi_{\mathrm{dR},W}(tw) = tw + t\phi_{\mathrm{dR},W}(w)$. This follows from

$$\begin{aligned}\phi_{\mathrm{dR},W}(tw) &= \lim_{\gamma \to 1} \frac{\gamma(tw) - tw}{\log \chi(\gamma)} \\ &= \lim_{\gamma \to 1} \frac{\chi(\gamma) - 1}{\log \chi(\gamma)} t\gamma(w) + t \lim_{\gamma \to 1} \frac{\gamma(w) - w}{\log \chi(\gamma)} \\ &= tw + t\phi_{\mathrm{dR},W}(w).\end{aligned}$$

□

Lemma-Definition 2.27. *Let $W$ be a finite free $A_\infty[[t]]$-representation of $\Gamma_k$. Then $W[t^{-1}] := W \otimes_{A_\infty[[t]]} A_\infty((t))$ is a finite free $A_\infty((t))$-module equipped with $\Gamma_k$-action and $\Gamma_k$-stable decreasing filtration defined by $\mathrm{Fil}^j W[t^{-1}] := t^j W$. Moreover, the $A_\infty$-linear endomorphism $\phi_{\mathrm{dR},W[t^{-1}]} : W[t^{-1}] \to W[t^{-1}]$ sending $w \in \mathrm{Fil}^j W[t^{-1}]$ to*

$$\phi_{\mathrm{dR},W[t^{-1}]}(w) := jw + t^j \phi_{\mathrm{dR},W}(t^{-j}w)$$

*is well-defined and satisfies $\phi_{\mathrm{dR},W[t^{-1}]}|_W = \phi_{\mathrm{dR},W}$.*

*Proof.* This follows from Proposition 2.26. □



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Definition 2.28. Let $V$ be a finite free $A \hat{\otimes} L_{\mathrm{dR}}$-module equipped with $\Gamma_k$-action and $\Gamma_k$-stable decreasing filtration $\mathrm{Fil}^j V$ such that $\mathrm{Fil}^0 V$ is a finite free $A \hat{\otimes} L_{\mathrm{dR}}^+$-representation of $\Gamma_k$ and $\mathrm{Fil}^j V = t^j \mathrm{Fil}^0 V$ for all $j \in \mathbb{Z}$. Define

$$V_{\mathrm{fin}} := (\mathrm{Fil}^0 V)_{\mathrm{fin}}[t^{-1}].$$

By Lemma-Definition 2.27, $V_{\mathrm{fin}}$ is a finite free $A_\infty((t))$-module equipped with $\Gamma_k$-action, $\Gamma_k$-stable decreasing filtration $\mathrm{Fil}^j V_{\mathrm{fin}}$, and $\phi_{\mathrm{dR},V_{\mathrm{fin}}}$. Since $\phi_{\mathrm{dR},V_{\mathrm{fin}}}$ preserves the filtration, it defines an $A_\infty$-linear endomorphism on $\mathrm{gr}^0 V_{\mathrm{fin}}$, which we denote by $\mathrm{Res}_{\mathrm{Fil}^0 V_{\mathrm{fin}}} \phi_{\mathrm{dR},V_{\mathrm{fin}}}$. It follows from the definition that

$$\mathrm{Res}_{\mathrm{Fil}^0 V_{\mathrm{fin}}} \phi_{\mathrm{dR},V_{\mathrm{fin}}} = \phi_{\mathrm{gr}^0 V}$$

as endomorphisms on the finite free $A_\infty$-module $\mathrm{gr}^0(V_{\mathrm{fin}}) = (\mathrm{gr}^0 V)_{\mathrm{fin}}$.

## 3. The arithmetic Sen endomorphism of a *p*-adic local system

From this section, we study relative $p$-adic Hodge theory in geometric families. Let $k$ be a finite field extension of $\mathbb{Q}_p$ and let $X$ be an $n$-dimensional smooth rigid analytic variety over $\mathrm{Spa}(k, \mathcal{O}_k)$. Let $K$ be the $p$-adic completion of $k_\infty := \bigcup_n k(\mu_{p^n})$ and let $X_K$ denote the base change of $X$ to $\mathrm{Spa}(K, \mathcal{O}_K)$. We denote by $\Gamma_k$ the Galois group $\mathrm{Gal}(k_\infty/k)$.

Based on the recent progresses on relative $p$-adic Hodge theory [KL15, KL16, Sch12, Sch13], Liu and Zhu attached to an étale $\mathbb{Q}_p$-local system $\mathbb{L}$ a nilpotent Higgs bundle $\mathcal{H}(\mathbb{L})$ on $X_K$ equipped with $\Gamma_k$-action [LZ17]. Our goal is to define an endomorphism $\phi_{\mathbb{L}}$ on $\mathcal{H}(\mathbb{L})$ by decompleting the $\Gamma_k$-action. The endomorphism $\phi_{\mathbb{L}}$, which we will call the arithmetic Sen endomorphism, is a natural generalization of the Sen endomorphism of a $p$-adic Galois representation of $k$.

### 3.1 Review of the *p*-adic Simpson correspondence à la Liu and Zhu

First let us briefly recall the sites and sheaves that we use. Let $X_{\mathrm{pro\acute{e}t}}$ be the pro-étale site on $X$ in the sense of [Sch13, Sch16]. The pro-étale site is equipped with a natural projection to the étale site on $X$

$$\nu : X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}.$$

Let $\nu' : X_{\mathrm{pro\acute{e}t}}/X_K \to (X_K)_{\mathrm{\acute{e}t}}$ be the restriction of $\nu$ and we identify $X_{\mathrm{pro\acute{e}t}}/X_K$ with $(X_K)_{\mathrm{pro\acute{e}t}}$ (see a discussion before [Sch13, Proposition 6.10]).

We denote by $\hat{\mathbb{Z}}_p$ (respectively $\hat{\mathbb{Q}}_p$) the constant sheaf on $X_{\mathrm{pro\acute{e}t}}$ associated to $\mathbb{Z}_p$ (respectively $\mathbb{Q}_p$). For a $\mathbb{Z}_p$-local system $\mathbb{L}$ (respectively $\mathbb{Q}_p$-local system) on $X_{\mathrm{\acute{e}t}}$, let $\hat{\mathbb{L}}$ denote the $\hat{\mathbb{Z}}_p$-module (respectively $\hat{\mathbb{Q}}_p$-module) on $X_{\mathrm{pro\acute{e}t}}$ associated to $\mathbb{L}$ (see [Sch13, § 8.2]).

We define sheaves on $X_{\mathrm{pro\acute{e}t}}$ as follows. We set

$$\mathcal{O}_X^+ := \nu^* \mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+, \quad \mathcal{O}_X := \nu^* \mathcal{O}_{X_{\mathrm{\acute{e}t}}} \quad \text{and} \quad \hat{\mathcal{O}}_X := \left( \varprojlim_n \mathcal{O}_X^+/p^n \right)[p^{-1}].$$

We also set $\Omega_X^1 = \nu^* \Omega_{X_{\mathrm{\acute{e}t}}}^1$ and we denote its $i$th exterior power by $\Omega_X^i$. Moreover, Scholze introduced the de Rham period sheaves $\mathbb{B}_{\mathrm{dR}}^+$, $\mathbb{B}_{\mathrm{dR}}$, $\mathcal{O}\mathbb{B}_{\mathrm{dR}}^+$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ on $X_{\mathrm{pro\acute{e}t}}$ in [Sch13, § 6] and [Sch16]. The structural de Rham sheaf $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ has the following properties: it is a sheaf of $\mathcal{O}_X$-algebras equipped with a decreasing filtration $\mathrm{Fil}^\bullet \mathcal{O}\mathbb{B}_{\mathrm{dR}}$ and an integrable connection

$$\nabla : \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega_X^1$$

satisfying the Griffiths transversality. Since $X$ is assumed to be smooth of dimension $n$, this gives rise to the following exact sequence of sheaves on $X_{\mathrm{proét}}$:

$$0 \to \mathbb{B}_{\mathrm{dR}} \longrightarrow \mathcal{O}\mathbb{B}_{\mathrm{dR}} \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \otimes_{\mathcal{O}_X} \Omega_X^n \longrightarrow 0.$$

Finally, we set $\mathcal{O}\mathbb{C} := \mathrm{gr}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Taking the associated graded connection of $\nabla$ on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ equips $\mathcal{O}\mathbb{C}$ with a Higgs field

$$\mathrm{gr}^0 \nabla : \mathcal{O}\mathbb{C} \to \mathcal{O}\mathbb{C} \otimes_{\mathcal{O}_X} \Omega_X^1(-1),$$

where $(-1)$ stands for the $(-1)$st Tate twist.

We review the formulation of the $p$-adic Simpson correspondence by Liu and Zhu. Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\mathrm{ét}}$ of rank $r$. We define

$$\mathcal{H}(\mathbb{L}) = \nu'_*(\hat{\mathbb{L}} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}\mathbb{C}).$$

Then Liu and Zhu proved the following theorem.

THEOREM 3.1 (Rough form of [LZ17, Theorem 2.1]). *$\mathcal{H}(\mathbb{L})$ is a vector bundle on $X_K$ of rank $r$ equipped with a nilpotent Higgs field $\vartheta_{\mathbb{L}}$ and a semilinear action of $\Gamma_k$. The functor $\mathcal{H}$ is a tensor functor from the category of $\mathbb{Q}_p$-local systems on $X_{\mathrm{ét}}$ to the category of nilpotent Higgs bundles on $X_K$. Moreover, $\mathcal{H}$ is compatible with pullback and smooth proper pushforward.*[1]

*Remark* 3.2. For our purpose, we use the $p$-adic Simpson correspondence formulated by Liu and Zhu as their output is a Higgs bundle over $X_K$ with a $\Gamma_k$-action. See [Fal05] and [AGT16] for the $p$-adic Simpson correspondence by Faltings and Abbes–Gros–Tsuji in a more general setting, and see [AB08, AB10] for the one over a pro-étale cover of $X_K$ by Andreatta and Brinon.

To define the arithmetic Sen endomorphism on $\mathcal{H}(\mathbb{L})$ and discuss its properties, let us recall Liu and Zhu's arguments in the proof of Theorem 3.1.

We follow the notation on base changes of adic spaces and rings in [LZ17]. We denote by $\mathbb{T}^n$ the $n$-dimensional rigid analytic torus

$$\mathrm{Spa}(k\langle T_1^{\pm}, \dots, T_n^{\pm}\rangle, \mathcal{O}_k\langle T_1^{\pm}, \dots, T_n^{\pm}\rangle).$$

For $m \geqslant 0$, we set

$$\mathbb{T}^n_m = \mathrm{Spa}(k_m\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle, \mathcal{O}_{k_m}\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle).$$

We denote by $\tilde{\mathbb{T}}^n_\infty$ the affinoid perfectoid $\varprojlim_m \mathbb{T}^n_m$ in $X_{\mathrm{proét}}$.

To study properties of $\mathcal{H}(\mathbb{L})$, we introduce the following base $\mathcal{B}$ for $(X_K)_{\mathrm{ét}}$: objects of $\mathcal{B}$ are the étale maps to $X_K$ that are the base changes of standard étale morphisms $Y \to X_{k'}$ defined over some finite extension $k'$ of $k$ in $K$ where $Y$ is affinoid admitting a toric chart after some finite extension of $k'$. Recall that an étale morphism between adic spaces is called standard étale if it is a composite of rational localizations and finite étale morphisms and that a toric chart means a standard étale morphism to $\mathbb{T}^n$. Morphisms of $\mathcal{B}$ are the base changes of étale morphisms over some finite extension of $k$ in $K$. We equip $\mathcal{B}$ with the induced topology from $(X_K)_{\mathrm{ét}}$. Then the associated topoi $(X_K)^\sim_{\mathrm{ét}}$ and $\mathcal{B}^\sim$ are equivalent [LZ17, Lemma 2.5].

[1] In the smooth proper pushforward case, we need to assume that $\mathbb{L}$ admits a global $\mathbb{Z}_p$-lattice. See [LZ17, Theorem 2.1(v)] and [SW18, Theorem 10.5.1].



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When $Y = \mathrm{Spa}(B, B^+)$ admits a toric chart over $k$, we use the following notation: we set

$$Y_m = \mathrm{Spa}(B_m, B_m^+) := Y \times_{\mathbb{T}^n} \mathbb{T}_m^n.$$

Then $\tilde{Y}_\infty := Y \times_{\mathbb{T}^n} \tilde{\mathbb{T}}_\infty^n$ is the affinoid perfectoid in $Y_{\text{proét}}$ represented by the relative toric tower $(Y_n)$. We denote by $(\hat{B}_\infty, \hat{B}_\infty^+)$ the perfectoid affinoid completed direct limit of the affinoid rings $(B_m, B_m^+)$ and set $\hat{Y}_\infty := \mathrm{Spa}(\hat{B}_\infty, \hat{B}_\infty^+)$, the affinoid perfectoid space associated to $Y_\infty$. We also set $B_{k_m} = B \otimes_k k_m$ as in § 2.1. When $Y$ admits a toric chart over a finite extension of $k$ in $K$, we similarly define these objects using the rigid analytic torus over the field.

