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DECOMPOSITION BASED GENERATING FUNCTIONS FOR SEQUENCES

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PART I: General Theorems.

1. Introduction. Numerous combinatorial enumeration problems may be reduced to equivalent problems of enumerating sequences with prescribed restrictions. For example, the expression, given by Tutte [**38**], for the number of planar maps may be derived (see Cori and Richard [**12**]) by essentially a sequence enumeration technique. The correspondence between a set of configurations which are to be enumerated and an appropriate set of sequences is often complicated. Indeed, the existence of such a correspondence has occasionally only been discovered fortuitously by observing the equality of two counting series (see, for example, Klarner [**25**]). However, sequence enumeration has, in principle, wider application than simply to what may be loosely termed the "classical" problems which date back to the beginning of combinatorial analysis. In general the classical sequence enumeration problems concern the determination of the number of sequences which possess or lack certain specific subsequences. However, even these problems typically have been treated by non-uniform methods which are specific to each case. In this paper we demonstrate that a decomposition of sequences into *maximal paths* leads to a uniform treatment of these problems and we present an elementary enumeration theorem which, when specialised appropriately, provides solutions to a considerable number of sequence problems. Some of these have, of course, been treated before, but the solutions of the remaining ones are, to the authors' present knowledge, new.

Let $\tilde{P}$ be a distinguished set of sequences which we will call *paths*. We are concerned with the significance of generating functions of the form

$$\sum_{k\geqq 0}\left\{\sum_{p\in\tilde{P}}\delta(p)\right\}^k$$

where δ is some function. It will be seen that many problems which involve enumerating sequences with respect to the presence or absence of subsequences which are members of $\tilde{P}$, have generating functions of this form. As an example, let $\tilde{P}$ be the set of strictly increasing subsequences. Then the generating function for the set of all sequences with no strictly increasing subsequences of length three has the above form.

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It is not true that all subsets $\tilde{P}$ can act in this capacity, and Section 2 characterises them, while Proposition 2.8 helps to identify an important subclass.

We now return to the reason why generating functions of this form are significant at all. If multiplication is viewed as concatenation of corresponding subsequences in the above form, then this sum represents an enumeration of all of the sequences which can be obtained by joining together elements of $\tilde{P}$. The terms $\delta(p)$ are simply markers for the fact that p was used in the construction. However, any given path may be made up of other paths in several ways. For example the *run* 1234 can be constructed using 1,234 or 12,34 or 1,2,3,4 and so on. Some technique is required for suppressing the contribution of these kinds of decompositions to the overall generating function. This difficulty is also described in Section 2. The solution yields an interpretation for generating functions of the above form and is given as Theorem 3.1. It makes use of a Möbius inversion formula, in fact the one connected with the lattice of compositions of an integer ordered by refinement.

The principal impediment to the solution of actual enumeration problems is that the Möbius inversion may be difficult to carry out in practice. However, certain simplifying assumptions may be made which, in a commutative ring of weights, reduce the problems to inverting power series or matrices. Although the applicability of the method is wide, problems of increasing "complexity" may well involve algebra which becomes increasingly difficult and perhaps intractable.

The material falls into three parts. Part I, consisting of the first five sections, deals with the decomposition, the main theorem and its principal corollaries. Part II, consisting of Section 6 to Section 9, deals with applications of the theory to a number of sequence enumeration problems. More specifically, Section 2 defines the decomposition of sequences into paths which is used throughout the paper. The main theorem is given in Section 3, and its two principal corollaries are presented as theorems in Section 4 and Section 5. In Part II, Section 6 deals with Andrews' refinement of the Simon Newcomb problem as an application of the theorem given in Section 5, while Section 7 contains applications of the theorem given in Section 4 to p-paths in sequences. The p-path problems for permutations are considered in Section 8 and some results relating to rises, levels, falls and the number of maximal runs within sequences are discussed in Section 9. In Part III, the method is extended to permit the extreme maximal paths to be differentiated from the remaining ones. Section 10 contains the extensions of the main theorem, while Section 11 gives certain specialisations. Section 12 gives a number of applications of the extended theorem to the enumeration of permutations with respect to *maxima* and *minima*. A brief discussion of some of the limitations of the method is given in Section 13. Also in this section we examine the possibility of establishing the "positivity" of rational functions by these techniques. For cross referencing purposes the Appendix contains a list of some of the more common problems which have been treated in this paper.

