ON REDUCIBLE BRAIDS AND COMPOSITE BRAIDS

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(Received 27 October, 1992)

1. Introduction. The braid group on n strings B_n has a presentation as a group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if} \quad 1 \le i, j \le n-1 \text{ and } |i-j| > 1;$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{for} \quad 1 \le i < n-1.$$

If $0 < m < n$, then we will think of B_m as being the subgroup of B_n generated by $\sigma_1, \ldots, \sigma_{m-1}$. If (α, n) is an n-braid, then its closure will be denoted by $\hat{\alpha}$. Markov's theorem [**1**] says that if two braids have the same closures, then they are equivalent under repetitions of the following two operations;

M_1: replace (α, n) by (β, n) where β is a conjugate of α.

M_2: replace (α, n) by $(\alpha\sigma_n^{\pm 1}, n+1)$ (or vice versa).

The algebraic link problem is to determine whether two braids are equivalent under the Markov "moves". The M_1 equivalence problem (i.e. the conjugacy problem in B_n) has been solved [**2**]. In Section 2 we give an algorithm which determines whether a braid γ in B_n can be written in the form $\alpha\sigma_{n-1}^{\pm 1}$, or, more generally, in the form $\alpha\sigma_{n-1}^{\pm 1}\beta$ with α and β in B_{n-1}. Braids conjugate to such braids are called *reducible*. This thus gives a partial answer to question 6 of [**1**, p. 216].

A braid γ in B_n is called *composite* if it can be represented by a word $U(\sigma_1, \ldots, \sigma_i)V(\sigma_{i+1}, \ldots, \sigma_{n-1})$ for some $0 < i < n-1$. In Section 3 we give an algorithm which determines whether a braid is composite. Note that if γ is a composite braid and U and V represent non-trivial knots, then $\hat{\gamma}$ is a composite knot.

2. The algorithm. Let $\pi_n : B_n \to S_n$ be the permutation representation of B_n onto the symmetric group. Let P_n, the group of *pure braids*, be the kernel of π_n. We will think of braids as "geometric braids" [**1**], where the composition $\alpha\beta$ consists of placing β on top of α. (See Figure 1.) For each subset $I \subset \{1, 2, \ldots, n\}$ there is a map $\varphi_I : B_n \to B_n$ which consists of taking a braid α in B_n and for each $i \in I$ pulling out the string which ends up in the ith position after doing α, and putting it back in as a straight string to the right of all the other strings (and then relabelling the strings accordingly). We have by [**3**]:

LEMMA 2.1. *Let $I \subset \{1, 2, \ldots, n\}$ and $\alpha, \beta \in B_n$. Then $\varphi_I(\alpha\beta) = \varphi_J(\alpha)\varphi_I(\beta)$, where $J = \pi(\beta^{-1})(I)$.*

The next result shows that a braid of the form $\alpha\sigma_{n-1}^{\pm 1}\beta$ can be recovered from its images under the functions φ_i.

LEMMA 2.2. *Let $\gamma = \alpha\sigma_{n-1}^{\epsilon}\beta$, where $\alpha, \beta \in P_{n-1}$ and ϵ is an odd integer, be a braid in B_n. Let $\gamma_1 = \varphi_n(\gamma)$, $\gamma_2 = \varphi_{n-1}(\gamma)$, $\gamma_{21} = \varphi_{n-1}(\gamma_2)$ and $\gamma_{12} = \varphi_{n-1}(\gamma_1)$. Let $\gamma_3 = \gamma_2\gamma_{12}^{-1}\gamma_1$. Then $\gamma_1 = \varphi_{n-1}(\alpha)\beta$, $\gamma_2 = \alpha\varphi_{n-1}(\beta)$, $\gamma_{12} = \gamma_{21} = \varphi_{n-1}(\alpha)\varphi_{n-1}(\beta)$, $\gamma_3 = \alpha\beta$ and*

$$\gamma = \gamma_3\gamma_1^{-1}\sigma_{n-1}^{\epsilon}\gamma_{12}\gamma_2^{-1}\gamma_3.$$

Proof. The first three equalities are easily seen from Figure 1. We thus have

$$\begin{aligned}\gamma_3 &= \gamma_2\gamma_{12}^{-1}\gamma_1 \\ &= (\alpha\varphi_{n-1}(\beta))(\varphi_{n-1}(\alpha)\varphi_{n-1}(\beta))^{-1}(\varphi_{n-1}(\alpha)\beta) = \alpha\beta.\end{aligned}$$

Glasgow Math. J. **36** (1994) 197–199.

