Focus on Fluids journals.cambridge.org/focus

Three coins in a fountain

H. K. Moffatt†

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

If, in a large expanse of fluid such as air or water, an object that is heavier than the fluid displaced is released from rest, it descends in a manner that can depend in a complex way on its geometry and density (relative to that of the fluid), and on the fluid viscosity, which, as in other fluid contexts, remains important no matter how small this viscosity may be. A major numerical attack on this problem for the case in which the object is a thin circular disc is presented by Auguste, Magnaudet & Fabre (*J. Fluid Mech.*, vol. 719, 2013, pp. 388–405).

Key words: flow–structure interactions, vortex shedding, wakes

1. Introduction

Anyone who, following tradition or superstition, has dropped a light coin into the Trevi Fountain in Rome will have noticed that the manner of its descent is highly irregular. Fairground operators frequently exploit this irregularity as a means of extracting money from their more gullible customers. To understand the dynamics of descent of a body of this kind through a fluid medium is the subject of a paper entitled 'Falling styles of discs' by Auguste, Magnaudet & Fabre (2013), which makes significant progress towards such understanding, although for ranges of parameters that do not as yet extend to the descent of coins in the Trevi Fountain. The spinning descent of a samara (the winged seed of trees such as the sycamore, *Acer pseudoplatanus*) is often cited as an example of this type of phenomenon, although here of course the geometry of the seed is responsible for the particular style of descent that is observed (Norberg 1973). At another extreme, the free fall of a spacecraft through a rarified atmosphere is governed essentially by the same principles of fluid and rigid-body dynamics.

This type of problem attracted the attention of no less than James Clerk Maxwell, who in 1853 (at the age of 22) published a paper whose opening paragraph reads:

Everyone must have observed that when a slip of paper falls through the air, its motion, though undecided and wavering at first, sometimes becomes regular. Its general path is not in the vertical direction, but inclined to it at an angle which remains nearly constant, and its fluttering appearance will be found to be due to a rapid rotation about a horizontal axis ....

† Email address for correspondence: hkm2@damtp.cam.ac.uk

Any stairwell provides an appropriate setting for repetition of this experiment: some height is needed for the 'regular' state to be established. Maxwell's discussion of the phenomenon and the physical insight that he thereby displayed was remarkable, given that the effects of the viscosity of fluids such as air were then as yet but dimly understood. It was his first venture into fluid mechanics, a field to which he later made a number of seminal contributions that deserve to be better known (Moffatt 2013).

The motion of a rigid body through a fluid in fact provides one of the central problems of fluid mechanics and aerodynamics, and as such has been studied intensively over the past 150 years and more. Viscous effects are recognized to be of crucial importance, even in the limit of vanishingly small fluid viscosity; without viscosity, the flight of bats, birds, bees and aircraft would be impossible! A rigid body such as a sphere or cylinder fixed in a uniform stream of air or water, or any other fluid, experiences a drag force associated with the rotational flow (or the 'vorticity') in the downstream wake. This vorticity is created through the agency of viscosity in a thin boundary layer on the surface of the body, and is then swept into the wake through a physically self-evident, though mathematically complex, 'separation' mechanism (Stewartson 1974). The famous 'von Kármán vortex street' shed from a cylinder is perhaps the best-known example – for beautiful images of this phenomenon, see Zdravkovich (1969), but there are many variants of this.

If the body is not fixed, but is free to move (as for the coin in the fountain) in response to the net force and torque (generally time-dependent) exerted on it by the fluid, together with the force of gravity, then there is a complex dynamical interaction between the fluid and the the body, governed, on the one hand, by the Navier–Stokes equations in the fluid, and, on the other, by the equations of rigid-body dynamics determining the motion of the body (generally unsteady) in response to the above force and torque. This problem, being fully three-dimensional, unsteady and nonlinear, is far beyond the reach of purely theoretical analysis, and has only recently become amenable to investigation through the use of powerful computational techniques. Much experimental evidence is available in a wide range of circumstances; the challenge is to reproduce experimental observations by computation, to investigate by this means the detailed flow structures, particularly in the vortical wake, and to extend such computations to regimes of parameter space where experimental results are not available, or extremely hard to come by.

