GALOIS CONNECTIONS AND PAIR ALGEBRAS

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1. Introduction. Unless further restricted, P, Q, and R denote arbitrary partially ordered sets whose order relations are all written "$\leqq$".

An isotone mapping $\phi\colon P \to Q$ is said to be *residuated* if there is an isotone mapping $\psi\colon Q \to P$ such that

(RM 1) $x\phi\psi \geqq x$ for all x in P;

(RM 2) $y\psi\phi \leqq y$ for all y in Q.

Let Q^* denote the partially ordered set with order relation dual to that of Q.

(A) The following conditions are equivalent:

(i) $\phi\colon P \to Q^*$ is a Galois connection;

(ii) $\phi\colon P \to Q$ is a residuated mapping;

(iii) $\operatorname{Max}\{z \in P\colon z y \leqq y\}$ exists for all y in Q and is equal to $y\psi$.

Since ψ is uniquely determined by ϕ, it will be denoted by ϕ^+.

(B) If $\phi\colon P \to Q$ and $\psi\colon Q \to R$ are residuated mappings, so is $\phi \circ \psi\colon P \to R$; moreover, $(\phi \circ \psi)^+ = \psi^+ \circ \phi^+$. Denote by $S(P, Q)$ the set of all residuated mappings $\phi\colon P \to Q$ and write $S(P)$ when $P = Q$. Note that $S(P)$ is a semigroup under composition of functions.

Though the theory of Galois connections is coextensive with that of residuated mappings, it is (B) that leads us to focus on the latter. More properties and examples can be found in (**2**; **3**).

We shall give a representation for residuated mappings (and hence Galois connections) that extends Ore's representation in the case when P and Q are power sets of some sets (**1**). Following this we shall show that P is a complete lattice if and only if $S(P)$ is a complete Baer semigroup.

2. Pair algebras. If X, Y, and Z are arbitrary sets and $B \subset X \times Y$, $C \subset Y \times Z$ binary relations, recall the following usual definitions: for $W \subset X$, $WB = \{y \in Y\colon (w, y) \in B \text{ for some } w \in W\}$, $B^{-1} = \{(y, x)\colon (x, y) \in B\}$, $xB = \{x\}B$, $B \circ C = \{(x, z)\colon (x, y) \in B \text{ and } (y, z) \in C \text{ for some } y \in Y\}$.

$J \subset P \times Q$ is called a *pair algebra* if:

(PA 1) $PJ = Q$;

(PA 1*) $QJ^{-1} = P$;

(PA 2) xJ is a principal dual ideal of Q for all $x \in P$;

(PA 2*) yJ^{-1} is a principal ideal of P for all $y \in Q$.

Denote the set of all pair algebras $J \subset P \times Q$ by $A(P, Q)$; if $P = Q$, write $A(P)$.

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THEOREM 2.1. (i) *For any pair algebra* $J \subset P \times Q$, *define* $\phi_J: P \to Q$ *by the rule* $x\phi_J = \operatorname{Min} xJ$, *then* ϕ_J *is residuated and its residual is given by* $y\phi_J^+ = \operatorname{Max} yJ^{-1}$.

(ii) *For any residuated mapping* $\phi: P \to Q$ *there is a unique pair algebra* $J(\phi)$ *such that* $\phi_{J(\phi)} = \phi$ *and* $J(\phi_J) = J$.

(iii) $J \subset K$ *if and only if* $\phi_J \geqq \phi_k$.

(iv) *If* $J \subset P \times Q$ *and* $K \subset Q \times R$ *are pair algebras, so is* $J \circ K$; *moreover,* $\phi_{J \circ K} = \phi_J \circ \phi_K$ *and* $J(\phi \circ \psi) = J(\phi) \circ J(\psi)$.

