MINIMUM AND CONJUGATE POINTS IN SYMMETRIC SPACES

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The purpose of this paper is to discuss conjugate points in symmetric spaces. Although the results are neither surprising nor altogether unknown, the author does not know of their explicit occurrence in the literature.

Briefly, conjugate points in the tangent bundle to the tangent space at a point of a symmetric space are characterized in terms of the algebraic structure of the symmetric space. It is then shown that in the simply connected case the first conjugate locus coincides with the minimum (cut) locus. The interest in this last fact lies in its identification of a more or less locally and analytically defined set with one which includes all the topological interest of the space.

1. Preliminaries. In this section the definitions of and basic results in symmetric spaces and their algebraization are given. This material is described in detail in **(1)** and, in part, in **(5)**.

By a symmetric space shall be meant a Riemannian manifold M with the property that about every point $m \in M$ the symmetry σ_m, taking each geodesic through m into itself by reversing its direction, is a well-defined isometry of M. Such a space is Riemannian homogeneous, and many of its properties can be realized in terms of the Lie algebra of the transitive group of isometries. In particular, a symmetric space gives rise to an *orthogonal involutive Lie algebra* (*oila*), which consists of the following objects: a real Lie algebra $\mathbf{G}$, an involution S of $\mathbf{G}$, and an inner product Q on $\mathbf{G}$, these objects being subject to the following conditions: if $\mathbf{H} = \{X \in \mathbf{G}|S(X) = X\}$, then Q is invariant under $\mathrm{ad}_{\mathbf{G}}\mathbf{H}$, and $\mathbf{H}$ contains no proper ideal of $\mathbf{G}$. Let $\mathbf{M} = \{X \in \mathbf{G}|S(X) = -X\}$, so that $\mathbf{G} = \mathbf{H} + \mathbf{M}$.

If $M = G/H$ is a symmetric space viewed as a homogeneous space of its group of isometries, $m = eH$, then the corresponding *oila* consists of $\mathbf{G}$, the Lie algebra of G, $\mathbf{H}$, the Lie algebra of H, S, the differential of the inner automorphism of G by σ_m, that is, $S = \mathrm{ad}_{\mathbf{G}}\sigma_m$, and Q, the inner product on M_m (the tangent space to M at m) pulled back to $\mathbf{M}$ and extended to $\mathbf{G}$. Geodesics through m are orbits of m under one-parameter groups tangent to $\mathbf{M}$. Let β be the Killing form on $\mathbf{G}$. Since $\mathbf{H}$ is compact, β is negative definite on $\mathbf{H}$. If M_m and $\mathbf{M}$ are identified, and if $\mathbf{G}$ is semi-simple and $\mathbf{M}$ is irreducible under $\mathrm{ad}\mathbf{H}$, then the Riemannian curvature K of M at m is given by:

Received February 16, 1961. This work was supported in part by Air Force Contracts at the Massachusetts Institute of Technology.

$$K(X, Y) = \frac{1}{\lambda} \frac{\beta([X, Y], [X, Y])}{\{Q(X,X)Q(Y, Y) - Q(X, Y)^2\}^{1/2}} \tag{1}$$

where $X, Y \in \mathbf{M} \approx M_m$ and λ is a constant. It turns out that $\mathbf{G}$ is compact if and only if λ is negative, and hence $\mathbf{G}$ is compact if and only if M has non-negative curvature. Therefore, it will be assumed henceforth that $\mathbf{G}$ is semi-simple and compact.

2. Cartan subalgebras

DEFINITION. A *Cartan subalgebra* of an *oila* $\mathbf{G} = \mathbf{H} + \mathbf{M}$ is a maximal subalgebra $\mathbf{P}$ of $\mathbf{M}$. Of necessity, $\mathbf{P}$ is abelian. An element $X \in \mathbf{P}$ is called *regular* if $\mathbf{C}(X) \cap \mathbf{M} = \mathbf{P}$, where $\mathbf{C}(X)$ is the centralizer of X. Regular elements always exist. If $X \in \mathbf{P}$ is not regular, then it is called *singular*.