Let $Y_{K,m} := \mathrm{Spa}(B_{K,m}, B_{K,m}^+)$ be the base change of $Y_m$ from $k_m$ to $K$ and let $\tilde{Y}_{K,\infty}$ be the affinoid perfectoid represented by the toric tower $(Y_{K,m})$. We denote the associated affinoid perfectoid space by $\hat{Y}_{K,\infty} = \mathrm{Spa}(\hat{B}_{K,m}, \hat{B}_{K,m}^+)$. The cover $\tilde{Y}_{K,\infty}/Y$ is Galois. We denote its Galois group by $\Gamma$. Then $\Gamma$ fits into a splitting exact sequence

$$1 \to \Gamma_{\text{geom}} \to \Gamma \to \Gamma_k \to 1.$$

To prove Theorem 3.1, Liu and Zhu gave a simple description of

$$\mathcal{H}(\mathbb{L})(Y_K) = H^0(X_{\text{proét}}/Y_K, \hat{\mathbb{L}} \otimes \mathcal{O}\mathbb{C})$$

for $(Y = \mathrm{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}$, which we recall now.

Proposition 3.3 [LZ17, Proposition 2.8]. *Put $\mathcal{M} = \hat{\mathbb{L}} \otimes_{\hat{\mathbb{Q}}_p} \hat{\mathcal{O}}_X$. Then there exists a unique finite projective $B_K$-submodule $M_K(Y)$ of $\mathcal{M}(\tilde{Y}_{K,\infty})$, which is stable under $\Gamma$, such that*

(i) *$M_K(Y) \otimes_{B_K} \hat{B}_{K,\infty} = \mathcal{M}(\tilde{Y}_{K,\infty})$, and*
(ii) *the $B_K$-linear representation of $\Gamma_{\text{geom}}$ on $M_K(Y)$ is unipotent.*

*In addition, the module $M_K(Y)$ has the following properties.*

(P1) *There exist some positive integer $j_0$ and some finite projective $B_{k_{j_0}}$-submodule $M(Y)$ of $M_K(Y)$ stable under $\Gamma$ such that $M(Y) \otimes_{B_{k_{j_0}}} B_K = M_K(Y)$. Moreover, the construction of $M(Y)$ is compatible with base change along standard étale morphisms.*

(P2) *The natural map*

$$M_K(Y)^{\Gamma_{\text{geom}}} \to \mathcal{M}(\tilde{Y}_{K,\infty})^{\Gamma_{\text{geom}}}$$

*is an isomorphism.*

Once this proposition is proved, we can describe $\mathcal{H}(\mathbb{L})(Y_K)$ in terms of $M_K(Y)$ as follows: the vanishing theorem on affinoid perfectoid spaces [Sch12, Proposition 7.13] implies the degeneration of the Cartan–Leray spectral sequence to the Galois cover $\{\tilde{Y}_{K,\infty} \to Y_K\}$ with Galois group $\Gamma_{\text{geom}}$, and thus we have

$$H^i(\Gamma_{\text{geom}}, \mathcal{M}(\tilde{Y}_{K,\infty})) \xrightarrow{\cong} H^i(X_{\text{proét}}/Y_K, \mathcal{M}),$$
$$H^i(\Gamma_{\text{geom}}, (\mathcal{M} \otimes \mathcal{O}\mathbb{C})(\tilde{Y}_{K,\infty})) \xrightarrow{\cong} H^i(X_{\text{proét}}/Y_K, \mathcal{M} \otimes \mathcal{O}\mathbb{C}).$$

Moreover, we know that $\mathcal{O}\mathbb{C}|_{\tilde{Y}_{K,\infty}} \cong (\hat{\mathcal{O}}_X|_{\tilde{Y}_{K,\infty}})[V_1, \ldots, V_n]$, where $V_i = t^{-1}\log([T_i^\flat]/T_i)$ for a fixed compatible sequence of $p$-power roots of the coordinate $T_i^\flat = (T_i, T_i^{1/p}, \ldots)$. It follows from these results and a simple argument on the direct limit of sheaves on $X_{\text{proét}}$ that the natural $\Gamma_k$-equivariant map

$$(M_K(Y)[V_1, \ldots, V_n])^{\Gamma_{\text{geom}}} \to \mathcal{H}(\mathbb{L})(Y_K)$$



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is an isomorphism. A simple computation shows that the map $M_K(Y)[V_1,\dots,V_n] \to M_K(Y)$ sending $V_i$ to 0 induces a $\Gamma_k$-equivariant isomorphism

$$(M_K(Y)[V_1,\dots,V_n])^{\Gamma_{\mathrm{geom}}} \xrightarrow{\cong} M_K(Y).$$

Thus we have a $\Gamma_k$-equivariant isomorphism

$$\mathcal{H}(\mathbb{L})(Y_K) \cong M_K(Y).$$

The above discussion is summarized in the following commutative diagram.

$$\begin{tikzcd}
M_K(Y)\otimes_{B_K}\hat{B}_{K,\infty} \arrow[r, hook] & (M_K(Y)\otimes_{B_K}\hat{B}_{K,\infty})[V_1,\dots,V_n] & \\
M_K(Y) \arrow[u, hook] \arrow[r, hook] & M_K(Y)[V_1,\dots,V_n] \arrow[u, hook] \arrow[r, "V_i=0"] & M_K(Y) \\
M_K(Y)^{\Gamma_{\mathrm{geom}}} \arrow[u, hook] \arrow[r, hook] & (M_K(Y)[V_1,\dots,V_n])^{\Gamma_{\mathrm{geom}}} \arrow[u, hook] \arrow[r, "\cong"] & M_K(Y) \arrow[u, equal]
\end{tikzcd}$$

Finally, we recall the Higgs field $\vartheta_{\mathbb{L}}$. This is defined to be

$$\vartheta_{\mathbb{L}} := \nu'_*(\mathrm{gr}\,\nabla : \hat{\mathbb{L}}\otimes\mathcal{O}\mathbb{C} \to \hat{\mathbb{L}}\otimes\mathcal{O}\mathbb{C}\otimes\Omega^1_X(-1))$$

under the identification $\nu'_*(\hat{\mathbb{L}}\otimes\mathcal{O}\mathbb{C}\otimes\Omega^1_X(-1)) \cong \mathcal{H}(\mathbb{L})\otimes\Omega^1_{X/k}(-1)$. Here $\mathcal{H}(\mathbb{L})\otimes\Omega^1_{X/k}(-1)$ denotes the $\mathcal{O}_{X_K}$-module $\mathcal{H}(\mathbb{L})\otimes_{\mathcal{O}_X}\Omega^1_{X/k}(-1) = \mathcal{H}(\mathbb{L})\otimes_{\mathcal{O}_{X_K}}\Omega^1_{X_K/K}(-1)$ equipped with a natural $\Gamma_k$-action.

We have another description under the isomorphism $\mathcal{H}(\mathbb{L})(Y_K)\cong M_K(Y)$, which proves that $\vartheta_{\mathbb{L}}$ is nilpotent. Namely, let $\rho_{\mathrm{geom}}$ denote the action of $\Gamma_{\mathrm{geom}}$ on $M_K(Y)$ and let $\chi_i : \Gamma_{\mathrm{geom}} \cong \mathbb{Z}_p(1)^n \to \mathbb{Z}_p(1)$ denote the composite of the natural identification and projection to the $i$th component. We can take the logarithm of $\rho_{\mathrm{geom}}$ on $M(Y)\subset M_K(Y)$ since the action is unipotent. Suppose the logarithm is written as

$$\log\rho_{\mathrm{geom}} = \sum_{i=1}^n \vartheta_i\otimes\chi_i\otimes t^{-1},$$

where $\vartheta_i \in \mathrm{End}(M(Y))$. Then $\vartheta_i$ can be regarded as an endomorphism on $M_K(Y)$ by extension of scalars and we define

$$\vartheta_{M_K(Y)} := \sum_{i=1}^n \vartheta_i\otimes d\log T_i\otimes t^{-1} = \sum_{i=1}^n \vartheta_i\otimes\frac{dT_i}{T_i}\otimes t^{-1} \in \mathrm{End}(M_K(Y))\otimes_B\Omega^1_{B/k'}(-1). \quad (3.1)$$

We can check $\vartheta_{M_K(Y)}\wedge\vartheta_{M_K(Y)} = 0$ and this defines a Higgs field on $M_K(Y)$. It turns out that $\vartheta_{\mathbb{L}}(Y_K) = \vartheta_{M_K(Y)}$ under the $\Gamma_k$-equivariant isomorphism $\mathcal{H}(\mathbb{L})(Y_K)\cong M_K(Y)$. See [LZ17, § 2] for the detail.

### 3.2 Definition and properties of the arithmetic Sen endomorphism

We will define the arithmetic Sen endomorphism $\phi_{\mathbb{L}} \in \mathrm{End}\,\mathcal{H}(\mathbb{L})$. Let $\mathcal{B}_{\mathbb{L}}$ be the refinement of the base $\mathcal{B}$ for $(X_K)_{\text{ét}}$ whose objects consist of $(Y = \mathrm{Spa}(B,B^+)\to X_{k'})\in\mathcal{B}$ such that $\mathcal{H}(\mathbb{L})(Y_K)$ is a finite free $B_K$-module.

For $(Y = \mathrm{Spa}(B,B^+)\to X_{k'})\in\mathcal{B}_{\mathbb{L}}$, $\mathcal{H}(\mathbb{L})(Y_K)$ is a $B_K$-representation of $\Gamma_{k'} := \mathrm{Gal}(K/k')$ in the sense of Definition 2.2. Thus Proposition 2.20 and Definition 2.23 equip $\mathcal{H}(\mathbb{L})(Y_K)$ with the $B_K$-linear endomorphism

$$\phi_{\mathcal{H}(\mathbb{L})(Y_K)} : \mathcal{H}(\mathbb{L})(Y_K) \to \mathcal{H}(\mathbb{L})(Y_K).$$

Lemma-Definition 3.4. *The assignment of endomorphisms*

$$\mathcal{B}_{\mathbb{L}} \ni (Y = \mathrm{Spa}(B, B^+) \to X_{k'}) \longmapsto \phi_{\mathcal{H}(\mathbb{L})(Y_K)} \in \mathrm{End}_{B_K} \mathcal{H}(\mathbb{L})(Y_K)$$

*defines an endomorphism $\phi_{\mathbb{L}}$ of the vector bundle $\mathcal{H}_{\mathbb{L}}$ on $(X_K)_{\text{ét}}$. We call $\phi_{\mathbb{L}}$ the arithmetic Sen endomorphism of $\mathbb{L}$.*

*Proof.* We need to check the compatibility of $\phi_{\mathbb{L},Y_K}$ via the pullback $Y''_K \to Y_K$ for

$$(Y = \mathrm{Spa}(B, B^+) \to X_{k'}), (Y'' = \mathrm{Spa}(B'', B''^+) \to X_{k''}) \in \mathcal{B}_{\mathbb{L}}.$$

For this it suffices to prove that

$$\mathcal{H}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes_{B_\infty} B''_\infty \cong \mathcal{H}(\mathbb{L})(Y''_K)_{\mathrm{fin}}$$

as $B''_\infty$-representation of $\mathrm{Gal}(k_\infty/k'')$, where $B_\infty$ and $B''_\infty$ are defined as in § 2.1.