Proof. (i) If $x_1 \leqq x_2$ and $x_2 \in yJ^{-1}$, then $x_1 \in yJ^{-1}$; thus,

$$x_1\phi_J = \operatorname{Min}\{y: x_1 \in yJ^{-1}\} \leqq \operatorname{Min}\{y: x_2 \in yJ^{-1}\} = x_2\phi_J.$$

Therefore, ϕ_J is isotone, and analogously so is ϕ_J^+. Now, since $x\phi_J = \operatorname{Min} xJx \in x\phi_J J^{-1}$; therefore, $x\phi_J\phi_J^+ = \operatorname{Max} x\phi_J J^{-1} \geqq x$. The other inequality follows analogously; thus, ϕ_J is residuated.

(ii) Let $J = J(\phi) = \{(x, y): y \geqq x\phi\}$. Note that $(x, x\phi) \in J$ for all $x \in P$, therefore $QJ^{-1} = P$; also, since $y\phi^+\phi \leqq y$, $(y\phi^+, y) \in J$ for all $y \in Q$, therefore $PJ = Q$. Now, $xJ = \{y: y \geqq x\phi\}$ is a principal daul ideal of Q for all $x \in P$, and, since $y \geqq x\phi$ is equivalent to $y\phi^+ \geqq x$, $yJ^{-1} = \{x: x \leqq y\phi^+\}$ is a principal ideal of P for all $y \in Q$. Note that $x\phi_{J(\phi)} = \operatorname{Min}\{y: y \geqq x\phi\} = x\phi$ for all $x \in P$. Also, $J(\phi_J) = \{(x, y): y \geqq x\phi_J\} = J$ since, in view of $x\phi_J \in xJ$, $y \geqq x\phi_J$ is equivalent to $y \in xJ$.

(iii) is straightforward with pointwise ordering on $S(P, Q)$.

(iv) If there is $z \in Q$ such that $y \geqq z\psi$ and $z \geqq x\phi$, then $y \geqq x\psi \geqq y\phi\psi$, thus $J(\phi) \circ J(\psi) \subset J(\phi \circ \psi)$. Conversely, suppose that $y \geqq x\phi\psi$ and put $z = y\psi^+$, then $z \geqq x\phi\psi\psi^+ \geqq x\phi$ and $y \geqq y\psi^+\psi = z\psi$; therefore, $J(\phi \circ \psi) = J(\phi) \circ J(\psi)$. It follows that $\phi_{J \circ K} = \phi_J \circ \phi_K$.

Thus, for any partially ordered set, $A(P)$ and $S(P)$ are order anti-isomorphic and isomorphic as semigroups.

LEMMA 2.2. *Let* P *and* Q *be partially ordered sets with least and greatest elements.* $J \subset P \times Q$ *is a pair algebra if and only if it satisfies* (PA 2), (PA 2*), *and*

(PA 3) $0_P J = Q$,

(PA 3*) $1_Q J^{-1} = P$.

Proof. These conditions are obviously sufficient. We show their necessity: since xJ is a dual principal ideal of Q, $1_Q \in xJ$; i.e., $x \in 1_Q J^{-1}$ for all $x \in P$, therefore $1_Q J^{-1} = P$; similarly, we have that $0_P J = Q$.

LEMMA 2.3. *Let* P *and* Q *be complete lattices.* $J \subset P \times Q$ *is a pair algebra if and only if it satisfies* (PA 3), (PA 3*), *and*

(PA 4) J *is a complete sublattice of* $P \times Q$.

Proof. Necessity. Let J be a pair algebra and $\phi = \phi_J$. Suppose that $(x_\lambda, y_\lambda) \in J$ for $\lambda \in \Lambda$; this is equivalent to $y_\lambda \geqq x_\lambda\phi$ for $\lambda \in \Lambda$. However,

$\bigvee_{\lambda\in\Lambda} y_\lambda \geqq \bigvee_{\lambda\in\Lambda} (x_\lambda\phi) = (\bigvee_{\lambda\in\Lambda} x_\lambda)\phi$ and $\bigwedge_{\lambda\in\Lambda} y_\lambda \geqq \bigwedge_{\lambda\in\Lambda}(x_\lambda\phi) \geqq (\bigwedge_{\lambda\in\Lambda} x_\lambda)\phi$; therefore $(\bigvee_{\lambda\in\Lambda} x_\lambda, \bigvee_{\lambda\in\Lambda} y_\lambda)$, $(\bigwedge_{\lambda\in\Lambda} x_\lambda, \bigwedge_{\lambda\in\Lambda} y_\lambda) \in J$.