2. How thin is 'thin'?

The three essential parameters of the disc problem are: (i) the ratio $\chi = d/h$ of the diameter d of the disc to its thickness h; (ii) the inertia ratio $I^* = \pi\rho_s/64\rho\chi$, where ρ and ρ_s are the fluid and solid densities, respectively; and (iii) the 'Archimedes' number defined by Auguste *et al.* (2013) as $Ar = \sqrt{3|\rho_s/\rho - 1|gh}(d/4\nu)$, where ν is the kinematic viscosity of the fluid. The Archimedes number, like a Reynolds number, here provides the relevant measure of inertial, as compared with viscous, forces. (The disc, of course, rises rather than falls if $\rho_s < \rho$.) Auguste *et al.* focus for the most part on thin discs ($\chi \geqslant 10$), and on the parameter ranges $0.0015 < I^* < 0.5$ and $0 < Ar < 110$. For a one-pound (UK) coin falling through water, $\chi \approx 10$ and $I^* \sim 0.35$; however, $Ar \sim 3 \times 10^3$, and this, as indicated above, is well beyond the range covered by the computations. (A deep ocean rather than the Trevi Fountain would in fact be required for the coin to reach anything like Maxwell's 'regular' state of motion.) At the much smaller Archimedes number $Ar = 80$ (i.e. much increased viscosity, other things being equal), the rise of a buoyant disc for which $\rho_s/\rho = 0.99$ and $\chi = 3$, as computed by Auguste (2010), is shown in figure 1. This rise is periodic in character, and is described, for obvious reasons, as the 'zig-zag' mode. The first question addressed by Auguste *et al.* (2013) is this: How large must χ be for the disc to be treated as 'infinitely thin'? In comparing computed results for $\chi = 10$ and $\chi = \infty$ (again, other things being equal), significant differences are found. For example, when $\chi = 10$, the zig-zag mode appears only for $Ar > 63.5$, whereas when $\chi = \infty$, the zig-zag mode is present for $33 \leqslant Ar \leqslant 45$, and for $Ar \geqslant 46$, a 'hula-hoop' mode (a combination of zig-zag and precession about a vertical axis) takes over. Thus, contrary to previous beliefs, it would appear that χ must be considerably greater than 10 before the behaviour becomes 'χ-independent'. A possible reason for this, as argued by the authors, is that the low-Reynolds-number flow round a sharp disc edge (when $\chi = \infty$) is quite different from that round the 90° corner encountered when $\chi < \infty$. Just how large χ has to be before the 'infinitely thin' idealization becomes accurate remains an intriguing problem.

FIGURE 1. Zig-zag mode (Auguste 2010).

3. From zig-zag to hula-hoop

The next questions addressed are these: For the infinitely thin disc ($\chi = \infty$), what 'mode of motion' is realized for given values of Ar and I^*? And what transitions between different modes are possible as we change these parameters continuously? It appears that there are at least six distinct modes, the first four of which have been identified experimentally in previous work (e.g. Field *et al.* 1997): (i) the straight vertical (SV) mode, realized at sufficiently small Ar, in which the disc is horizontal and the centre descends on a straight vertical path; (ii) the zig-zag (ZZ) mode shown in figure 1; (iii) an autorotation (AR) mode, for which the disc rotates about a horizontal axis and drifts on an inclined path (this is apparently the mode discovered by Maxwell with his slip of paper); (iv) an intermediate chaotic regime (ZZ/AR), being a mixture of periods of ZZ and AR; (v) a hula-hoop mode (HH), as described above; and (vi) a 'helical autorotation' (HA) mode. These last two have been discovered computationally and described by Auguste *et al.* (2013) as *characterized by a slow precession of the plane within which the disc falls, while zig-zagging in the former case or tumbling in the latter*. Auguste *et al.* (2013) have explored the (Ar, I^*) plane, mapping out the regions in which each of the above six modes is realized. For example, if Ar is increased at constant $I^* \approx 0.03$, the disc undergoes the successive mode transitions SV $\rightarrow$ ZZ $\rightarrow$ ZZ/AR $\rightarrow$ AR. Moreover, the first of these transitions SV $\rightarrow$ ZZ is subcritical in character: it occurs for finite-amplitude perturbations, and the SV mode remains stable to infinitesimal disturbances for a range of Ar beyond the bifurcation value Ar_c.