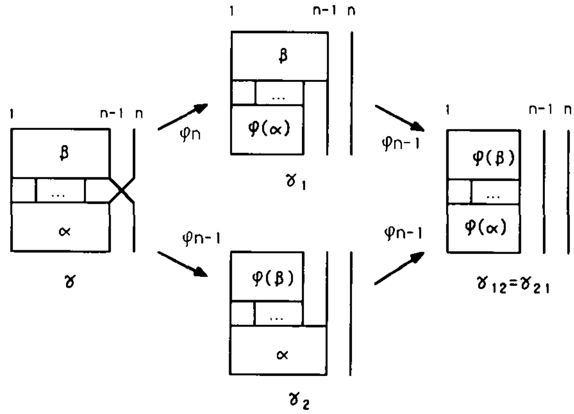

Figure 1.

Note that $\varphi_{n-1}(\alpha)$ and $\varphi_{n-1}(\beta)$ are both in the subgroup of B_n generated by $\sigma_1, \ldots, \sigma_{n-3}$ and so commute with σ_{n-1}. We now have

$$\gamma_3\gamma_1^{-1}\sigma_{n-1}^{\epsilon}\gamma_{12}\gamma_2^{-1}\gamma_3 = (\alpha\beta)(\varphi_{n-1}(\alpha)\beta)^{-1}\sigma_{n-1}^{\epsilon}(\varphi_{n-1}(\alpha)\varphi_{n-1}(\beta))(\alpha\varphi_{n-1}(\beta))^{-1}(\alpha\beta) = \alpha\sigma_{n-1}^{\epsilon}\beta = \gamma.$$

Corollary 2.3. *Let* $\{y_1, \ldots, y_{(n-1)!}\}$ *be a set of coset representatives for* P_{n-1} *in* B_{n-1}. *If* $\delta = \alpha\sigma_{n-1}^{\epsilon}\beta$, *where* $\alpha, \beta \in B_{n-1}$ *and* $\epsilon = \pm 1$ *then there are* $1 \le i, j \le (n-1)!$ *such that if we let* $\gamma = y_i\delta y_j$, $\gamma_1 = \varphi_n(\gamma)$, $\gamma_2 = \varphi_{n-1}(\gamma)$, $\gamma_{21} = \varphi_{n-1}(\gamma_2)$, $\gamma_{12} = \varphi_{n-1}(\gamma_1)$, $\gamma_3 = \gamma_2\gamma_{12}^{-1}\gamma_1$, *then* $\gamma = \gamma_3\gamma_1^{-1}\sigma_{n-1}^{\epsilon}\gamma_{12}\gamma_2^{-1}\gamma_3$.

Proof. Suppose that the cosets y_i, y_j represent the cosets of α^{-1}, β^{-1} respectively. Then $y_i\alpha$ and βy_j are both in P_{n-1} and the result now follows from Lemma 2.2.

We now describe our algorithm for deciding if a braid has the form $\alpha\sigma_{n-1}^{\epsilon}\beta$ for some $\alpha, \beta \in B_{n-1}$. This is based on the above results and the fact that the word problem is solved in B_n.

Step 1. Find a set $\{y_1, \ldots, y_{(n-1)!}\}$ of coset representatives for P_{n-1} in B_{n-1}.

Do Steps $2(i, j)$ for each choice of $i, j = 1, \ldots, (n-1)!$. If δ has the form $\alpha\sigma_{n-1}^{\epsilon}\beta$ for some $\alpha, \beta \in B_{n-1}$, then we shall succeed in showing this for some value of i and j, by Corollary 2.3. If not, then δ does not have this form.

Step 2(i, j). Let $\gamma_{i,j} = y_i\delta y_j$. Calculate $\gamma_1 = \varphi_n(\gamma_{i,j})$, $\gamma_2 = \varphi_{n-1}(\gamma_{i,j})$, $\gamma_{12} = \varphi_{n-1}(\gamma_1)$, $\gamma_3 = \gamma_2\gamma_{12}^{-1}\gamma_1$ and $\tau(\epsilon) = \gamma_3\gamma_1^{-1}\sigma_{n-1}^{\epsilon}\gamma_{12}\gamma_2^{-1}\gamma_3$. Here we calculate the braids $\varphi_n(\gamma_{i,j})$, $\varphi_{n-1}(\gamma_{i,j})$, etc. using Proposition 2.3 of **[3]**. Apply the solution of the work problem found in **[2]** or **[1]** to determine whether the words $\tau(\epsilon)$ and $\gamma_{i,j}$ are equal for $\epsilon = \pm 1$.

Remark. The above algorithm can easily be modified to detect words of the form (a) $\alpha\sigma_{n-1}^{\epsilon}\beta$ for any odd ϵ where $\alpha, \beta \in B_{n-1}$; or (b) $\alpha\sigma_i^{\epsilon}\beta$ for any odd ϵ, $0 < i < n$, and words α, β not involving σ_i.