A principal fact concerning Cartan subalgebras is contained in the following **(1)**:

THEOREM 1. *If $M = G/H$, and $\mathbf{P}$ and $\mathbf{P}'$ are Cartan subalgebras of the corresponding oila, then there exists an element $k \in H$ such that* $\operatorname{ad}k(\mathbf{P}) = \mathbf{P}'$ *and* $\operatorname{ad}k|_{\mathbf{P}\cap\mathbf{P}'} =$ *identity.*

The dimension of a Cartan subalgebra is called the *rank* of the symmetric space **(1**, p. 41**)**.

Let G/H be a symmetric space, $\mathbf{G} = \mathbf{H} + \mathbf{M}$, and let S denote the involution both on G and $\mathbf{G}$, so $H =$ fixed points of S, although H is not necessarily connected. Denote the composition

$$\mathbf{G} \xrightarrow{\approx} G_e \xrightarrow{\exp_e} G$$

by $e^{\cdot}$. Let $M = e^{\mathbf{M}}$. Then it turns out that M is a symmetric space as a homogeneous space of G under the action $g(m) = gm(S(g))^{-1}$, where $g \in G$, $m \in M$. H is the isotropy group at e, and it acts by inner automorphism. M acts on itself with the action $m(n) = mnm$, $m, n \in M$. M is, of course, the same symmetric space as G/H, but the identification is not by means of the restriction of the projection $G \to G/H$ to M. Rather, it is given by:μ: $G/H \xrightarrow{\approx} M$, where $\mu(gH) = g(S(g))^{-1}$. Equivalently, if $m \in M$ and if $m^{1/2}$ is any square root of m, then $\mu^{-1}(m) = m^{1/2}H$. Geodesics in M through e are one-parameter subgroups of G lying in M. So the following diagram is commutative **(1)**:

$$\begin{array}{ccc} (G/H)_{eH} & \approx & \mathbf{M} \\ \downarrow \exp_{eH} & & \downarrow e^{\cdot}|_{\mathbf{M}} \\ G/H & \xrightarrow[\mu]{\approx} & M \end{array} \tag{2}$$

Assume now that G is compact and semi-simple, as usual, and that G/H is a symmetric space, so that $\mathbf{G} = \mathbf{H} + \mathbf{M}$. Let $\mathbf{P}$ be a Cartan subalgebra. Then $\mathbf{G}$ admits the following decomposition with respect to $\operatorname{ad}_{\mathbf{G}}\mathbf{P}$:

$$\mathbf{G} = \mathbf{Z}_0 + \mathbf{P} + \sum_{\alpha} V_\alpha, \tag{3}$$

where $\mathbf{Z}_0 = \mathbf{C}(\mathbf{P}) \cap \mathbf{H}$, the α are not necessarily distinct linear functionals on $\mathbf{P}$, and each V_α is a 2-dimensional invariant subspace such that with respect to a suitable basis

$$\mathrm{ad}_{\mathbf{G}}\mathbf{P}|_{V\alpha} = \begin{pmatrix} 0 & 2\pi\alpha \\ -2\pi\alpha & 0 \end{pmatrix}. \tag{4}$$

The α are *roots* of the Cartan subalgebra $\mathbf{P}$. Since $\mathbf{G}$ is semi-simple, the α span $\mathbf{P}^*$, the space dual to $\mathbf{P}$. The following facts will be used:

(i) If $P = e^{\mathbf{P}}$, then

$$\mathrm{ad}_{\mathbf{G}}P|_{V\alpha} = \begin{pmatrix} \cos(2\pi\alpha \log) & \sin(2\pi\alpha \log) \\ -\sin(2\pi\alpha \log) & \cos(2\pi\alpha \log) \end{pmatrix}. \tag{5}$$

(ii) If

$$\mathbf{V}_\alpha = \sum_{\beta=\pm\alpha} V_\beta,$$

then

$$\mathbf{V}_\alpha = \mathbf{V}_\alpha \cap \mathbf{H} + \mathbf{V}_\alpha \cap \mathbf{M}. \tag{6}$$