Since $\mathcal{H}(\mathbb{L})$ is a vector bundle on $X_K$, we have the natural isomorphisms

$$\begin{aligned}(\mathcal{H}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes_{B_\infty} B''_\infty) \otimes_{B''_\infty} B''_K &\cong (\mathcal{H}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes_{B_\infty} B_K) \otimes_{B_K} B''_K \\ &\cong \mathcal{H}(\mathbb{L})(Y_K) \otimes_{B_K} B''_K \cong \mathcal{H}(\mathbb{L})(Y''_K).\end{aligned}$$

On the other hand, we see from definition $\mathcal{H}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes_{B_\infty} B''_\infty \subset \mathcal{H}(\mathbb{L})(Y''_K)_{\mathrm{fin}}$. Hence the lemma follows from the faithful flatness of $B''_\infty \to B''_K$. □

Proposition 3.5. *The following diagram commutes.*

$$\begin{tikzcd} \mathcal{H}(\mathbb{L}) \arrow[r, "\vartheta_{\mathbb{L}}"] \arrow[d, "\phi_{\mathbb{L}}"'] & \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \arrow[d, "\phi_{\mathbb{L}} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id}"] \\ \mathcal{H}(\mathbb{L}) \arrow[r, "\vartheta_{\mathbb{L}}"] & \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \end{tikzcd}$$

*In particular, the endomorphisms $\phi_{\mathbb{L}} \otimes \mathrm{id} - i(\mathrm{id} \otimes \mathrm{id})$ on $\mathcal{H}(\mathbb{L}) \otimes \Omega^i_{X/k}(-i)$ give rise to an endomorphism on the complex of $\mathcal{O}_{X_K}$-modules on $X_K$*

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\vartheta_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \xrightarrow{\vartheta_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/k}(-2) \longrightarrow \cdots$$

*induced by the Higgs field.*

*Proof.* It is enough to check the commutativity of the diagram evaluated at $Y_K$ for each $(Y = \mathrm{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}_{\mathbb{L}}$. In this setting, we can use the identification

$$(\mathcal{H}(\mathbb{L})(Y_K), \vartheta_{\mathbb{L}}(Y_K), \phi_{\mathbb{L}}(Y_K)) \cong (M_K(Y), \vartheta_{M_K(Y)}, \phi_{M_K(Y)}).$$

So it suffices to show the commutativity of the following diagram.

$$\begin{tikzcd} M_K(Y) \arrow[r, "\vartheta_{M_K(Y)}"] \arrow[d, "\phi_{M_K(Y)}"'] & M_K(Y) \otimes_B \Omega^1_{B/k'}(-1) \arrow[d, "\phi_{M_K(Y)} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id}"] \\ M_K(Y) \arrow[r, "\vartheta_{M_K(Y)}"] & M_K(Y) \otimes_B \Omega^1_{B/k'}(-1) \end{tikzcd}$$

Moreover, since $M(Y) \otimes_{B_{k_{j0}}} B_K = M_K(Y)$, we only need to check the commutativity on $M(Y) \subset M_K(Y)$.

We use the notation in (3.1). Then we have

$$\vartheta_{M_K(Y)} \circ \phi_{M_K(Y)} = \sum_{i=1}^{n} (\vartheta_i \circ \phi_{M_K(Y)}) \otimes \frac{dT_i}{T_i} \otimes t^{-1}$$

and

$$(\phi_{M_K(Y)} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id}) \circ \vartheta_{M_K(Y)} = \sum_{i=1}^{n} (\phi_{M_K(Y)} \circ \vartheta_i - \vartheta_i) \otimes \frac{dT_i}{T_i} \otimes t^{-1}.$$

Thus we need to show that $[\phi_{M_K(Y)}, \vartheta_i] = \vartheta_i$ for each $i$. To see this, take a topological generator $\gamma_i$ of the $i$th component of $\Gamma_{\mathrm{geom}} \cong \mathbb{Z}_p(1)^n$. Let $\rho_{\mathrm{geom}}$ denote the action of $\Gamma_{\mathrm{geom}}$ on $M_K(Y)$ and write $\log \rho_{\mathrm{geom}} = \sum_{i=1}^{n} \vartheta_i \otimes \chi_i \otimes t^{-1}$ as before. Since $\gamma\gamma_i\gamma^{-1} = \gamma_i^{\chi(\gamma)}$ for $\gamma \in \Gamma_k$, we have

$$\gamma(\log \rho_{\mathrm{geom}}(\gamma_i))\gamma^{-1} = \log \rho_{\mathrm{geom}}(\gamma\gamma_i\gamma^{-1}) = \chi(\gamma) \log \rho_{\mathrm{geom}}(\gamma_i).$$

Hence $\gamma\vartheta_i = \chi(\gamma)\vartheta_i\gamma$ for $\gamma \in \Gamma_k$.

For $m \in M(Y)$, we compute

$$\begin{aligned}
\phi_{M_K(Y)}\vartheta_i m &= \lim_{j\to\infty} \frac{1}{\log \chi(\gamma)} \frac{\gamma^{p^j}\vartheta_i m - \vartheta_i m}{p^j} \\
&= \lim_{j\to\infty} \frac{1}{\log \chi(\gamma)} \frac{(\chi(\gamma)^{p^j} - 1)\vartheta_i\gamma^{p^j} m + \vartheta_i(\gamma^{p^j} m - m)}{p^j} \\
&= \vartheta_i m + \vartheta_i \phi_{M_K(Y)} m.
\end{aligned}$$

Hence $[\phi_{M_K(Y)}, \vartheta_i] = \vartheta_i$. □

*Remark* 3.6. Brinon generalized Sen's theory to the case of $p$-adic fields with imperfect residue fields in [Bri03]. Analogues of $\phi_{\mathbb{L}}$ and $\vartheta_i$ have already appeared in his work.

We discuss properties of the arithmetic Sen endomorphism along the lines of Theorem 3.1 (i.e. [LZ17, Theorem 2.1]).

Theorem 3.7. (i) *There are canonical isomorphisms*

$$(\mathcal{H}(\mathbb{L}_1 \otimes \mathbb{L}_2), \vartheta_{\mathbb{L}_1\otimes\mathbb{L}_2}, \phi_{\mathbb{L}_1\otimes\mathbb{L}_2}) \cong (\mathcal{H}(\mathbb{L}_1) \otimes \mathcal{H}(\mathbb{L}_2), \vartheta_{\mathbb{L}_1} \otimes \mathrm{id} + \mathrm{id} \otimes \vartheta_{\mathbb{L}_2}, \phi_{\mathbb{L}_1} \otimes \mathrm{id} + \mathrm{id} \otimes \phi_{\mathbb{L}_2})$$

*and*

$$(\mathcal{H}(\mathbb{L}^\vee), \vartheta_{(\mathbb{L}^\vee)}, \phi_{(\mathbb{L}^\vee)}) \cong (\mathcal{H}(\mathbb{L})^\vee, (\vartheta_{\mathbb{L}})^\vee, (\phi_{\mathbb{L}})^\vee).$$

(ii) *Let $f: Y \to X$ be a morphism between smooth rigid analytic varieties over $k$ and $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. Then there is a canonical isomorphism*

$$f^*(\mathcal{H}(\mathbb{L}), \vartheta_{\mathbb{L}}, \phi_{\mathbb{L}}) \cong (f^*\mathcal{H}(\mathbb{L}), \vartheta_{f^*\mathbb{L}}, \phi_{f^*\mathbb{L}}).$$

*Proof.* Part (i) follows from [LZ17, Theorem 2.1(iv)] and Lemma 2.22. Part (ii) follows from [LZ17, Theorem 2.1(iii)] and Proposition 2.20 (functoriality of $\phi_W$). □

By construction, we also have the following in the case of points.

Proposition 3.8. *If $X$ is a point, then $\phi_{\mathbb{L}}$ coincides with the Sen endomorphism attached to the Galois representation $\mathbb{L}$.*

For pushforwards, we have the following theorem (the notation is explained after the statement).

Theorem 3.9. *Let $f : X \to Y$ be a smooth proper morphism between smooth rigid analytic varieties over $k$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Then we have*

$$(\mathcal{H}(R^i f_*\mathbb{L}), \vartheta_{R^i f_*\mathbb{L}}) \cong R^i f_{\text{Higgs},*}(\mathcal{H}(\mathbb{L}) \otimes \Omega^{\bullet}_{X/Y}(-\bullet), \bar{\vartheta_{\mathbb{L}}}).$$

*Moreover, under this isomorphism, we have*

$$\phi_{R^i f_*\mathbb{L}} = R^i f_{K,\text{ét},*}(\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\,(\mathrm{id} \otimes \mathrm{id})).$$

Let us explain the notation in the theorem. Recall the complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\vartheta_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \xrightarrow{\vartheta_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/k}(-2) \longrightarrow \cdots.$$

This has an $\mathcal{O}_{X_K}$-linear endomorphism $\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\,(\mathrm{id} \otimes \mathrm{id})$ by Proposition 3.5. The complex yields a complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\bar{\vartheta}_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\bar{\vartheta}_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots$$

by composing with the projection $\Omega^i_{X/k} \to \Omega^i_{X/Y}$. The new complex has an induced $\mathcal{O}_{X_K}$-linear endomorphism, which we still denote by $\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\,(\mathrm{id} \otimes \mathrm{id})$.

We denote by $f_K : X_K \to Y_K$ the base change of $f$. Then $R^i f_{\text{Higgs},*}$ is the $i$th derived pushforward of the complex with the Higgs field. In particular, $R^i f_{\text{Higgs},*}(\mathcal{H}(\mathbb{L}) \otimes \Omega^{\bullet}_{X/Y}(-\bullet), \bar{\vartheta_{\mathbb{L}}})$ is the $\mathcal{O}_{X_K,\text{ét}}$-module $R^i f_{K,\text{ét},*}(\mathcal{H}(\mathbb{L}) \otimes \Omega^{\bullet}_{X/Y}(-\bullet))$ together with a Higgs field.

*Proof.* The first part is [LZ17, Theorem 2.1(v)] (see Theorem 3.1). Note that [LZ17, Theorem 2.1(v)] has an additional assumption that $R^i f_*\mathbb{L}$ is a $\mathbb{Z}_p$-local system on $Y_{\text{ét}}$ for every $i$, but this is always the case; to see this, it suffices to check that $(R^i f_*\mathbb{L})|_{Y_{\hat{\bar{k}}},\text{ét}}$ is a $\mathbb{Z}_p$-local system, which follows from [SW18, Theorem 10.5.1]. So we will prove the statement on arithmetic Sen endomorphisms.

Since the statement is local on $Y$, we may assume that $Y$ is an affinoid $\mathrm{Spa}(A, A^+)$ and that $\mathcal{H}(R^i f_*\mathbb{L})$ is a globally free vector bundle on $Y_K$. So $\mathcal{H}(R^i f_*\mathbb{L})$ is associated to a finite free $A_K$-module (say $V$). Then $V$ is an $A_K$-representation of $\Gamma_k$ and the endomorphism $\phi_{R^i f_*\mathbb{L}}$ is associated to $\phi_V$.

Since $X$ is quasi-compact, there exists a finite affinoid open cover $X = \bigcup_{i\in I} U^{(i)}$ with $U^{(i)} = \mathrm{Spa}(B^{(i)}, B^{(i),+})$ such that $\mathcal{H}(\mathbb{L})|_{U^{(i)}_K}$ is a globally finite free vector bundle for each $i$. So $\mathcal{H}(\mathbb{L})|_{U^{(i)}_K}$ gives rise to a $B^{(i)}_K$-representation of $\Gamma_k$ and the latter is defined over $B^{(i)}_{k_m}$ for a sufficiently large $m$ (cf. the proof of Theorem 2.9). Since the same holds for $\mathcal{H}(\mathbb{L})|_{U^{(i)}_K \cap U^{(j)}_K}$, there exists a large integer $m$ such that the complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\bar{\vartheta}_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\bar{\vartheta}_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots$$

with the $\Gamma_k$-action and the endomorphism $\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\,(\mathrm{id} \otimes \mathrm{id})$ descends to a complex of $\mathcal{O}_{X_{k_m}}$-modules

$$\mathcal{H}(\mathbb{L})_{k_m} \xrightarrow{\bar{\vartheta}_{\mathbb{L}}} \mathcal{H}(\mathbb{L})_{k_m} \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\bar{\vartheta}_{\mathbb{L}}} \mathcal{H}(\mathbb{L})_{k_m} \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots$$



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on $X_{k_m}$ equipped with $\Gamma$-action and an endomorphism $\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet(\mathrm{id} \otimes \mathrm{id})$ such that $\mathcal{H}(\mathbb{L})_{k_m}|_{U_{k_m}^{(i)}}$ is a globally finite free vector bundle for each $i$. We denote by $\mathcal{F}^\bullet$ the complex on $X_{k_m}$ and by $\phi^\bullet$ the descended endomorphism.