3. The algorithm for composite braids. If $m < n$, then we will think of B_m as being a subgroup of B_n. If $n > i + j - 1$, then we let $\Pi_i : B_j \to B_n$ be the monomorphism determined by $\Pi_i(\sigma_k) = \sigma_{k+i-1}$, for all $k = 1, \ldots, j-1$.

LEMMA 3.1. *Let* $\gamma = U(\sigma_1, \ldots, \sigma_i)V(\sigma_{i+1}, \ldots, \sigma_{n-1})$ *for some* $0 < i < n-1$ *be a composite braid where* U, V *are pure braids. Let* $\gamma_1 = \varphi_{\{i+2,\ldots,n\}}(\gamma)$, $\gamma_2 = \varphi_{\{1,\ldots,i\}}(\gamma)$. *Then* $\gamma_1 = U(\sigma_1, \ldots, \sigma_i)$, $\Pi_{i+1}(\gamma_2) = V(\sigma_{i+1}, \ldots, \sigma_{n-1})$ *and* $\gamma = \gamma_1 \Pi_{i+1}(\gamma_2)$.

Proof. By Lemma 2.1 we have

$$\gamma_1 = \varphi_{\{i+2,\ldots,n\}}(\gamma) = \varphi_{\{i+2,\ldots,n\}}(U(\sigma_1, \ldots, \sigma_i))\varphi_{\{i+2,\ldots,n\}}(V(\sigma_{i+1}, \ldots, \sigma_{n-1})) = U(\sigma_1, \ldots, \sigma_i);$$

$$\gamma_2 = \varphi_{\{1,\ldots,i\}}(\gamma) = \varphi_{\{1,\ldots,i\}}(U(\sigma_1, \ldots, \sigma_i))\varphi_{\{1,\ldots,i\}}(V(\sigma_{i+1}, \ldots, \sigma_{n-1})) = \Pi_{i+1}^{-1}(V(\sigma_{i+1}, \ldots, \sigma_{n-1})).$$

As in 2.3 above we now obtain

COROLLARY 3.2. *Let* $0 < i < n-1$ *and let* $\{y_1, \ldots, y_{(i+1)!}\}$ *be a set of coset representatives for* P_{i+1} *in* B_{i+1} *and let* $\{z_1, \ldots, z_{(n-i)!}\}$ *be a set of coset representatives for* $\Pi_{i+1}(P_{n-i})$ *in* $\Pi_{i+1}(B_{n-i})$. *Suppose that* $\gamma = U(\sigma_1, \ldots, \sigma_i)V(\sigma_{i+1}, \ldots, \sigma_{n-1})$. *Then there are* j, k *such that if* $\delta = y_j \gamma z_k$, $\delta_1 = \varphi_{\{i+2,\ldots,n\}}(\delta)$, *and* $\delta_2 = \varphi_{\{1,\ldots,i\}}(\delta)$, *then* $\delta_1 = U(\sigma_1, \ldots, \sigma_i)$, $\Pi_{i+1}(\delta_2) = V(\sigma_{i+1}, \ldots, \sigma_{n-1})$ *and* $\delta = \delta_1 \Pi_{i+1}(\delta_2)$.

We now describe our algorithm for deciding if a braid is composite. Do Step 1(i) for each $0 < i < n-1$.

Step 1(i). Find a set $\{y_1, \ldots, y_{(i+1)!}\}$ of coset representatives for P_{i+1} in B_{i+1} and a set $\{z_1, \ldots, z_{(n-i)!}\}$ of coset representatives for $\Pi_{i+1}(P_{n-i})$ in $\Pi_{i+1}(B_{n-i})$.

For fixed i do Steps $2(k, j)$ for each $0 < k \leq (i+1)!$ and each $0 < j \leq (n-i)!$. If γ is composite, then we shall succeed for some values of i, k, j, by Corollary 3.2. If we do not succeed at all, then γ is not composite.

Step 2(k, j). Let $\delta_{k,j} = y_k \gamma z_j$. Calculate $\delta_1 = \varphi_{\{i+2,\ldots,n\}}(\delta_{k,j})$, and $\delta_2 = \varphi_{\{1,\ldots,i\}}(\delta_{k,j})$. Again we calculate the braids $\varphi_{\{i+2,\ldots,n\}}(\delta_{k,j})$, etc. using Proposition 2.3 of [**3**]. Apply the solution of the word problem found in [**2**] or [**1**] to determine whether the words $\delta_1 \Pi_{i+1}(\delta_2)$ and $\delta_{k,j}$ are equal.

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