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LETTERS TO THE EDITOR

ANOTHER LOOK AT THE EHRENFEST URN VIA ELECTRIC NETWORKS

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Abstract

Using the electric network approach, we give closed-form formulas for the expected hitting times in the Ehrenfest urn model.

FIRST PASSAGE; EFFECTIVE RESISTANCE

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1. Introduction

The Ehrenfest model consists of d balls distributed between two urns, I and II; at each time $n = 0, 1, \cdots$ a ball is chosen—each with probability $1/d$—and changed to the other urn. If we use a 'full description' with 2^d states representing the possible configurations by d-tuples $i = (i_1, \cdots, i_d)$, where $i_k = 1, 0$ if ball k is in urn I, II, one may identify the states with the vertices of the d-cube, and the evolution of the Ehrenfest model is that of a simple random walk on the cube.

In general, a simple random walk on a finite connected undirected graph, $G = (V, \boldsymbol{E})$, is the Markov chain X_n, $n \geqq 0$, that from its current vertex v jumps to one of the $d(v)$ neighboring vertices with uniform probability.

The hitting time (or first passage) T_v of the vertex v is the minimum number of steps the random walk takes to reach that vertex: $T_v = \inf\{n \geqq 0 : X_n = v\}$. The expected value of T_v when the walk starts at the vertex w is denoted by $\boldsymbol{E}_w T_v$. The commute time between vertices i and j is $\boldsymbol{E}_i T_j + \boldsymbol{E}_j T_i$.

We presented in Palacios (1993) an alternative way (see Bingham (1991)) to compute the hitting time $\boldsymbol{E}_0 T_d$ for the unit cube and other graphs, based on the electric approach (basic reference Doyle and Snell (1984)) to random walks on graphs. Here is a summary of the method. First we use the following formula due to Chandra et al. (1989) involving the commute time between a and b and the effective resistance R_{ab} between those two vertices when every edge of the graph is considered to be a unit resistor:

$$\boldsymbol{E}_a T_b + \boldsymbol{E}_b T_a = 2\,|\boldsymbol{E}|\,R_{ab}. \tag{1}$$

If we can ensure further that $\boldsymbol{E}_a T_b = \boldsymbol{E}_b T_a$, for instance under some symmetry assumption, then (1) simplifies to

$$\boldsymbol{E}_a T_b = |\boldsymbol{E}|\,R_{ab}. \tag{2}$$

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Next we argue that (1) and (2) enable one to compute the commute time—and hitting time in the presence of symmetry—not only of a single vertex but also of a *set* of vertices because one can short together those vertices (they have 'zero voltage', see Doyle and Snell (1984), p. 53) into a single vertex. Finally we argue that one can short together all vertices sharing the same potential in order to take advantage of the symmetry of the problem.

Now we look at the d-cube: if we start the walk at any vertex that we relabel as the 'origin' $(0, \cdots, 0)$, the vertex it takes longest to hit (the 'opposite' vertex) is $(1, \cdots, 1)$. If we apply a unit voltage between these two vertices so that the voltage at $(0, \cdots, 0)$ is 1 and the voltage at $(1, \cdots, 1)$ is 0, then all vertices having the same number of 1's share the same voltage and can be shorted. For a picture of the effect of doing this on the 3-cube, and for further details, see Palacios (1993).

After shorting, we obtain a new graph with $d+1$ vertices, where the kth new vertex consists of the shorting of all vertices in the unit cube with k 1's. Since every vertex in the unit cube with k 1's is connected to $d-k$ vertices with $k+1$ 1's, there are $(d-k)\binom{d}{k} = d\binom{d-1}{k}$ resistors between vertex k and $k+1$ in the new graph, $0 \leqq k \leqq d-1$. Then (2) yields that the expected hitting time of the opposite vertex in the cube (i.e. the expected time to move all balls from urn II to urn I) is:

$$\mathbf{E}_{(0,\cdots,0)}T_{(1,\cdots,1)} = \mathbf{E}_0 T_d = |\mathbf{E}|\, R_{0d} = d2^{d-1} \sum_{k=0}^{d-1} \frac{1}{d\binom{d-1}{k}} = 2^{d-1} \sum_{k=0}^{d-1} \frac{1}{\binom{d-1}{k}}. \tag{3}$$