Let $f_{k_m} : X_{k_m} \to Y_{k_m}$ denote the base change of $f$. Set

$$\mathcal{H}_{Y,k_m} := R^i f_{k_m,\text{ét}} \mathcal{F}^\bullet.$$

Since $f_{k_m}$ is proper, this is a coherent $\mathcal{O}_{Y_{k_m}}$-module by Kiehl's finiteness theorem.[2] We have

$$\mathcal{H}_{Y,k_m}|_{Y_K} = (R^i f_{k_m,\text{ét}} \mathcal{F}^\bullet)|_{Y_K} = R^i f_{K,\text{ét}}(\mathcal{F}^\bullet|_{X_K}) = R^i f_{K,\text{ét}}(\mathcal{H}(\mathbb{L}) \otimes \Omega^\bullet_{X/Y}(-\bullet)),$$

and this is isomorphic to $\mathcal{H}(R^i f_* \mathbb{L})$ by the first assertion. Thus (after increasing $m$) $\mathcal{H}_{Y,k_m}$ is globally finite free and associated to a finite free $A_{k_m}$-module (say $V_{k_m}$) with $\Gamma_k$-action satisfying $V_{k_m} \otimes_{A_{k_m}} A_K = V$. By construction, $V_{k_m}$ is contained in $V_{\text{fin}}$ and the $A_K$-linear endomorphism $\phi_V$ on $V$ is uniquely characterized by the following property: for each $v \in V_{k_m}$, there exists an open subgroup $\Gamma'_k \subset \Gamma_k$ such that

$$\exp(\log \chi(\gamma)\phi_V)v = \gamma v$$

for all $\gamma \in \Gamma'_k$.

We will show that $R^i f_{K,\text{ét},*}(\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\, (\mathrm{id} \otimes \mathrm{id}))$ defines an $A_K$-linear endomorphism on $V$ with the same property. To see this, we compute $V_{k_m}$ via the Čech-to-derived functor spectral sequence. Note that

$$V_{k_m} = \Gamma(Y_{k_m,\text{ét}}, \mathcal{H}_{Y,k_m}) = R^i\Gamma(X_{k_m,\text{ét}}, \mathcal{F}^\bullet)$$

by definition.

Let us briefly recall the Čech-to-derived functor spectral sequence. Set $\mathcal{U} := \{U_{k_m}^{(i)}\}_{i\in I}$. For $i_0, \ldots, i_a \in I$, we denote by $U_{k_m}^{(i_0\cdots i_a)}$ the affinoid open $U_{k_m}^{(i_0)} \cap \cdots \cap U_{k_m}^{(i_a)}$. Consider the Čech double complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ associated to the complex $\mathcal{F}^\bullet$; this is defined by

$$\check{C}^a(\mathcal{U}, \mathcal{F}^b) := \prod_{i_0,\ldots,i_a \in I} \mathcal{F}^b(U_{k_m}^{(i_0\cdots i_a)}).$$

Let $\underline{H}^b$ be the $b$th right derived functor of the forgetful functor from the category of abelian sheaves on $X_{k_m,\text{ét}}$ to the category of abelian presheaves on $X_{k_m,\text{ét}}$; for an abelian sheaf $\mathcal{G}$, $\underline{H}^b(\mathcal{G})$ associates to $(U \to X_{k_m})$ the abelian group $H^b(U, \mathcal{G})$. Then the Čech-to-derived functor spectral sequence is a spectral sequence with

$$E_2^{a,b} = H^a(\mathrm{Tot}(\check{C}^\bullet(\mathcal{U}, \underline{H}^b(\mathcal{F}^\bullet))))$$

converging to $R^{a+b}\Gamma(X_{k_m,\text{ét}}, \mathcal{F}^\bullet)$. Moreover, this is functorial in $\mathcal{F}^\bullet$.

In our case, $\mathcal{F}^\bullet$ consists of coherent $\mathcal{O}_{k_m}$-modules and $U_{k_m}^{(i_0\cdots i_a)}$ are all affinoid. So $\underline{H}^b(\mathcal{F}^c)(U_{k_m}^{(i_0\cdots i_a)}) = 0$ for each $b > 0$ and any $a$ and $c$ by Kiehl's theorem. Thus the spectral sequence yields an isomorphism

$$H^i(\mathrm{Tot}(\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \xrightarrow{\cong} R^i\Gamma(X_{k_m,\text{ét}}, \mathcal{F}^\bullet) = V_{k_m}.$$

Moreover, this isomorphism is $\Gamma_k$-equivariant as the construction is functorial in $\mathcal{F}^\bullet$.

[2] For a coherent $\mathcal{O}_{k_m}$-module $\mathcal{F}$, we have $(R^i f_{k_m} \mathcal{F})_{\text{ét}} = R^i f_{k_m,\text{ét}} \mathcal{F}_{\text{ét}}$ [Sch13, Proposition 9.2]. So we simply write $\mathcal{F}$ for the sheaf $\mathcal{F}_{\text{ét}}$ on $X_{k_m,\text{ét}}$.

Let us unwind the definition of $\check{C}^a(\mathcal{U},\mathcal{F}^b)$:

$$\check{C}^a(\mathcal{U},\mathcal{F}^b) = \prod_{i_0,\dots,i_a\in I} \Gamma(U_{k_m}^{(i_0\cdots i_a)}, \mathcal{H}(\mathbb{L})_{k_m}\otimes \Omega^b_{X/Y}(-b)).$$

Set

$$W^{(i_0\cdots i_a),b} := \Gamma(U_K^{(i_0\cdots i_a)}, \mathcal{H}(\mathbb{L})\otimes \Omega^b_{X/Y}(-b))$$

and

$$W_{k_m}^{(i_0\cdots i_a),b} := \Gamma(U_{k_m}^{(i_0\cdots i_a)}, \mathcal{H}(\mathbb{L})_{k_m}\otimes \Omega^b_{X/Y}(-b)).$$

They have a natural $\Gamma_k$-action, and $W_{k_m}^{(i_0\cdots i_a),b}$ is contained in $(W^{(i_0\cdots i_a),b})_{\mathrm{fin}}$. In particular, the restriction of $\phi_{W^{(i_0\cdots i_a),b}}$ to $W_{k_m}^{(i_0\cdots i_a),b}$ satisfies the following property: for each $w\in W_{k_m}^{(i_0\cdots i_a),b}$, there exists an open subgroup $\Gamma'_k\subset\Gamma_k$ such that

$$\exp(\log\chi(\gamma)\phi_{W^{(i_0\cdots i_a),b}})w=\gamma w$$

for all $\gamma\in\Gamma'_k$.

It follows from our construction that under the isomorphism $H^i(\mathrm{Tot}(\check{C}^\bullet(\mathcal{U},\mathcal{F}^\bullet)))\cong R^i\Gamma(X_{k_m,\text{ét}},\mathcal{F}^\bullet)=V_{k_m}$, the endomorphism $R^if_{K,\text{ét},*}(\phi_{\mathbb{L}}\otimes\mathrm{id}-\bullet\,(\mathrm{id}\otimes\mathrm{id}))|_{V_{k_m}}=R^if_{k_m,\text{ét},*}\phi^\bullet$ corresponds to

$$H^i\biggl(\mathrm{Tot}\biggl(\prod_{i_0,\dots,i_a\in I}\phi_{W^{(i_0\cdots i_a),b}}\biggr)\biggr).$$

Since differentials in the complex $\mathrm{Tot}(\check{C}^\bullet(\mathcal{U},\mathcal{F}^\bullet))$ are all $\Gamma_k$-equivariant, we see that $H^i(\mathrm{Tot}(\prod_{i_0,\dots,i_a\in I}\phi_{W^{(i_0\cdots i_a),b}}))$ satisfies the above-mentioned characterizing property of $\phi_V$. This completes the proof. □

## 4. Constancy of generalized Hodge–Tate weights

In this section, we prove the multiset of eigenvalues of $\phi_{\mathbb{L}}$ is constant on $X_K$ (Theorem 4.8). For this we give a description of $\phi_{\mathbb{L}}$ as the residue of a formal connection in § 4.1. Then the constancy is proved by the theory of formal connections developed in § 4.2.

### 4.1 The decompletion of the geometric Riemann–Hilbert correspondence

We review the geometric Riemann–Hilbert correspondence by Liu and Zhu and discuss its decompletion.

Keep the notation in § 3. Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$ of rank $r$. Following [LZ17], we define

$$\mathcal{RH}(\mathbb{L})=\nu'_*(\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{B}_{\mathrm{dR}}).$$

In order to state their theorem, let us recall a ringed space $\mathcal{X}$ introduced in [LZ17, § 3.1]. Let $L^+_{\mathrm{dR}}$ denote the de Rham period ring $\mathbb{B}^+_{\mathrm{dR}}(K,\mathcal{O}_K)$ as before ([LZ17] uses $B^+_{\mathrm{dR}}$ but we prefer to use $L^+_{\mathrm{dR}}$). Define a sheaf $\mathcal{O}_X\,\hat{\otimes}\,(L^+_{\mathrm{dR}}/t^i)$ on $X_{K,\text{ét}}$ by assigning

$$(Y=\mathrm{Spa}(B,B^+)\to X_{k'})\in\mathcal{B}\longmapsto B\,\hat{\otimes}_{k'}\,(L^+_{\mathrm{dR}}/t^i).$$

This defines a sheaf by the Tate acyclicity theorem. We also set

$$\mathcal{O}_X\,\hat{\otimes}\,L^+_{\mathrm{dR}}=\varprojlim_i\mathcal{O}_X\,\hat{\otimes}\,(L^+_{\mathrm{dR}}/t^i)$$

and

$$\mathcal{O}_X \hat{\otimes} L_{\mathrm{dR}} = (\mathcal{O}_X \hat{\otimes} L_{\mathrm{dR}}^+)[t^{-1}].$$

We denote the ringed space $(X_K, \mathcal{O}_X \hat{\otimes} L_{\mathrm{dR}})$ by $\mathcal{X}$. We have a natural base change functor $\mathcal{E} \mapsto \mathcal{E} \hat{\otimes} L_{\mathrm{dR}}$ from the category of vector bundles on $X$ to the category of vector bundles on $\mathcal{X}$. We set

$$\Omega^1_{\mathcal{X}/L_{\mathrm{dR}}} = \Omega^1_{X/k} \hat{\otimes} L_{\mathrm{dR}}.$$

Theorem 4.1 [LZ17, Theorem 3.8].

(i) *$\mathcal{RH}(\mathbb{L})$ is a filtered vector bundle on $\mathcal{X}$ of rank $r$ equipped with an integrable connection*

$$\nabla_{\mathbb{L}} : \mathcal{RH}(\mathbb{L}) \longrightarrow \mathcal{RH}(\mathbb{L}) \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/L_{\mathrm{dR}}}$$

*that satisfies the Griffiths transversality. Moreover,* $\mathrm{Gal}(K/k)$ *acts on* $\mathcal{RH}(\mathbb{L})$ *semilinearly, and the action preserves the filtration and commutes with* $\nabla_{\mathbb{L}}$*.*

(ii) *There is a canonical isomorphism*

$$(\mathrm{gr}^0 \mathcal{RH}(\mathbb{L}), \mathrm{gr}^0(\nabla_{\mathbb{L}})) \cong (\mathcal{H}(\mathbb{L}), \vartheta_{\mathbb{L}}).$$

We want to consider a decompletion of $\mathcal{RH}(\mathbb{L})$. Here we only develop an *ad hoc* local theory that is sufficient for our purpose.

Take $(Y = \mathrm{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}_{\mathbb{L}}$ and consider $\mathrm{Fil}^0 \mathcal{RH}(\mathbb{L})(Y_K)$. Since $\mathrm{gr}^0 \mathcal{RH}(\mathbb{L})(Y_K)$ is a finite free $B_K$-module by the definition of $\mathcal{B}_{\mathbb{L}}$, the $B \hat{\otimes}_{k'} L^+_{\mathrm{dR}}$-module $\mathrm{Fil}^0 \mathcal{RH}(\mathbb{L})(Y_K)$ is also finite free. Thus $\mathrm{Fil}^0 \mathcal{RH}(\mathbb{L})(Y_K)$ is a $B \hat{\otimes}_{k'} L^+_{\mathrm{dR}}$-representation of $\mathrm{Gal}(K/k')$, and $\mathcal{RH}(\mathbb{L})(Y_K) = (\mathrm{Fil}^0 \mathcal{RH}(\mathbb{L})(Y_K))[t^{-1}]$.