(iii) There is a connection between the roots of a Cartan subalgebra of an *oila* $\mathbf{G} = \mathbf{H} + \mathbf{M}$ and the roots of the Lie algebra $\mathbf{G}$ (see **6**, Exposé 9). Let $\mathbf{P}'$ be a Cartan subalgebra of the Lie algebra $\mathbf{G}$ containing the Cartan subalgebra $\mathbf{P}$ of the *oila* $\mathbf{G} = \mathbf{H} + \mathbf{M}$, and let $\{\theta_i\}$ be the roots of $\mathbf{P}'$, $\{\alpha\}$ the roots of $\mathbf{P}$. Then the α are the restrictions to $\mathbf{P}$ of those θ_i which do not vanish on $\mathbf{P}$, and the multiplicity of α equals the number of θ_i that restrict to α on $\mathbf{P}$.

(iv) There exists a set of (dim $\mathbf{P}$) linearly independent roots such that any root is an integral linear combination of the roots of this set with either all positive or all negative coefficients. Such a set is called a *simple system* of roots (**6**, Exposé 10).

DEFINITIONS. *The diagram $D(G,H)$ of the oila* $\mathbf{G} = \mathbf{H} + \mathbf{M}$ is a subset of $\mathbf{P}$ given by

$$D(G,H) = \{X \in \mathbf{P} | \alpha(X) \equiv 0 \bmod 1, \text{ for some } \alpha\}.$$

For every α, let h_α be the element of $\mathbf{P}$ such that the ray from 0 to h_α is perpendicular to the hyperplane $\alpha^{-1}(0)$ and such that $\alpha(h_\alpha) = 2$. Let Γ be the lattice generated by the h_α.

THEOREM 2 (**1**; **3**). *Let $X \in \mathbf{P}$. Then $e^{tX}(0 \leqslant t \leqslant 1)$ is a null-homotopic loop in M if and only if $X \in \Gamma$.*

3. Conjugate points. For the moment let M be any C^∞ Riemannian manifold, $m \in M$. Let $p \in M_m$, ρ be the ray in M_m from 0 through p, $n = \exp_m(p)$, and $\sigma = \exp \circ \rho$. Then p is a *conjugate point* of m and n is a *conjugate* *point* of m along σ if $d\exp_m$ is singular at p, that is, if there exists a tangent $t \in (M_m)_p$ such that $d\exp_m(t) = 0$.

Jacobi vector fields which vanish at two points are useful in the study of conjugate points. Let $m \in M$, and σ be any geodesic from m. We shall define only Jacobi fields along σ which vanish at m. Let $e_1, \ldots, e_d$ be a basis of M_m, $x_1, \ldots, x_d$ a dual basis, and let $p \in M_m$ be such that the ray ρ from 0 through p goes into σ under $\exp_m$, and finally take any $t \in (M_m)_p$.

Then

$$t = \sum_{i=1}^{d} a_i \frac{\partial}{\partial x_i}(p), \tag{7}$$

where the a_i are real. Consider the not necessarily globally well-defined vector field along σ defined by:

$$d\exp_m\left(r \sum_{i=1}^{d} a_i \frac{\partial}{\partial x_i}\right) \circ \rho, \tag{8}$$

where $r^2 = \sum_{i=1}^{d} x_i^2$. Then this vector field is a Jacobi vector field along σ, and all Jacobi fields vanishing at m are of this form. In general, a Jacobi field along σ will be the sum of two vector fields of the form of (8).

It is easy to see from this definition that n is conjugate to m along σ if and only if there is a Jacobi field along σ which vanishes at both m and n.

We shall also use an alternative approach to Jacobi fields. A *2-cube*, $\sigma(s,t)$, is a C^∞ mapping of a rectangle $[a,b] \times [c,d]$ into M. By holding alternately s and t fixed, we have curves σ_s and σ^t, called *transverse* and *longitudinal* curves, respectively. A vector field X along σ^c is associated with the 2-cube as follows:

$$\begin{aligned} X_s &= \text{tangent to } \sigma_s \text{ at } t = c \\ &= \sigma_{s*}(c). \end{aligned}$$

Assume that σ^c is a geodesic and that σ_a is constant. Then X is a Jacobi field as above if the longitudinal curves σ^t are geodesics.