2. Hitting times of the Ehrenfest urn

The only drawback of formula (2) is that it requires symmetry, and thus can only be applied to 'opposite vertices' of the unit cube. However, there are more general one-sided hitting time formulas, also in terms of effective resistances, due to Tetali (1991):

$$\mathbf{E}_a T_b = \tfrac{1}{2} \sum_z d(z)[R_{ab} + R_{bz} - R_{az}] \tag{4}$$

or equivalently

$$\mathbf{E}_a T_b = |\mathbf{E}|\, R_{ab} + \tfrac{1}{2} \sum_z d(z)[R_{bz} - R_{az}]. \tag{5}$$

Here R_{xy} is the effective resistance between vertices x and y and $d(z)$ is the degree of vertex z. (R_{yy} is taken to be 0 for all y.)

If we are interested in the 'reduced description' of the Ehrenfest model, with $d+1$ states representing the possible numbers of balls in urn I, instead of looking at the cube (the full description model) we can look at the shorted graph, described in the introduction, and notice that the evolution of the reduced Ehrenfest model is that of the random walk on the (non-simple) shorted graph.

Let us recall from the introduction that there are $d\binom{d-1}{k}$ edges (resistors) between vertices k and $k+1$ in the shorted graph, $0 \leqq k \leqq d-1$, and therefore the degree of vertex k is $d(k) = d\binom{d}{k}$. It is also plain to see that for the shorted graph, the total number of edges is $|\mathbf{E}| = d2^{d-1}$. These observations yield immediately Kac's formulas for the recurrence times

$\mathbf{E}_k T_k$, because these are the inverses of the values of the stationary distribution at the vertex k, and in the case of random walks on graphs, this is $2|\mathbf{E}|/d(k)$, so that we have

$$\mathbf{E}_k T_k = \frac{\binom{d}{k}}{2^d} \qquad 0 \leqq k \leqq d.$$

Next we will compute $\mathbf{E}_{k-1} T_k$, $1 \leqq k \leqq d$. Since the objective of the walk started at $k-1$ is to hit vertex k, we can short everything beyond vertex k (i.e. we short vertices k, $k+1, \cdots, d$), and we apply formula (5) taking into consideration that in this new graph

$$d(z) = d\binom{d}{z} \qquad \text{for } 1 \leqq z \leqq k-1,$$

$$d(k) = d\binom{d-1}{k-1} \quad \text{(because of the shorting)},$$

$$|\mathbf{E}| = \sum_{j=0}^{k-1} d\binom{d-1}{j},$$

$$R_{kz} - R_{k-1,z} = \begin{cases} R_{k-1,k} & \text{if } 0 \leqq z \leqq k-1 \\ -R_{k-1,k} & \text{if } z = k \end{cases},$$

and

$$R_{k-1,k} = \frac{1}{d\binom{d-1}{k-1}}.$$

Then (5) yields

$$\mathbf{E}_{k-1} T_k = \sum_{j=0}^{k-1} \binom{d-1}{j} \frac{1}{\binom{d-1}{k-1}} + \frac{1}{2}\left[\sum_{j=0}^{k-1} \binom{d}{j} \frac{1}{\binom{d-1}{k-1}} - 1\right].$$

The above expression can be reduced by taking everything into a single summation yielding

$$\mathbf{E}_{k-1} T_k = \frac{\sum_{j=0}^{k-1} \binom{d}{j}}{\binom{d-1}{k-1}}. \tag{6}$$

In particular, taking $k = d$ in (6) tells us how hard it is for an urn, full but for one ball, to get that very last ball:

$$\mathbf{E}_{d-1} T_d = 2^d - 1. \tag{7}$$

One could derive this last result also by using the fact (see Palacios (1992)) that the hitting time between adjacent vertices in a symmetric graph, such as the cube, with N vertices is $N-1$. Equation (6) allows us to obtain an explicit formula for any hitting time $\mathbf{E}_i T_j$, $i \neq j$ for the Ehrenfest urn simply by noticing that, for $i < j$

$$\mathbf{E}_i T_j = \sum_{k=i}^{j-1} \mathbf{E}_k T_{k+1},$$

and that by exchanging the roles of urns I and II we have:

$$\boldsymbol{E}_i T_j = \boldsymbol{E}_{d-i} T_{d-j}.$$

Thus, for instance:

$$\boldsymbol{E}_0 T_r = \sum_{k=1}^{r} \frac{\sum_{j=0}^{k-1} \binom{d}{j}}{\binom{d-1}{k-1}}. \tag{8}$$

These results for the reduced description model also provide some limited information for the random walk on the cube: formula (8) for example, provides the expected time, starting from a given vertex v, to hit the set of vertices at a distance r from v.