Definition 2.28 yields the $B_\infty((t))$-module $\mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}$ and the $B_\infty$-linear endomorphism

$$\phi_{\mathrm{dR}, \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}} : \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}} \to \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}.$$

For simplicity, we denote $\phi_{\mathrm{dR}, \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}}$ by $\phi_{\mathrm{dR}, \mathbb{L}, Y_K}$. It satisfies

$$\phi_{\mathrm{dR}, \mathbb{L}, Y_K}(\alpha m) = t\partial_t(\alpha) m + \alpha \phi_{\mathrm{dR}, \mathbb{L}, Y_K}(m)$$

for every $\alpha \in B_\infty((t))$ and $m \in \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}$. Note that $\nabla_{\mathbb{L}}$ is $\mathrm{Gal}(K/k)$-equivariant. Hence under the identification

$$\mathcal{RH}(\mathbb{L}) \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega^1_{\mathcal{X}/L_{\mathrm{dR}}} \cong \mathcal{RH}(\mathbb{L}) \otimes_{\mathcal{O}_X} \Omega^1_{X/k},$$

we have

$$\nabla_{\mathbb{L}, Y_K}(\mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}) \subset (\mathcal{RH}(\mathbb{L})(Y_K) \otimes \Omega^1_{B/k'})_{\mathrm{fin}} = \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes \Omega^1_{B/k'}.$$

Proposition 4.2. *The following diagram commutes.*

$$\begin{tikzcd}
\mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}} \arrow[r, "\nabla_{\mathbb{L}, Y_K}"] \arrow[d, "\phi_{\mathrm{dR}, \mathbb{L}, Y_K}"'] & \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes_B \Omega^1_{B/k'} \arrow[d, "\phi_{\mathrm{dR}, \mathbb{L}, Y_K} \otimes \mathrm{id}"] \\
\mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}} \arrow[r, "\nabla_{\mathbb{L}, Y_K}"] & \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}} \otimes_B \Omega^1_{B/k'}
\end{tikzcd}$$

*Moreover, we have*

$$\mathrm{Res}_{\mathrm{Fil}^0 \mathcal{RH}(\mathbb{L})(Y_K)_{\mathrm{fin}}} \phi_{\mathrm{dR}, \mathbb{L}, Y_K} = \phi_{\mathbb{L}, Y_K}.$$

*Proof.* The commutativity of the diagram follows from the fact that $\nabla_{\mathbb{L}}$ is $\mathrm{Gal}(K/k)$-equivariant. The second assertion is a consequence of Theorem 4.1(ii) (cf. Definition 2.28). □

*Remark* 4.3. In [AB10], Andreatta and Brinon developed a Fontaine-type decompletion theory in the relative setting. Roughly speaking, they associated to a local system on $X$ a formal connection over the pro-étale cover $\widetilde{X}_{K,\infty}$ over $X_K$ when $X$ is an affine scheme admitting a toric chart.

### 4.2 Theory of formal connections

To study $\phi_{\mathrm{dR},\mathbb{L},Y_K}$ in the previous subsection, we develop a theory of formal connections. We work on the following general setting: let $R$ be an integral domain of characteristic 0 (e.g. $R = B_\infty$ in the previous subsection) and fix an algebraic closure of the fraction field of $R$. Consider the ring of Laurent series $R((t))$ and define the $R$-linear derivation $d_0 : R((t)) \to R((t))$ by

$$d_0\left(\sum_{j\in\mathbb{Z}} a_j t^j\right) = \sum_{j\in\mathbb{Z}} j a_j t^{j-1}.$$

Let $M$ be a finite free $R((t))$-module of rank $r$ and let $D_0 : M \to M$ be an $R$-linear map which satisfies the Leibniz rule

$$D_0(\alpha m) = \alpha D_0(m) + d_0(\alpha) m \quad (\alpha \in R((t)), m \in M).$$

Definition 4.4. A *$tD_0$-stable lattice* of $M$ is a finite free $R[[t]]$-submodule $\Lambda$ of $M$ that satisfies

$$\Lambda \otimes_{R[[t]]} R((t)) = M \quad \text{and} \quad tD_0(\Lambda) \subset \Lambda.$$

For a $tD_0$-stable lattice $\Lambda$ of $M$, we have $tD_0(t\Lambda) \subset t\Lambda$ by the Leibniz rule. Thus $tD_0 : \Lambda \to \Lambda$ induces an $R$-linear endomorphism on $\Lambda/t\Lambda$. We denote this endomorphism by $\mathrm{Res}_\Lambda D_0$. Since $\Lambda/t\Lambda$ is a finite free $R$-module of rank $r$, the endomorphism $\mathrm{Res}_\Lambda D_0$ has $r$ eigenvalues (counted with multiplicity) in the algebraic closure of the fraction field of $R$.

The following is known for $tD_0$-stable lattices.

Theorem 4.5. *Assume that $R$ is an algebraically closed field.*

(i) *There exists a finite subset $\mathcal{A}$ of $R$ such that the submodule*

$$\Lambda_{\mathcal{A}} := \bigoplus_{\alpha\in\mathcal{A}} \mathrm{Ker}(tD_0 - \alpha)^r \otimes_R R[[t]]$$

*is a $tD_0$-stable lattice of $M$. In particular, the eigenvalues of $\mathrm{Res}_{\Lambda_{\mathcal{A}}} D_0$ lie in $\mathcal{A}$.*

(ii) *For any $tD_0$-stable lattices $\Lambda$ and $\Lambda'$ of $M$, the eigenvalues of $\mathrm{Res}_\Lambda D_0$ and those of $\mathrm{Res}_{\Lambda'} D_0$ differ by integers. Namely, for each eigenvalue $\alpha$ of $\mathrm{Res}_\Lambda D_0$, there exists an eigenvalue $\alpha'$ of $\mathrm{Res}_{\Lambda'} D_0$ such that $\alpha - \alpha' \in \mathbb{Z}$.*

See [DGS94, III.8 and V. Lemma 2.4] and [AB01, ch. 1, Proposition 3.2.2] for details.

We now turn to the following multivariable situation: Let $R$ be an integral domain of characteristic 0 as before. Suppose that $R$ is equipped with pairwise commuting derivations $d_1, \ldots, d_n$; this means that for each $i = 1, \ldots, n$, the map $d_i : R \to R$ is additive and satisfies the Leibniz rule

$$d_i(ab) = d_i(a)b + a d_i(b) \quad (a, b \in R),$$



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and $d_i \circ d_j = d_j \circ d_i$ for each $i$ and $j$. Since $R$ is an integral domain of characteristic 0, the derivations $d_1, \dots, d_n$ extend uniquely over the algebraic closure of the fraction field of $R$.

For each $i = 1, \dots, n$, we extend $d_i : R \to R$ to an additive map $d_i : R((t)) \to R((t))$ by

$$d_i\left(\sum_{j\in\mathbb{Z}} a_j t^j\right) = \sum_{j\in\mathbb{Z}} d_i(a_j)t^j.$$

Then endomorphisms $d_0, d_1, \dots, d_n$ on $R((t))$ commute with each other. Moreover, $d_1, \dots, d_n$ commute with $td_0$.

Let $M$ be a finite free $R((t))$-module of rank $r$ together with pairwise commuting additive endomorphisms $D_0, D_1, \dots, D_n : M \to M$ satisfying the Leibniz rule

$$D_i(\alpha m) = \alpha D_i(m) + d_i(\alpha)m \quad (\alpha \in R((t)), m \in M, 0 \leqslant i \leqslant n).$$

Note that $D_0$ is $R$-linear and $D_1, \dots, D_n$ commute with $tD_0$.

The following proposition is the key to the constancy of generalized Hodge–Tate weights.

Proposition 4.6. *With the notation as above, let $\Lambda$ be a $tD_0$-stable lattice of $M$. Then each eigenvalue $\alpha$ of $\mathrm{Res}_\Lambda D_0$ in the algebraic closure of the fraction field of $R$ satisfies*

$$d_1(\alpha) = \cdots = d_n(\alpha) = 0.$$

*Proof.* By extending scalars from $R$ to the algebraic closure of its fraction field, we may assume that $R$ is an algebraically closed field. By Theorem 4.5(i), there exists a finite subset $\mathcal{A}$ of $R$ such that the submodule

$$\Lambda_{\mathcal{A}} := \bigoplus_{\alpha\in\mathcal{A}} \mathrm{Ker}(tD_0 - \alpha)^r \otimes_R R[[t]]$$

is a $tD_0$-stable lattice of $M$.

By Theorem 4.5(ii), the eigenvalues of $\mathrm{Res}_\Lambda D_0$ and those of $\mathrm{Res}_{\Lambda_{\mathcal{A}}} D_0$ differ by integers. Since every integer $a$ satisfies $d_1(a) = \cdots = d_n(a) = 0$, it suffices to treat the case where $\Lambda = \Lambda_{\mathcal{A}}$.

Lemma 4.7. *The finite free $R[[t]]$-submodule $\Lambda_{\mathcal{A}}$ is stable under $D_1, \dots, D_n$.*

Note that Lemma 4.7 says that the connection $(\Lambda_{\mathcal{A}}, D_0, \dots, D_n)$ is regular singular along $t = 0$. In this case, Proposition 4.6 is easy to prove. In fact, this is an algebraic analogue of the following fact: let $X$ be the complex affine space $\mathbb{A}^{n+1}_{\mathbb{C}}$ and $D$ the divisor $\{0\} \times \mathbb{A}^n_{\mathbb{C}}$. Consider a vector bundle $\Lambda$ on $X$ and an integrable connection $\nabla$ on $\Lambda|_{X\backslash D}$ that admits logarithmic poles along $D$. Let $T$ be the monodromy transformation of $(\Lambda|_{X\backslash D})^{\nabla=0}$ defined by the positive generator of $\pi_1(X\backslash D) = \mathbb{Z}$. Then $T$ extends to an automorphism $\widetilde{T}$ of $\Lambda$ and satisfies

$$\widetilde{T}|_D = \exp(-2\pi i \,\mathrm{Res}_D \nabla).$$

See [Del70, Proposition 3.11].

*Proof of Lemma 4.7.* This is [AB01, Lemma 3.3.2]. For the convenience of the reader, we reproduce the proof here. Fix $1 \leqslant i \leqslant n$ and $\alpha \in \mathcal{A}$. It is enough to show that for each $0 \leqslant j \leqslant r$,

$$D_i \,\mathrm{Ker}(tD_0 - \alpha)^j \subset \mathrm{Ker}(tD_0 - \alpha)^{j+1}.$$

We prove this inclusion by induction on $j$. The assertion is trivial when $j = 0$. Assume $j > 0$ and take $m \in \operatorname{Ker}(tD_0 - \alpha)^j$. Then $(tD_0 - \alpha)m \in \operatorname{Ker}(tD_0 - \alpha)^{j-1}$, and thus $D_i(tD_0 - \alpha)m \in \operatorname{Ker}(tD_0 - \alpha)^j$ by the induction hypothesis. We need to show $(tD_0 - \alpha)^{j+1}D_i m = 0$. Since $D_i$ commutes with $tD_0$ and satisfies $D_i(\alpha m) = \alpha D_i(m) + d_i(\alpha)m$, we have

$$(tD_0 - \alpha)D_i m = D_i(tD_0 - \alpha)m + d_i(\alpha)m.$$

Therefore

$$\begin{aligned}(tD_0 - \alpha)^{j+1}D_i m &= (tD_0 - \alpha)^j(tD_0 - \alpha)D_i m \\ &= (tD_0 - \alpha)^j D_i(tD_0 - \alpha)m + (tD_0 - \alpha)^j d_i(\alpha)m \\ &= (tD_0 - \alpha)^j D_i(tD_0 - \alpha)m + d_i(\alpha)(tD_0 - \alpha)^j m.\end{aligned}$$

For the third equality, note that $d_i(\alpha) \in R$ and $D_0$ is $R$-linear. Since $D_i(tD_0-\alpha)m \in \operatorname{Ker}(tD_0-\alpha)^j$ and $m \in \operatorname{Ker}(tD_0 - \alpha)^j$, the last sum is zero. □

We continue the proof of Proposition 4.6. Fix an $R[[t]]$-basis of $\Lambda_{\mathcal{A}}$ and identify $\Lambda_{\mathcal{A}}$ with $R[[t]]^r$. Note that $R[[t]]^r$ has natural differentials $d_0, d_1, \ldots, d_n : R[[t]]^r \to R[[t]]^r$. Consider the map

$$t(D_0 - d_0) : R[[t]]^r \to R[[t]]^r.$$

This is $R[[t]]$-linear. We denote the corresponding $r \times r$ matrix by $C_0 \in M_r(R[[t]])$.

Fix $1 \leqslant i \leqslant n$. By Lemma 4.7, the map $D_i$ gives an endomorphism on $R[[t]]^r$ that satisfies the Leibniz rule, and thus $D_i - d_i$ is an $R[[t]]$-linear endomorphism on $R[[t]]^r$. We denote the corresponding $r \times r$ matrix by $C_i \in M_r(R[[t]])$.

We have $[tD_0, D_i] = 0$ and $[td_0, d_i] = 0$ in $\operatorname{End}(R[[t]]^r)$. Plugging $tD_0 = td_0 + C_0$ and $D_i = d_i + C_i$ into $[tD_0, D_i] = 0$ yields

$$[C_0, C_i] = d_i C_0 - td_0 C_i, \tag{4.1}$$

where $d_0$ and $d_i$ are derivatives acting on the matrices entrywise.