One more general concept that we shall employ is that of a minimum point. Let $m \in M$ and let σ be a geodesic from m. Let n be a point on σ. Then n is a *minimum point* of m if σ minimizes arc length up to n but no further. It is not hard to see that if n is a minimum point to m and if there is a unique geodesic from m to n which minimizes arc length, then n is conjugate to m along this geodesic.

As mentioned above, the importance of the locus of all minimum points of m is that it contains all the topological interest of M in the sense that the complement in M is homeomorphic to a cell.

We now shift our attention back to symmetric spaces. Let G/H be symmetric with $\mathbf{G} = \mathbf{H} + \mathbf{M}$, and $M = e^{\mathbf{M}}$ the realization of the symmetric space with which we shall be concerned.

H is a group of automorphisms acting on M, and hence any $Y \in \mathbf{H}$ gives rise to a vector field $\bar{Y}$ on M. In fact, if $m \in M$, then $\bar{Y}_m$ is the tangent to the curve $e^{tY}me^{-tY} = \sigma(t)$, say, at m. That is, $\bar{Y}_m$ is tangent to the orbit of m under the action of e^{tY}. Now,

$$(9) \qquad \bar{Y}_m = \left.\frac{d\sigma}{dt}\right|_{t=0} = dR_m Y_e - dL_m Y_e,$$

where R_m and L_m denote right and left translation, respectively, in G; and hence $\bar{Y}$ is the difference between right and left translation. In particular, $\bar{Y}_e = 0$.

Now fix $X \in \mathbf{M}$, and consider the vector field $\bar{Y}$ restricted to the geodesic e^{sX}. By definition of $\bar{Y}$, a 2-cube associated with this vector field is

$$(10) \qquad \sigma(s,t) = e^{tY}e^{sX}e^{-tY}.$$

Now for fixed t, $\sigma(s,t)$ is a geodesic, since it is the image of e^{sX} under the isometry e^{tY}. So the longitudinal curves of the 2-cube $\sigma(s,t)$ are geodesics through e, and hence $\bar{Y}$ is a Jacobi field along the geodesic e^{sX}. Since $\bar{Y}_e = 0$, from the definition of Jacobi fields it follows that there exists a vector field A on $\mathbf{M}$ such that $\bar{Y} = de^{rA}$ along e^{sX}. A is now determined.

$\sigma_*{}^t(0)$ is the tangent at e to the longitudinal curve σ^t. Now

$$(11) \qquad \sigma_*^t(0) = dL_{e^{tY}} \circ dR_{e^{-tY}}(X_e),$$

which is the adjoint of e^{tY} operating on X.

So to every t there corresponds this element of $\mathbf{M}$, and A will be given by taking the tangent to this curve at $t = 0$ and extending it as in the definition of Jacobi field. Before doing this, a general remark is made concerning the differential of a representation of a Lie group.

Let

$$G \times R^d \xrightarrow{r} R^d$$

be a representation of a Lie group G on R^d. Then

$$\mathbf{G} \times R^d \xrightarrow{dr} R^d$$

is defined and is a representation of $\mathbf{G}$ on R^d. For example, if $r = \mathrm{ad}: G \times \mathbf{G} \to \mathbf{G}$, then $dr: \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ is given by: $X, Y \in \mathbf{G}$, $dr(X, Y) = [X, Y]$. A definition of dr is given here which is well adapted to the above situation. Let $X \in \mathbf{G}$, $x \in R^d$, and $dr(X, x)$ is to be defined as an element of R^d again. Let $n: (R^d)_0 \xrightarrow{\approx} R^d$ be the natural identification, and let $p: (R^d)_x \xrightarrow{\approx} (R^d)_0$ be given by parallel translation in R^d. Now $r\ (e^{tX}, x)$ is a curve in R^d, call it $\gamma(t)$. Then $\gamma_*(0) \in (R^d)_x$, since $\gamma(0) = x$. Define dr by:

$$(12) \qquad dr(X, x) = n \circ p(\gamma_*(0)).$$

Combining this remark with the above example of the adjoint and (11), we see that $A_X = p^{-1} \circ n^{-1}\ ([Y, X])$. So A may be taken as the vector field generated by Euclidean parallel translation of $n^{-1}\ ([Y, X])$.