Sufficiency. We shall show that xJ is a principal dual ideal of Q. Since J is a complete sublattice of $P \times Q$, $y_0 = \bigwedge\{y \in Q: y \in xJ\} \in xJ$; hence, $xJ \subset [y_0, 1]$. Now suppose that $y_0 \leqq y$; by (PA 3), $(0, y) \in J$, hence $(x_1, y) = (x \vee 0, y_0 \vee y) = (x_1, y_0) \vee (0, y) \in J$, i.e. $y \in xJ$. Therefore, $xJ = [y_0, 1]$.

Thus, we see that the concept of pair algebra coincides with that introduced for finite lattices in (**4**); residuated mappings coinciding with *admissible functions* are defined in (**7**).

COROLLARY. *If P and Q are complete lattices, $A(P, Q)$ is a complete ring of sets; hence, $S(P, Q)$ is a complete completely distributive lattice. Furthermore, if $P = Q$, then $A(P)$ are lattice-ordered semigroups.*

3. Coordinatization of complete lattices. Let S be a semigroup with zero. Consider the following mappings $R, L: \mathscr{P}(S) \to \mathscr{P}(S)$, for $X \subset S$,

$$\begin{aligned} R(X) &= \{y \in S: xy = 0 \text{ for all } x \in X\},\\ L(X) &= \{y \in S: yx = 0 \text{ for all } x \in X\},\\ R'(X) &= \{e \in S: e = e^2 \text{ and } eS = R(X)\},\\ L'(X) &= \{f \in S: f = f^2 \text{ and } Sf = L(X)\}.\end{aligned}$$

We say that S is a *complete Baer semigroup* if $R'(X) \neq \emptyset \neq L'(X)$ for all $X \subset S$. This is an extension of notions in (**5**; **6**). Let $\mathscr{L}$ ($\mathscr{R}$) denote the collection of all left (right) annihilating ideals of elements of S; i.e.,

$$L = \{L(\{x\}): x \in S\} \quad (\mathscr{R} = \{R(\{x\}): x \in S\}).$$

S is said to coordinatize a lattice P if $P \simeq \mathscr{L}$.

THEOREM 3.1. *Let P be a partially ordered set with least and greatest element. The following conditions are equivalent:*

(i) *P is a complete lattice;*

(ii) *$S(P)$ is a complete Baer semigroup;*

(iii) *P can be coordinatized by a complete Baer semigroup.*

Proof. In view of (**5**, Theorem 2.3), it suffices to establish the following two propositions:

(1) *If S is a complete Baer semigroup, then $\mathscr{L}$ is a complete lattice;*

(2) *If P is a complete lattice, then $S(P)$ is a complete Baer semigroup.*

First, note that R and L are a pair of Galois connection on the complete lattice $\mathscr{P}(S)$; hence, the Galois-closed elements form complete sublattices: $\{R(X): X \subset S\}$ and $\{L(X): X \subset S\}$, respectively. However, $\mathscr{L} = \{L(X): X \subset S\}$, since $R(X) = RLE(X) = R(Sf) = R(\{f\})$ for $f \in L'(R(X))$; therefore, $\mathscr{L}$ is a complete lattice.

Second, note that $S(P)$ is a Baer semigroup, since P is a lattice, which is completely lattice-ordered since P is complete. For any $F \subset S(P)$ we have that $R(F) = R(\vee_{\phi \in F} \phi)$ and $L(F) = L(\vee_{\phi \in F} \phi)$; therefore, $S(P)$ is in fact a complete Baer semigroup.

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