Consider the surjection $R[[t]] \to R$ evaluating $t$ by 0. We denote the image of $C_0$ (respectively $C_i$) in $M_r(R)$ by $\overline{C}_0$ (respectively $\overline{C}_i$). By construction $\overline{C}_0$ is the matrix corresponding to $\operatorname{Res}_{\Lambda_{\mathcal{A}}} D_0$. Thus it suffices to show that each eigenvalue of $\overline{C}_0$ is killed by $d_i$. This is standard. Namely, by (4.1), we have

$$[\overline{C}_0, \overline{C}_i] = d_i \overline{C}_0.$$

This implies that

$$d_i(\overline{C}_0^2) = \overline{C}_0 d_i(\overline{C}_0) + d_i(\overline{C}_0)\overline{C}_0 = \overline{C}_0[\overline{C}_0, \overline{C}_i] + [\overline{C}_0, \overline{C}_i]\overline{C}_0 = [\overline{C}_0^2, \overline{C}_i].$$

Similarly, for each $j \in \mathbb{N}$,

$$d_i(\overline{C}_0^j) = [\overline{C}_0^j, \overline{C}_i].$$

In particular, we get

$$d_i(\operatorname{tr}(\overline{C}_0^j)) = 0.$$

This implies that each eigenvalue of $\overline{C}_0$ is killed by $d_i$. □



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### 4.3 Constancy of generalized Hodge–Tate weights

Here is the key theorem of this paper.

Theorem 4.8. *Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a smooth rigid analytic variety over $k$ and $\mathbb{L}$ a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. Consider the arithmetic Sen endomorphism $\phi_{\mathbb{L}} \in \operatorname{End}(\mathcal{H}(\mathbb{L}))$. Then eigenvalues of $\phi_{\mathbb{L},x} \in \operatorname{End}(\mathcal{H}(\mathbb{L})_x)$ for $x \in X_K$ are algebraic over $k$ and constant on each connected component of $X_K$.*

We call these eigenvalues *generalized Hodge–Tate weights* of $\mathbb{L}$.

*Proof.* Since $\phi_{\mathbb{L}}$ is an endomorphism on the vector bundle $\mathcal{H}(\mathbb{L})$ on $X_K$, it suffices to prove the statement étale locally on $X$. Thus we may assume that $X$ is an affinoid $\operatorname{Spa}(B, B^+)$ which admits a toric chart $X_{k'} \to \mathbb{T}^n_{k'}$ over some finite extension $k'$ of $k$ in $K$.

Take $(Y = \operatorname{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}_{\mathbb{L}}$. We may assume that $B_\infty$ is connected, hence an integral domain. Note that $Y$ admits a toric chart

$$Y_{k''} \to \mathbb{T}^n_{k''} = \operatorname{Spa}(k''\langle T_1^{\pm}, \dots, T_n^{\pm}\rangle, \mathcal{O}_{k''}\langle T_1^{\pm}, \dots, T_n^{\pm}\rangle)$$

after base change to a finite extension $k''$ of $k'$ in $K$. Then the derivations $\partial/\partial T_1, \dots, \partial/\partial T_n$ on $k''\langle T_1^{\pm}, \dots, T_n^{\pm}\rangle$ extends over $B_\infty$. We also denote the extensions by $\partial/\partial T_1, \dots, \partial/\partial T_n$.

We set

$$R = B_{k_\infty}, \quad d_0 = \partial_t \quad \text{and} \quad d_i = \frac{\partial}{\partial T_i} \quad (1 \leqslant i \leqslant n).$$

Consider the $R((t))$-module

$$M = \mathcal{RH}(\mathbb{L})(Y_K)_{\text{fin}}$$

equipped with endomorphisms

$$D_0 = t^{-1}\phi_{\text{dR},\mathbb{L},Y_K} \quad \text{and} \quad D_i = (\nabla_{\mathbb{L},Y_K})_{\partial/\partial T_i} \quad (1 \leqslant i \leqslant n).$$

By Proposition 4.2, they satisfy the assumptions in the previous subsection.

Consider the $R[[t]]$-submodule of $M$

$$\Lambda = (\operatorname{Fil}^0 \mathcal{RH}(\mathbb{L})(Y_K))_{\text{fin}}.$$

Then $\Lambda$ is $tD_0$-stable, and $\operatorname{Res}_\Lambda D_0$ is $\phi_{\mathbb{L},Y_K}$. Thus by Proposition 4.6, each eigenvalue $\alpha$ of $\operatorname{Res}_\Lambda D_0$ in an algebraic closure $L$ of $\operatorname{Frac} R$ satisfies

$$d_1(\alpha) = \cdots = d_n(\alpha) = 0.$$

On the other hand, we can check that

$$L^{d_1 = \cdots = d_n = 0} = (\overline{\operatorname{Frac} k''\langle T_1^{\pm}, \dots, T_n^{\pm}\rangle})^{\partial/\partial T_1 = \cdots = \partial/\partial T_n = 0} = \bar{k}.$$

Therefore the eigenvalues of $\phi_{\mathbb{L},Y_K}$ are algebraic over $k$ and constant on $Y_K$. □

Corollary 4.9. *Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and $\mathbb{L}$ a $\mathbb{Q}_p$-local system on $X$. Then the multiset of generalized Hodge–Tate weights of the $p$-adic representations $\mathbb{L}_{\overline{x}}$ of $\operatorname{Gal}(\overline{k(x)}/k(x))$ does not depend on the choice of a classical point $x$ of $X$.*

*In particular, if $\mathbb{L}_{\overline{x}}$ is presque Hodge–Tate for one classical point $x$ of $X$ (i.e. generalized Hodge–Tate weights are all integers), $\mathbb{L}_{\overline{y}}$ is presque Hodge–Tate for every classical point $y$ of $X$.*

*Proof.* This follows from Theorem 4.8. □



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## 5. Applications and related topics

We study properties of Hodge–Tate sheaves using the arithmetic Sen endomorphism. We keep the notation in § 3.

Consider the Hodge–Tate period sheaf on $X_{\text{proét}}$:

$$\mathcal{O}\mathbb{B}_{\mathrm{HT}} := \mathrm{gr}^{\bullet}\,\mathcal{O}\mathbb{B}_{\mathrm{dR}} = \bigoplus_{j\in\mathbb{Z}} \mathcal{O}\mathbb{C}(j).$$

For a $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{ét}}$, we define a sheaf $D_{\mathrm{HT}}(\mathbb{L})$ on $X_{\text{ét}}$ by

$$D_{\mathrm{HT}}(\mathbb{L}) := \nu_*(\hat{\mathbb{L}} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{HT}}).$$

Proposition 5.1. *The sheaf $D_{\mathrm{HT}}(\mathbb{L})$ is a coherent $\mathcal{O}_{X_{\text{ét}}}$-module. Moreover, for every affinoid $Y \in X_{\text{ét}}$,*

$$\Gamma(Y, D_{\mathrm{HT}}(\mathbb{L})) = \bigoplus_{j\in\mathbb{Z}} H^0(\Gamma_k, \mathcal{H}(\mathbb{L})(Y)(j)).$$

*Proof.* This follows from the proof of [LZ17, Theorem 3.9(i)]. □

*Remark* 5.2. In [KL16, Theorem 8.6.2(a)], Kedlaya and Liu proved this statement for pseudocoherent modules over a pro-coherent analytic field.

We are going to study the relation between $D_{\mathrm{HT}}(\mathbb{L})$ and $\phi_{\mathbb{L}} \in \mathrm{End}\,\mathcal{H}(\mathbb{L})$. For each $j \in \mathbb{Z}$, we set

$$\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j} := \mathrm{Ker}(\phi_{\mathbb{L}} - j\,\mathrm{id} : \mathcal{H}(\mathbb{L}) \to \mathcal{H}(\mathbb{L})).$$

This is a coherent $\mathcal{O}_{X_{K,\text{ét}}}$-module. We denote by $D_{\mathrm{HT}}(\mathbb{L})|_{X_K}$ the coherent $\mathcal{O}_{X_K,\text{ét}}$-module associated to the pullback of $D_{\mathrm{HT}}(\mathbb{L})$ on $X$ to $X_K$ as coherent sheaves.

Proposition 5.3. *Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system of rank $r$ on $X_{\text{ét}}$. Assume that $\mathbb{L}$ satisfies one of the following conditions:*

(i) *$\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}$ is a vector bundle on $X_{K,\text{ét}}$ for each $j \in \mathbb{Z}$;*

(ii) *$D_{\mathrm{HT}}(\mathbb{L})$ is a vector bundle of rank $r$ on $X_{\text{ét}}$.*

*Then we have*

$$D_{\mathrm{HT}}(\mathbb{L})|_{X_{K,\text{ét}}} \cong \bigoplus_{j\in\mathbb{Z}} \mathcal{H}(\mathbb{L}(j))^{\phi_{\mathbb{L}(j)}=0}.$$

*Moreover, this is isomorphic to $\bigoplus_{j\in\mathbb{Z}} \mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}$. In particular, $D_{\mathrm{HT}}(\mathbb{L})$ is a vector bundle on $X_{\text{ét}}$ and $\bigoplus_{j\in\mathbb{Z}} \mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}$ is a vector bundle on $X_{K,\text{ét}}$.*

*Proof.* The statement is local. So it suffices to prove that for each affinoid $Y = \mathrm{Spa}(B, B^+) \in X_{\text{ét}}$ such that $\mathcal{H}(\mathbb{L})|_{Y_K}$ is associated to a finite free $B_K$-module (say $V$), we have

$$\Gamma(Y, D_{\mathrm{HT}}(\mathbb{L}))\,\hat{\otimes}_B\, B_K \cong \bigoplus_{j\in\mathbb{Z}} V(j)^{\phi_{V(j)}=0}.$$

Note $\Gamma(Y, D_{\mathrm{HT}}(\mathbb{L})) = \bigoplus_{j\in\mathbb{Z}} (V(j))^{\Gamma_k}$. Moreover, it follows from the Tate–Sen method [LZ17, Lemma 3.10] that

$$(V_{\mathrm{fin}}(j))^{\Gamma_k} \xrightarrow{\cong} (V(j))^{\Gamma_k}.$$



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Lemma 5.4.

(i) $(V_{\mathrm{fin}}^{\phi_V=0})\otimes_{B_\infty} B_K \cong V^{\phi_V=0}$.

(ii) *The natural map*

$$(V_{\mathrm{fin}}^{\Gamma_k})\otimes_B B_\infty \to V_{\mathrm{fin}}^{\phi_V=0}$$

*is injective.*

*Proof.* Part (i) follows from the flatness of $B_\infty \to B_K$ and $V_{\mathrm{fin}}\otimes_{B_\infty} B_K \cong V$.

We prove part (ii). By the definition of $\phi_V$, the natural map

$$(V_{\mathrm{fin}}^{\Gamma_k})\otimes_B B_\infty \to V_{\mathrm{fin}}$$

factors through $V_{\mathrm{fin}}^{\phi_V=0}$. So we show that the above map is injective.

We denote the total fraction ring of $B$ (respectively $B_\infty$) by $\mathrm{Frac}\,B$ (respectively $\mathrm{Frac}\,B_\infty$). We first claim that the natural map

$$V_{\mathrm{fin}}^{\Gamma_k} \to V_{\mathrm{fin}}^{\Gamma_k}\otimes_B \mathrm{Frac}\,B$$

is injective. To see this, note that $V_{\mathrm{fin}}$ is a finite free $B_\infty$-module. Hence the composite

$$V_{\mathrm{fin}} \to V_{\mathrm{fin}}\otimes_B \mathrm{Frac}\,B = V_{\mathrm{fin}}\otimes_{B_\infty}(B_\infty\otimes_B \mathrm{Frac}\,B) \to V_{\mathrm{fin}}\otimes_{B_\infty}\mathrm{Frac}\,B_\infty$$

is injective, and thus so is the first map. Since the composite

$$V_{\mathrm{fin}}^{\Gamma_k} \to V_{\mathrm{fin}}^{\Gamma_k}\otimes_B \mathrm{Frac}\,B \to V_{\mathrm{fin}}\otimes_B \mathrm{Frac}\,B$$

coincides with the composite of injective maps $V_{\mathrm{fin}}^{\Gamma_k}\to V_{\mathrm{fin}}$ and $V_{\mathrm{fin}}\to V_{\mathrm{fin}}\otimes_B\mathrm{Frac}\,B$, the map $V_{\mathrm{fin}}^{\Gamma_k}\to V_{\mathrm{fin}}^{\Gamma_k}\otimes_B\mathrm{Frac}\,B$ is also injective.