So certain Jacobi fields along e^{sX} arise from elements of $\mathbf{H}$ in this way, and in fact all Jacobi fields vanishing at e and at some other point of e^{sX} are of this type. This follows from the fact that n^{-1} ($[X,\mathbf{H}]$) covers all the directions at $X \in \mathbf{M}$ which could give rise to such Jacobi fields, the others being too flat for Jacobi fields to vanish. More precisely, if $Z \in \mathbf{C}(X) \cap \mathbf{M}$ and if we consider the Jacobi field arising from $n^{-1}(Z)$ by parallel translation, then $K(X,Z) = 0(1)$ implies that this Jacobi field cannot vanish except at e. On the other hand, if $Z \notin \mathbf{C}(X)$, then by (3) there exists a $Y \in \mathbf{H}$ such that $[Y,X] = Z$, and so $\bar{Y}$ is the Jacobi field.

The above is a statement of the variational completeness of the action of H on M **(2; 4)**.

THEOREM 3. *Let $X \in \mathbf{P}$. e^X is conjugate to e along e^{sX} if and only if there exists a root α such that α $(X) \equiv 0$ mod 1, α $(X) \neq 0$. If*

$$\mathbf{V}_\alpha = \sum_{\beta=\pm\alpha} V_\beta,$$

then the Euclidean parallel translate of

$$n^{-1}\left(\sum_{\substack{\alpha(X) \equiv 0(1)\\ \alpha(X) \neq 0}} \mathbf{V}_\alpha \cap \mathbf{M}\right)$$

to X constitutes the kernel of the differential of the exponential map at X. In particular, the order of e^X as a conjugate point of e is

$$\sum_{\substack{\alpha(X) \equiv 0(1)\\ \alpha(X) \neq 0\\ \text{no repetitions.}}} \dim (\mathbf{V}_\alpha \cap \mathbf{M}).$$

Proof. $\alpha(X) \equiv 0$ (mod 1) $\Leftrightarrow \mathrm{ad} e^X|_{\mathbf{V}_\alpha} =$ identity map, by (5). Hence, if $Y \in \mathbf{V}_\alpha \cap \mathbf{H}$, then $e^X e^{tY} e^{-X} = e^{tY}$, all t, or

$$e^{-tY} e^X e^{tY} = e^X.$$

That is, $-\bar{Y}_{e^X} = 0$. $\alpha(X) \neq 0 \Rightarrow \bar{Y} \not\equiv 0$. Therefore, since $\bar{Y}$ is a non-zero Jacobi field vanishing at e and at e^X, e^X is conjugate to e along e^{sX}. And the vector annihilated is $p^{-1}n^{-1}$ ($[Y,X]$), where $[Y,X] \in \mathbf{V}_\alpha \cap \mathbf{M}$ and p is parallel translation from $X \in \mathbf{M}$ to $0 \in \mathbf{M}$.

Conversely, suppose e^X is conjugate to e along e^{sX}. Then there exists a Jacobi field J along e^{sX} such that $J_e = J_{e^X} = 0$. Variational completeness then implies that there exists a $Y \in \mathbf{H}$ such that $\bar{Y} = J$ on e^{sX}. Let

$$Y = \sum_\alpha a_\alpha Y_\alpha,\ Y_\alpha \in V_\alpha.$$

Then $\mathrm{ad} e^X(tY_\alpha) = tY_\alpha$ by (3), all α. But $\mathrm{ad} e^X$ is a rotation of V_α, by (5), and hence must be the identity. So $\alpha(X) \equiv 0$ (1), all α. Also there exists some α such that $\alpha(X) \neq 0$; for if $a_\alpha \alpha(X) = 0$, all α, then $a_\alpha \alpha(sX) = 0$, all α, and hence $\bar{Y}_{e^{sX}} = 0$, which contradicts the choice of $\bar{Y}$.