Notice that from (6) we can derive the following recurrence for these 'consecutive' hitting times:

$$\boldsymbol{E}_k T_{k+1} = \frac{k}{d-k} \boldsymbol{E}_{k-1} T_k + \frac{d}{d-k}, \tag{9}$$

with the initial condition $\boldsymbol{E}_0 T_1 = 1$. Of course, one can derive (9) from first principles, but it seems harder to try to derive the solution to this recurrence, as is expressed in (6), from first principles than to proceed as we did with the electric approach. In fact, Blom (1989) expressed the solution to (9) as an integral, and thereby he obtained explicit expressions for $\boldsymbol{E}_0 T_{d/2}$ and $\boldsymbol{E}_0 T_d$, but not for $\boldsymbol{E}_{k-1} T_k$. Aldous (1982) studied the continuous version of the random walk on the d-cube and found the mean hitting times to be of the form $\boldsymbol{E}_i T_j = f(|i-j|)$, where $|i-j|$ denotes the number of coordinates where i and j differ, and f can be found from the recursion (beware of the typo):

$$f(0) = 0, \qquad f(1) = 2^d - 1, \qquad f(r+1) = [d(f(r) - 1) - rf(r-1)]/(d-r).$$

This recursion is reminiscent of our (9), though more involved because it deals with the full description model; since the mean holding time at each vertex is 1, the values of the mean hitting times are the same for the continuous and discrete models, and thus the formula $f(1) = 2^d - 1$ of Aldous implies (7).

From formula (6) we can deduce that the expected times to increase by one the count of balls in urn I, $E_{k-1} T_k$, form a strictly increasing sequence in k. Simply notice that if $0 \leqq s \leqq k-2$:

$$\frac{\binom{d}{s}}{\binom{d-1}{k-1}} < \frac{\binom{d}{s+1}}{\binom{d-1}{k}},$$

and for $s = k-1$ the inequality becomes an equality. This increase is slow for $1 \leqq k \leqq d/2$ because both numerator and denominator in (6) increase, whereas for $k > d/2$ the increase is fast because while the numerator increases, the denominator decreases. Thus $\boldsymbol{E}_0 T_{d/2}$ is much smaller than $\boldsymbol{E}_0 T_k$ for $k > d/2$, as observed by Blom (1989) for $k = d$. This fact reflects the rapidly-mixing character of random walk on the d-cube for large d, as is explained in detail in example 5.1 of Aldous (1983).

Kemperman (1961) was the first to give a complete treatment of the hitting times of the Ehrenfest urn model, although his formulas in terms of Kac's coefficients are far less explicit than formula (6) and his derivations more involved. The coefficients c_{β_j} of Kac are defined by the equation

$$(1-w)^{\beta}(1+w)^{d-\beta} = \sum_j c_{\beta_j} w^j \qquad (|w| < 1, \beta \text{ real}),$$

the expected hitting times are expressed as

$$\boldsymbol{E}_i T_k = \frac{1}{\binom{d}{k}} \sum_{r=1}^{d} (c_{kr} - c_{ir}) c_{rk} \binom{d}{2r},$$

and explicit formulas can be derived only when both i and k assume one of the values 0, d or $d/2$.

The Kac coefficients are related to the *Krawtchouk polynomials,* defined by the formula

$$K_n(x, p, N) = \sum_{\nu=0}^{n} (-1)^{\nu} \frac{\binom{n}{\nu}\binom{x}{\nu}}{\binom{N}{\nu}} \frac{1}{p^{\nu}}, \qquad n = 0, 1, \cdots, N.$$

Indeed, one has

$$c_{\beta_j} = \binom{d}{j} K_j(\beta, \tfrac{1}{2}, d).$$

The Krawtchouk polynomials are orthogonal polynomials of interest in their own right (see Stanton (1984) for background, and Karlin and McGregor (1965) for expressions of the expected hitting times of Ehrenfest urns in terms of Krawtchouk polynomials), and perhaps the connection between these and the electric approach used here deserves further examination.

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