By the above claim, it suffices to show the injectivity of the natural map

$$(V_{\mathrm{fin}}^{\Gamma_k}\otimes_B \mathrm{Frac}\,B)\otimes_{\mathrm{Frac}\,B}\mathrm{Frac}\,B_\infty \to V_{\mathrm{fin}}\otimes_{B_\infty}\mathrm{Frac}\,B_\infty.$$

Now that $\mathrm{Frac}\,B$ and $\mathrm{Frac}\,B_\infty$ are products of fields, this follows from standard arguments; we may assume that $\mathrm{Frac}\,B$ is a field. Replacing $k$ by an algebraic closure in $\mathrm{Frac}\,B$, we may further assume that $\mathrm{Frac}\,B_\infty$ is also a field. Note that $\mathrm{Frac}\,B_\infty = (\mathrm{Frac}\,B)\otimes_k k_\infty$ and thus $(\mathrm{Frac}\,B_\infty)^{\Gamma_k} = \mathrm{Frac}\,B$.

Assume the contrary. Let $a>0$ be the minimal positive integer such that there exist $v_1,\dots,$ $v_a\in V_{\mathrm{fin}}^{\Gamma_k}\otimes_B\mathrm{Frac}\,B$ that are linearly independent over $\mathrm{Frac}\,B$ and non-zero $b_1,\dots,b_a\in\mathrm{Frac}\,B_\infty$ satisfying $b_1v_1+\cdots+b_av_a=0$. By replacing $b_i$ by $b_1^{-1}b_i$, we may further assume $b_1=1$. Take any $\gamma\in\Gamma_k$. As $v_1,\dots,v_a\in V_{\mathrm{fin}}^{\Gamma_k}\otimes_B\mathrm{Frac}\,B$, we have $v_1+\gamma(b_2)v_2+\cdots+\gamma(b_a)v_a=0$ and thus $(\gamma(b_2)-b_2)v_2+\cdots+(\gamma(b_a)-b_a)v_a=0$. By the minimality, we have $\gamma(b_i)=b_i$ for each $2\leqslant i\leqslant a$ and $\gamma\in\Gamma_k$. Therefore we have $b_i\in\mathrm{Frac}\,B$ for all $i$, which contradicts the linear independence of $v_1,\dots,v_a$ over $\mathrm{Frac}\,B$. □

We continue the proof of Proposition 5.3. By Lemma 5.4 and discussions above, it is enough to show $V_{\mathrm{fin}}^{\Gamma_k}\otimes_B B_\infty\cong V_{\mathrm{fin}}^{\phi_V=0}$ assuming either condition (i) or (ii). In fact, the Tate twist of this isomorphism implies $(V(j))^{\Gamma_k}\hat{\otimes}_B B_K\cong (V(j))^{\phi_{V(j)}=0}$, and a choice of a generator of $\mathcal{O}_{X_{K,\mathrm{\acute{e}t}}}(j)$ yields $\mathcal{H}(\mathbb{L}(j))^{\phi_{\mathbb{L}(j)}=0}\cong\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=-j}$.

We show that condition (ii) implies condition (i). For each $j\in\mathbb{Z}$, let $V_{\mathrm{fin}}^{(j)}$ denote the generalized eigenspace of $\phi_V$ on $V_{\mathrm{fin}}$ with eigenvalue $j$. By the constancy of $\phi_V$, $V_{\mathrm{fin}}^{(j)}$ is a direct



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summand of $V_{\mathrm{fin}}$ and thus a finite projective $B_\infty$-module. By Lemma 5.4(ii), we have injective $B_\infty$-linear maps

$$(V(j)_{\mathrm{fin}})^{\Gamma_k} \otimes_B B_\infty \hookrightarrow (V(j)_{\mathrm{fin}})^{\phi_{V(j)}=0} \cong V_{\mathrm{fin}}^{\phi_V=-j} \hookrightarrow V_{\mathrm{fin}}^{(-j)}$$

for each $j \in \mathbb{Z}$. From this we obtain

$$\operatorname{rank} D_{HT}(\mathbb{L}) = \sum_{j\in\mathbb{Z}} \operatorname{rank}_B (V_{\mathrm{fin}}(j))^{\Gamma_k} \leqslant \sum_{j\in\mathbb{Z}} \operatorname{rank}_{B_\infty} V_{\mathrm{fin}}^{(-j)} \leqslant \operatorname{rank} \mathcal{H}(\mathbb{L}) = r.$$

Hence it follows from condition (ii) that $(V_{\mathrm{fin}}(j))^{\Gamma_k} \otimes_B B_\infty$ and $V_{\mathrm{fin}}^{(-j)}$ are finite projective $B_\infty$-modules of the same rank. This implies $V_{\mathrm{fin}}^{\phi_V=-j} = V_{\mathrm{fin}}^{(-j)}$. So $V_{\mathrm{fin}}^{\phi_V=-j}$ is a finite projective $B_\infty$-module for every $j \in \mathbb{Z}$ and thus $\mathcal{H}(\mathbb{L})$ satisfies condition (i).

From now on, we assume that $\mathcal{H}(\mathbb{L})$ satisfies condition (i). By condition (i) and Lemma 5.4(i), $V_{\mathrm{fin}}^{\phi_V=0}$ is finite projective over $B_\infty$. So shrinking $Y$ if necessary, we may assume that $V_{\mathrm{fin}}^{\phi_V=0}$ is finite free over $B_\infty$. Note that we only concern the $B_\infty$-representation $V_{\mathrm{fin}}$ of $\Gamma_k$ and we have $(V_{\mathrm{fin}}^{\phi_V=0})^{\Gamma_k} = V_{\mathrm{fin}}^{\Gamma_k}$. Thus replacing $V_{\mathrm{fin}}$ by the subrepresentation $V_{\mathrm{fin}}^{\phi_V=0}$, we may further assume $\phi_V = 0$ on $V_{\mathrm{fin}}$. Under this assumption, it remains to prove $V_{\mathrm{fin}}^{\Gamma_k} \otimes_B B_\infty \cong V_{\mathrm{fin}}$.

Fix a $B_\infty$-basis $v_1, \ldots, v_r$ of $V_{\mathrm{fin}}$. Then there exists a large positive integer $m$ such that for each $\gamma \in \Gamma_k$ the matrix of $\gamma$ with respect to $(v_i)$ has entries in $\mathrm{GL}_r(B_{k_m})$. Since $\phi_V = 0$, by increasing $m$ if necessary, we may further assume that $\gamma v_i = v_i$ for each $1 \leqslant i \leqslant r$ and $\gamma \in \Gamma'_k := \mathrm{Gal}(k_\infty/k_m) \subset \Gamma_k$. Set $V_{k_m} := \bigoplus_{1\leqslant i\leqslant r} B_{k_m} v_i$. This is a $B_{k_m}$-representation of $\Gamma_k/\Gamma'_k = \mathrm{Gal}(k_m/k)$ and satisfies $V_{\mathrm{fin}} = V_{k_m} \otimes_{B_{k_m}} B_\infty$.

It follows from [BC08, Proposition 2.2.1] that $(V_{k_m})^{\Gamma_k/\Gamma'_k}$ is a finite projective $B$-module and that $(V_{k_m})^{\Gamma_k/\Gamma'_k} \otimes_B B_{k_m} \cong V_{k_m}$. As $V_{\mathrm{fin}}^{\Gamma_k} = (V_{k_m})^{\Gamma_k/\Gamma'_k}$, this yields

$$V_{\mathrm{fin}}^{\Gamma_k} \otimes_B B_\infty \cong V_{\mathrm{fin}}. \qquad \square$$

Theorem 5.5. *Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system of rank $r$ on $X_{\mathrm{\acute{e}t}}$. Then the following conditions are equivalent:*

(i) *$D_{\mathrm{HT}}(\mathbb{L})$ is a vector bundle of rank $r$ on $X_{\mathrm{\acute{e}t}}$;*
(ii) *$\nu^* D_{\mathrm{HT}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}\mathbb{B}_{\mathrm{HT}} \cong \hat{\mathbb{L}} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}\mathbb{B}_{\mathrm{HT}}$;*
(iii) *$\phi_{\mathbb{L}}$ is a semisimple endomorphism on $\mathcal{H}(\mathbb{L})$ with integer eigenvalues;*
(iv) *there exist integers $j_1 < \cdots < j_a$ such that if we set $F(s) := \prod_{1\leqslant i\leqslant a}(s - j_i) \in \mathbb{Z}[s]$, then*

$$F(\phi_{\mathbb{L}}) = 0$$

*as an endomorphism of $\mathcal{H}(\mathbb{L})$.*

Definition 5.6. A $\mathbb{Q}_p$-local system on $X_{\mathrm{\acute{e}t}}$ is a *Hodge–Tate sheaf* if it satisfies the equivalent conditions in Theorem 5.5.

*Remark* 5.7. Tsuji obtained Theorem 5.5 in the case of semistable schemes [Tsu11, Theorem 9.1]. He also gave a characterization of Hodge–Tate local systems in terms of restrictions to divisors. See [Tsu11, Theorem 9.1] for the detail.

*Proof of Theorem 5.5.* The equivalence of (iii) and (iv) is clear, and (iii) implies (i) by Proposition 5.3. Conversely, assume condition (i). Thus $D_{\mathrm{HT}}(\mathbb{L})|_{X_K}$ is a vector bundle of rank $r$ on $X_{K,\text{ét}}$. By Proposition 5.3, it is also isomorphic to $\bigoplus_{j\in\mathbb{Z}}\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}$. Thus $\bigoplus_{j\in\mathbb{Z}}\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}=\mathcal{H}(\mathbb{L})$, and there exist integers $j_1<\cdots<j_a$ such that $\bigoplus_{1\leqslant i\leqslant a}\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j_i}=\mathcal{H}(\mathbb{L})$. So $F(s):=\prod_{1\leqslant i\leqslant a}(s-j_i)$ satisfies $F(\phi_{\mathbb{L}})=0$, which is condition (iv).

Next we show that condition (iv) implies (ii). Obviously, there is a natural morphism

$$\nu^*D_{\mathrm{HT}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{HT}}\to\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{B}_{\mathrm{HT}} \tag{5.1}$$

on $X_{\text{proét}}$ and we will prove that this is an isomorphism. It is enough to check this on $X_{\text{proét}}/X_K\cong X_{K,\text{proét}}$. Recall a canonical isomorphism in [LZ17, Theorem 2.1(ii)]:

$$\nu'^*\mathcal{H}(\mathbb{L})\otimes_{\mathcal{O}_{X_K}}\mathcal{O}\mathbb{C}|_{X_{K,\text{proét}}}\cong(\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{C})|_{X_{K,\text{proét}}}.$$

Then the restriction of the morphism (5.1) to $X_{K,\text{proét}}$ is obtained as

$$\begin{aligned}(\nu^*D_{\mathrm{HT}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}}&\cong\nu^*D_{\mathrm{HT}}(\mathbb{L})|_{X_{K,\text{proét}}}\otimes_{\mathcal{O}_{X_K}}\mathcal{O}\mathbb{B}_{\mathrm{HT}}|_{X_{K,\text{proét}}}\\&\cong\nu'^*(D_{\mathrm{HT}}(\mathbb{L})|_{X_K})\otimes_{\mathcal{O}_{X_K}}\left.\left(\bigoplus_{j\in\mathbb{Z}}\mathcal{O}\mathbb{C}(j)\right)\right|_{X_{K,\text{proét}}}\\&\cong\nu'^*\left(D_{\mathrm{HT}}(\mathbb{L})|_{X_K}\otimes\bigoplus_{j\in\mathbb{Z}}\mathcal{O}_{X_K}(j)\right)\otimes_{\mathcal{O}_{X_K}}\mathcal{O}\mathbb{C}|_{X_{K,\text{proét}}}\\&\to\nu'^*\left(\bigoplus_{j\in\mathbb{Z}}\mathcal{H}(\mathbb{L})(j)\right)\otimes_{\mathcal{O}_{X_K}}\mathcal{O}\mathbb{C}|_{X_{K,\text{proét}}}\\&\cong\left.\left(\bigoplus_{j\in\mathbb{Z}}\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{C}(j)\right)\right|_{X_{K,\text{proét}}}\cong(\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}}.\end{aligned}$$

This can be checked by considering affinoid perfectoids represented by the toric tower, and the verification is left to the reader. It follows from condition (iii) and Proposition 5.3 that

$$\bigoplus_{j\in\mathbb{Z}}D_{\mathrm{HT}}(\mathbb{L})|_{X_K}(j)\cong\bigoplus_{j\in\mathbb{Z}}\mathcal{H}(\mathbb{L})(j).$$

Hence $(\nu^*D_{\mathrm{HT}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}}\cong(\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}}$.