https://doi.org/10.4153/CJM-1962-024-8 Published online by Cambridge University Press

Thus, the conjugate points in a Cartan subalgebra are those elements of the diagram lying on hyperplanes not through 0. In view of Theorem 1 of the previous section and the fact that every point of **M** lies in some Cartan subalgebra, the conjugate locus is determined by the diagram of a single Cartan subalgebra and the isotropy group. Actually, since the automorphisms of a Cartan subalgebra arising from the action of the isotropy group are transitive on the chambers of the diagram, that is, the connected components of the set of regular elements, one need only consider the intersection of the diagram with the closure of a chamber to describe the conjugate locus.

In the symmetric case, one sees that points are conjugate because an infinity of nearby geodesics of the same length pass through them. This is essentially what variational completeness says. In other words, the exponential map is many-to-one on the conjugate locus in the tangent space. Hence, one might expect the conjugate locus in M to be of dimension less than $d - 1$, where d is the dimension of M, and this is in fact true (**1**, p. 50). However, this is not true in general. For example, the conjugate locus of a point on an ellipsoid is homeomorphic to a circle.

The following theorem gives the relation between conjugate points and minimum points.

THEOREM 4. *Let M be a simply connected, compact symmetric space with a semi-simple group of isometries. Let $e \in M$. Then the minimum locus of e coincides with the first conjugate locus of e.*

Proof. Let G be the group of isometries, H the isotropy subgroup of G at e, and assume M is imbedded as above, with e = identity of G. Let $\mathbf{G} = \mathbf{H} + \mathbf{M}$ be the *oila*, and $\mathbf{P}$ be a Cartan subalgebra. The last theorem exhibits the conjugate points in $\mathbf{P}$.

Let $X \in \mathbf{P}$ be such that e^X is a minimum point of e along e^{sX}. Assume there exists another $Y \in \mathbf{M}$ such that $|Y| = |X|$ and $e^X = e^Y$, for otherwise e^X is already conjugate to e.

Case 1. Assume $[X, Y] = 0$ and assume e^X is not conjugate to e. Then X and Y both belong to some Cartan subalgebra, $\mathbf{P}'$ say. $e^Y = e^X \Rightarrow e^{Y-X} = e$, so $e^{s(Y-X)}$, $0 \leqslant s \leqslant 1$, is a loop at e. Since M is simply connected, $e^{s(Y-X)}$ is a null homotopic loop, so by Theorem 2 of the last section, $Y - X \in \Gamma$.

Let $\{\alpha'\}$ be the roots of $\mathbf{P}'$, and let

$$D = \{Z \in \mathbf{P}' | -1 < \alpha'(Z) < 1, \text{ all } \alpha'\}.$$

Then by Theorem 3, the assumption that e^X is not conjugate to e, and the fact that $e^X = e^Y$ is a minimum point, $X, Y \in D$. So the parallel translate of the vector $Y - X$ to X lies entirely within D, as D is convex.

Let S be a simple system of roots of $\mathbf{P}'$, so that by Theorem 2 of the last section, $e^{\mathbf{P}'}$ is a torus with generators in $\mathbf{P}'$ given by the elements $h_{\alpha'}$, for $\alpha' \in S$. If

$$D' = \{Z \in \mathbf{P}' | -1 < \alpha'(Z) < 1, \alpha' \in S\},$$

then $D \subset D'$, and no integral linear combination of the generators $h_{\alpha'}$, $\alpha' \in S$, is parallel to a vector lying entirely within D'. But since $Y - X \in \Gamma$ and S is a simple system of roots, $Y - X$ is an integral linear combination of the $h_{\alpha'}$, $\alpha' \in S$. Hence, $Y - X$ is parallel to no vector in D' and so to no vector in D. This is a contradiction to the assertion that $X, Y \in D$. Hence, the assumption that e^X is not conjugate to e is false.