A major conclusion of Auguste *et al.* (2013) is that the coupled disc–fluid system is much more unstable than the flow system in which a disc is fixed, and that there is a great richness and diversity in the various mode changes that can occur as either the inertia parameter I^* or the Archimedes number Ar increases. (Incidentally, one may question the attribution of this number to Archimedes, who was aware of buoyancy

FIGURE 2. Three coins: 1 pound (UK); 5 Yen (Japan); and 20 cents (Hong Kong, 1977).

but not of viscosity! The number is really a Reynolds number, Re_g say, based on the diameter d and a gravitational velocity U_g proportional to $\sqrt{gh}$.)

4. Why three? 'Disc'ussion

The song 'Three coins in a fountain' was a top hit, sung by Frank Sinatra, in 1954 (the lyrics may still be found on YouTube). The three coins illustrated in figure 2, representative of those that can be found in the Trevi Fountain, perhaps suggest three questions for future investigation. (i) Do the roughness of a disc and of its rim (if serrated) influence its mode of descent in a tank of water? (ii) What if the disc has a hole in it? It is to be expected that this may modify the wake, and so the various instabilities, quite dramatically. (iii) What if the disc has a wavy edge? Since the vortex shedding process occurs at the edge, this also may be expected to have a strong effect. Are there additional modes as yet undiscovered? That seems more than likely! With increasing Reynolds number, does the wake become fully turbulent? If so, does the disc respond perceptibly to this turbulence. Much work remains for continuing investigation of these variations on a fascinating theme!

References

AUGUSTE, F. 2010 Instabilité de sillage et trajectoires d'un corps solide cylindrique immergé dans un fluide visqueux. PhD thesis, Université Paul Sabatier, Toulouse, France. Available at https://www.imft.fr/Projet-ANR-OBLIC.

AUGUSTE, F., MAGNAUDET, J. & FABRE, D. 2013 Falling styles of discs. *J. Fluid Mech.* **719**, 388–405.

ERN, P., RISSO, F., FABRE, D. & MAGNAUDET, J. 2012 Wake-induced oscillatory paths of bodies freely rising or falling in fluids. *Annu. Rev. Fluid Mech.* **44**, 97–121.

FIELD, S., KLAUS, M., MOORE, M. & NORI, F. 1997 Chaotic dynamics of falling discs. *Nature* **388**, 252–254.

MAXWELL, J. C. 1853 On a particular case of the descent of a heavy body in a resisting medium. In *The Scientific Papers of James Clerk Maxwell*, vol. 1, pp. 115–118. Cambridge University Press (2010).

MOFFATT, H. K. 2013 The fluid dynamics of James Clerk Maxwell. In *James Clerk Maxwell* (ed. R. Flood, M. McCartney & A. Whitaker), (in press) Oxford University Press.

NORBERG, R. A. 1973 Autorotation, self-stability and structure of single-winged seeds (samaras) with comparative remarks on animal flight. *Biol. Rev.* **48**, 561–596.

STEWARTSON, K. 1974 Multistructured boundary layer on flat plates and related bodies. *Adv. Appl. Mech.* **14**, 145–239.

ZDRAVKOVICH, M. M. 1969 Smoke observations of the formation of a von Kármán vortex street. *J. Fluid Mech.* **37**, 491–496.