Finally we show that (ii) implies (i). By condition (ii), we have

$$\nu'_*((\nu^*D_{\mathrm{HT}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}})\cong\nu'_*((\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}}).$$

On the other hand, it is easy to check

$$\nu'_*((\nu^*D_{\mathrm{HT}}(\mathbb{L})\otimes_{\mathcal{O}_X}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}})\cong\bigoplus_{j\in\mathbb{Z}}D_{\mathrm{HT}}(\mathbb{L})|_{X_K}(j).$$

Since $\nu'_*((\hat{\mathbb{L}}\otimes_{\hat{\mathbb{Q}}_p}\mathcal{O}\mathbb{B}_{\mathrm{HT}})|_{X_{K,\text{proét}}})=\bigoplus_{j\in\mathbb{Z}}\mathcal{H}(\mathbb{L}(j))$, we have

$$\bigoplus_{j\in\mathbb{Z}}D_{\mathrm{HT}}(\mathbb{L})|_{X_K}(j)\cong\bigoplus_{j\in\mathbb{Z}}\mathcal{H}(\mathbb{L}(j)).$$

In particular, $D_{\mathrm{HT}}(\mathbb{L})|_{X_K}$ is a vector bundle on $X_{K,\text{ét}}$, and thus $D_{\mathrm{HT}}(\mathbb{L})$ is a vector bundle on $X_{\text{ét}}$. Moreover, condition (ii) implies rank $D_{\mathrm{HT}}(\mathbb{L})=r$. □

*Example* 5.8. Suppose that there exists a Zariski dense subset $T \subset X$ consisting of classical rigid points with residue field finite over $k$ such that the restriction of $\mathbb{L}$ to each $x \in T$ defines a Hodge–Tate representation. Then $\mathbb{L}$ is a Hodge–Tate sheaf by Theorems 4.8 and 5.5(iii). See [KL16, Theorem 8.6.6] for a generalization of this remark.

COROLLARY 5.9.

(i) *Hodge–Tate sheaves are stable under taking dual, tensor product, and subquotients.*
(ii) *Let $f : Y \to X$ be a morphism between smooth rigid analytic varieties over $k$. If $\mathbb{L}$ is a Hodge–Tate sheaf on $X_{\text{ét}}$, then $f^*\mathbb{L}$ is a Hodge–Tate sheaf on $Y_{\text{ét}}$.*

*Proof.* This follows from Proposition 2.20, Lemma 2.22, and Theorem 5.5(iii). □

We next turn to the pushforward of Hodge–Tate sheaves.

THEOREM 5.10. *Let $f : X \to Y$ be a smooth proper morphism between smooth rigid analytic varieties over $k$ of relative dimension $m$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$.*

(i) *If $\alpha \in \overline{k}$ is a generalized Hodge–Tate for $R^i f_*\mathbb{L}$, then $\alpha$ is of the form $\beta - j$ with a generalized Hodge–Tate weight $\beta$ of $\mathbb{L}$ and an integer $j \in [0, m]$.*
(ii) *If $\mathbb{L}$ is a Hodge–Tate sheaf on $X_{\text{ét}}$, then $R^i f_*\mathbb{L}$ is a Hodge–Tate sheaf on $Y_{\text{ét}}$.*[3]

*Remark* 5.11. Theorem 5.10(ii) is proved by Hyodo [Hyo86, § 3, Corollary] when $f : X \to Y$ and $\mathbb{L}$ are analytifications of corresponding algebraic objects.

*Proof.* Let $f_K : X_K \to Y_K$ denote the base change of $f$ over $K$.

Part (i) easily follows from Theorem 3.9. In fact, we have the isomorphism

$$\mathcal{H}(R^i f_*\mathbb{L}) \cong R^i f_{K,\text{ét},*}(\mathcal{H}(\mathbb{L}) \otimes \Omega^\bullet_{X/Y}(-\bullet)),$$

and under this identification $\phi_{R^i f_*\mathbb{L}}$ corresponds to $R^i f_{K,\text{ét},*}(\phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\,(\mathrm{id} \otimes \mathrm{id}))$. Consider the spectral sequence with

$$E_1^{a,b} = R^b f_{K,\text{ét},*}\mathcal{H}(\mathbb{L}) \otimes \Omega^a_{X/Y}(-a)$$

converging to $\mathcal{H}(R^{a+b} f_*\mathbb{L})$. Then the endomorphism $R^b f_{K,\text{ét},*}((\phi_{\mathbb{L}} - a) \otimes \mathrm{id})$ on $E_1^{a,b}$ converges to $\phi_{R^{a+b} f_*\mathbb{L}}$, and this implies part (i).

For part (ii), we need arguments similar to the proof of Theorem 3.9. We may assume that $Y$ is affinoid. Take a finite affinoid covering $\mathcal{U} = \{U_K^{(i)}\}$ of $X_K$. Let $\mathcal{F}^\bullet$ denote the complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\vartheta_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\vartheta_{\mathbb{L}}} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots$$

on $X_K$ equipped with the natural $\Gamma_k$-action and the endomorphism $\phi_{\mathcal{F}^\bullet} = \phi_{\mathbb{L}} \otimes \mathrm{id} - \bullet\,(\mathrm{id} \otimes \mathrm{id})$.

Recall also the Čech-to-derived functor spectral sequence with

$$E_2^{a,b} = H^a(\mathrm{Tot}(\check{C}^\bullet(\mathcal{U}, \underline{H}^b(\mathcal{F}^\bullet))))$$

converging to $R^{a+b}\Gamma(X_{K,\text{ét}}, \mathcal{F}^\bullet)$. This spectral sequence degenerates at $E_2$ and yields

$$H^i(\mathrm{Tot}(\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \xrightarrow{\cong} R^i\Gamma(X_{K,\text{ét}}, \mathcal{F}^\bullet) = \Gamma(Y_K, \mathcal{H}(R^i f_*\mathbb{L})). \tag{5.2}$$

[3] A $\mathbb{Z}_p$-local system $\mathbb{L}$ is called Hodge–Tate if the $\mathbb{Q}_p$-local system $\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is Hodge–Tate.

Note that both source and target in (5.2) have arithmetic Sen endomorphisms and they are compatible under the isomorphism.

Since $\mathbb{L}$ is a Hodge–Tate sheaf, there exist integers $j_1 < \cdots < j_a$ such that $F(\phi_{\mathbb{L}}) = 0$ with $F(s) := \prod_{1\leqslant i\leqslant a}(s - j_i)$. Set $J := \{j_1 - m, j_1 - m + 1, \ldots, j_a - 1, j_a\}$. This is a finite subset of $\mathbb{Z}$. We set $G(s) := \prod_{j\in J}(s - j) \in \mathbb{Z}[s]$. For each $0 \leqslant j \leqslant m$, the endomorphism $\phi_{\mathbb{L}} \otimes \mathrm{id} - j(\mathrm{id} \otimes \mathrm{id})$ on $\mathcal{H}(\mathbb{L}) \otimes \Omega^j_{X/Y}(-j)$ satisfies

$$G(\phi_{\mathbb{L}} \otimes \mathrm{id} - j(\mathrm{id} \otimes \mathrm{id})) = 0.$$

This implies $G(\phi_{\mathcal{F}^\bullet}) = 0$, and thus $G(\mathrm{Tot}(\check{C}^\bullet(\mathcal{U}, \phi_{\mathcal{F}^\bullet}))) = 0$. Therefore (5.2) yields

$$G(\phi_{R^i f_* \mathbb{L}}) = 0.$$

Hence $R^i f_* \mathbb{L}$ is a Hodge–Tate sheaf on $Y_{\mathrm{\acute{e}t}}$. □

We now turn to a rigidity of Hodge–Tate representations. Let us first recall Liu and Zhu's rigidity result for de Rham representations [LZ17, Theorem 1.3]: let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\mathrm{\acute{e}t}}$. If $\mathbb{L}_{\overline{x}}$ is a de Rham representation at a classical point $x \in X$, then $\mathbb{L}$ is a de Rham sheaf. In particular, $\mathbb{L}_{\overline{y}}$ is a de Rham representation at every classical point $y \in X$.

The same result holds for Hodge–Tate local systems of rank at most two. We do not know whether this is true for Hodge–Tate local systems of higher rank.

Theorem 5.12. *Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\mathrm{\acute{e}t}}$. Assume that* $\operatorname{rank}\mathbb{L}$ *is at most two. If $\mathbb{L}_{\overline{x}}$ is a Hodge–Tate representation at a classical point $x \in X$, then $\mathbb{L}$ is a Hodge–Tate sheaf. In particular, $\mathbb{L}_{\overline{y}}$ is a Hodge–Tate representation at every classical point $y \in X$.*

Before the proof, let us recall a remarkable theorem by Sen on Hodge–Tate representations of weight 0.

Theorem 5.13 [Sen81, § Corollary]. *Let $k$ be a finite extension of $\mathbb{Q}_p$ and let $\rho : G_k \to \mathrm{GL}_r(\mathbb{Q}_p)$ be a continuous representation of the absolute Galois group $G_k$ of $k$. Then $\rho$ is a Hodge–Tate representation with all the Hodge–Tate weights zero if and only if $\rho$ is potentially unramified, i.e. the image of the inertia subgroup of $k$ is finite.*

Note that $\rho$ being a Hodge–Tate representation with all the Hodge–Tate weights zero is equivalent to the Sen endomorphism of $\rho$ being zero. Since potentially unramified representations are de Rham and de Rham representations are stable under Tate twists, Theorem 5.13 implies that a Hodge–Tate representation with a single weight is necessarily de Rham.

*Proof of Theorem 5.12.* We check condition (iii) in Theorem 5.5. By Theorem 4.8 and assumption, all the eigenvalues of $\phi_{\mathbb{L}}$ are integers. So the statement is obvious either when $\operatorname{rank}\mathbb{L} = 1$ or when $\operatorname{rank}\mathbb{L} = 2$ and two eigenvalues are distinct integers.

Assume that $\operatorname{rank}\mathbb{L} = 2$ and two eigenvalues are the same integer. Then $\mathbb{L}_{\overline{x}}$ is de Rham by Theorem 5.13, and thus $\mathbb{L}$ is de Rham by the above-mentioned rigidity theorem for de Rham representations by Liu and Zhu [LZ17, Theorem 1.3]. In particular, $\mathbb{L}$ is a Hodge–Tate sheaf. □

*Remark* 5.14. The proof shows that Theorem 5.12 holds for $\mathbb{L}$ of an arbitrary rank if one of the following conditions holds.

(i) $\mathbb{L}_{\overline{x}}$ is a Hodge–Tate representation with a single weight at a classical point $x \in X$.

(ii) $\mathbb{L}_{\overline{x}}$ is a Hodge–Tate representation with rank $\mathbb{L}$ distinct weights at a classical point $x \in X$.

We end with another application of Sen's theorem in the relative setting.

THEOREM 5.15. *Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Assume that $\mathbb{L}$ is a Hodge–Tate sheaf with a single Hodge–Tate weight. Then there exists a finite étale cover $f : Y \to X$ such that $(f^*\mathbb{L})_{\overline{y}}$ is semistable at every classical point $y$ of $Y$.*

*Proof.* Since semistable representations are stable under Tate twists, we may assume that $\mathbb{L}$ is a Hodge–Tate sheaf with all the weights zero. Let $\overline{\mathbb{L}}$ denote the $\mathbb{Z}/p^2$-local system $\mathbb{L}/p^2\mathbb{L}$ on $X_{\text{ét}}$. Then there exists a finite étale cover $f : Y \to X$ such that $f^*\overline{\mathbb{L}}$ is trivial on $Y_{\text{ét}}$. We will prove that this $Y$ works.

Let $y$ be a classical point of $Y$. We denote by $k'$ the residue field of $y$. Let $\rho : G_{k'} \to \mathrm{GL}(V)$ be the Galois representation of $k'$ corresponding to the stalk $V := (f^*\mathbb{L})_{\overline{y}}$ at a geometric point $\overline{y}$ above $y$. By assumption, $\rho$ is a Hodge–Tate representation with all the weights zero, and thus it is potentially unramified by Theorem 5.13. Hence if we denote the inertia group of $k'$ by $I_{k'}$, $\rho(I_{k'})$ is finite.

By construction, the mod $p^2$ representation

$$G_{k'} \xrightarrow{\rho} \mathrm{GL}(V) \longrightarrow \mathrm{GL}(V/p^2V)$$

is trivial. On the other hand, $\mathrm{Ker}(\mathrm{GL}(V) \to \mathrm{GL}(V/p^2V))$ does not contain elements of finite order except the identity. Thus we see that $\rho(I_{k'})$ is trivial and hence $\rho$ is an unramified representation. In particular, $\rho$ is semistable. $\square$

*Remark* 5.16. As mentioned in the introduction, it is an interesting question whether one can extend Colmez's strategy [Col08] to prove the relative $p$-adic monodromy conjecture using Theorem 5.15.

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