Case 2. Assume now that $[X, Y] \neq 0$. Hence,

$$Y = Y_0 + \sum_{\alpha} Y_\alpha,$$

where $Y_0 \in \mathbf{C}(X)$, $Y_\alpha \in V_\alpha$, and not all $Y_\alpha = 0$.

Now $e^X = e^Y$, so $e^{sY} = e^Y e^{sY} e^{-Y} = e^X e^{sY} e^{-X}$,
so

$$\operatorname{ad} e^X(Y) = Y.$$

Therefore, $Y = \operatorname{ad} e^X(Y) = Y_0 + \sum_\alpha \operatorname{ad} e^X(Y_\alpha) = Y_0 + \sum_\alpha Y_\alpha$, and so $\operatorname{ad} e^X(Y_\alpha) = Y_\alpha \neq 0$ for some α by (3).

Hence, by (5),

$$\operatorname{ad} e^X|_{V_\alpha} = \text{identity, for some } \alpha.$$

So for some α, $\alpha(X) \equiv 0$ (1). Now if $\alpha(X) = 0$ for all α such that $Y_\alpha \neq 0$, then $[X, Y_\alpha] = 0$ for all such α, and hence $[X, Y] = 0$, contrary to assumption. Hence, there exists an α such that $\alpha(X) \equiv 0$ (1), $\alpha(X) \neq 0$, and so, by Theorem 3, e^X is conjugate to e along e^{sX}.

Therefore, it has been proven that every minimum point is a conjugate point, and so a first conjugate point, of e. There is a minimum point in every direction since M is compact, and hence this also proves that every first conjugate point is a minimum point.

4. The general case

THEOREM 5. *Let M be a simply connected complete symmetric space, $e \in M$. Then the minimum locus of e coincides with the first conjugate locus of e.*

This theorem follows from Theorem 4 and two remarks.

Remark 1. In **(1)** it is shown that if M is a complete simply connected symmetric space, then M admits a Riemannian product decomposition

$$M = M_0 \times M_1 \times \ldots \times M_s,$$

where M_0 is Euclidean space and the M_i, $i > 0$, are irreducible symmetric spaces with semi-simple groups of isometries. From the remarks following formula (1), it follows that the non-compact M_i are diffeomorphic to Euclidean spaces via the exponential maps, and so they contribute nothing to either the first conjugate or minimum loci of M. Theorem 4 applies to the compact M_i.

Remark 2. If $M = M_1 \times M_2$ is a Riemannian product of Riemannian manifolds and if the first conjugate and minimum loci coincide in M_i, $i = 1,2$, then the same is true of M. This follows from the

THEOREM. *If* $M = M_1 \times M_2$ *as above,* $(m_1, m_2) \in M$, L_i, C_i *are the first conjugate and minimum loci, respectively, of* m_i *in* M_i, $i = 1,2$, *then* $(M_1 \times L_2) \cup (L_1 \times M_2)$ *and* $(M_1 \times C_2) \cup (C_1 \times M_2)$ *are the first conjugate and minimum loci of* (m_1, m_2) *in* M.

These facts are more or less well known, and so only an indication of the proof is given.

Let $\rho_i : M \to M_i$, $i = 1,2$, be the usual projections of the product onto its factors. Then a curve σ in M is a geodesic in the Riemannian product structure if and only if $\rho_1 \circ \sigma$ and $\rho_2 \circ \sigma$ are geodesics, possibly one a point. This follows from the relation between the Riemannian connexion on M and the Riemannian connexions on the M_i.

Using the fact that Jacobi fields are precisely those fields attached to 2-cubes with geodesic longitudinal curves, a similar characterization of Jacobi fields in M obtains. The part of the theorem relating first conjugate loci now follows from the fact, stated above, that two points are conjugate if and only if there is a non-trivial Jacobi field vanishing at both points.

The remainder of the theorem follows from the above property of geodesics in M by a simple argument, using that a minimum point on a geodesic σ is that point beyond which σ ceases to minimize arc-length globally.

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https://doi.org/10.4153/CJM-1962-024-8 Published online by Cambridge University Press