These enumeration theorems fall into several broad classes, as follows. Sections 3 and 10 are fundamental and contain the primary results. The next level of results are corollaries to these, in one case dealing with length and type encoding (Sections 4 and 11), and in the other with initiator-terminator and type encoding (Section 5). The third level contains analyses of permutation problems, based on length and type encoding techniques (Sections 8 and 12), and also of Simon Newcomb type problems treated with initiator-terminator and type encodings (Section 6). Finally several areas of intersection between these diverse approaches and other extant work are examined in Sections 9, 11 and 12.

A number of investigations of decomposition based generating functions have already been carried out, but from different points of view. Among such constructions are the "prefab" of Bender and Goldman [**6**], and the "dissect" of Henle [**19**].

For convenience, the following notational conventions are observed:

(i) Let $\boldsymbol{x} = (x_1, x_2, \ldots)$ and $\boldsymbol{i} = (i_1, i_2, \ldots)$, both finite dimensional. Then $x_1^{i_1}x_2^{i_2}\ldots$ is denoted by $\boldsymbol{x}^{\boldsymbol{i}}$.

(ii) The empty sum and empty product are taken to be 0 and 1, respectively.

(iii) The (i, j)-element of a matrix A is denoted by $[A]_{ij}$.

(iv) The coefficient of $\boldsymbol{x}^{\boldsymbol{i}}$ in the formal power series $F(\boldsymbol{x})$ is denoted by $[\boldsymbol{x}^{\boldsymbol{i}}]F(\boldsymbol{x})$. We use $[\boldsymbol{x}]F(\boldsymbol{x})$ to denote $[x_1 x_2 \ldots]F(\boldsymbol{x})$.

(v) A *rise* in a sequence $i_1i_2\ldots i_n$ is defined to be any pair i_ki_{k+1} such that $i_k < i_{k+1}$. *Falls* and *levels* are defined analogously.

(vi) The symbol $\circ$ is used to denote function composition.

(vii) $\boldsymbol{i} = (i_1, i_2, \ldots)$ is the *type* of a sequence, where i_j is the frequency of occurrence of the element j.

To facilitate cross-referencing between theorems, most of them are preceded with single phrases which indicate the contexts in which the theorems apply.

2. Maximal decompositions. Let $\tilde{N}^*$ be the set of all sequences (free monoid with concatenation) over an arbitrary countable set $\tilde{N}$, which we identify with the positive integers. Let $\tilde{N}^+ = \tilde{N}^*\backslash\epsilon$ where ϵ is the empty word, and let $\tilde{P} \subset \tilde{N}^+$ be a distinguished subset called the set of *paths*. The following preliminary definitions are needed.

Definition 2.1. (i) The set of *sequences* over $\tilde{P}$ is the monoid $\tilde{P}^* = \bigcup_{i=0}^{\infty} \tilde{P}^i$ with concatenation as product.

(ii) The set of *decompositions* of $\tilde{P}^*$ is the monoid $\tilde{P}^{(*)} = \bigcup_{i=0}^{\infty} \tilde{P}^i$ with the product $(p_1, \ldots, p_m) \cdot (q_1, \ldots, q_n) = (p_1, \ldots, p_m, q_1, \ldots, q_n)$.

(iii) $c : \tilde{P}^{(*)} \to \tilde{P}^*$ is the *concatenation* homomorphism defined by $c(p_1, \ldots, p_n) = p_1p_2\ldots p_n$.

$\tilde{P}^{(*)}$ is the free monoid with letters being the elements of $\tilde{P}$. We will always write elements of $\tilde{P}^{(*)}$ with parentheses to distinguish them from the sequences in $\tilde{P}^*$.

PROPOSITION 2.2. *Let $\leqq$ be the order relation defined on the free monoid $\tilde{P}^{(*)}$ by $d_1 \leqq d_2 \Leftrightarrow d_1$ is a refinement of d_2, where d_1 and d_2 are decompositions. Then $c^{-1}(p) = \{d | d \leqq (p)\}$ for $p \in \tilde{P}$ and $c^{-1}(p)$ is finite.*

Proof. Immediate. The finiteness of $c^{-1}(p)$ follows from the presence of the underlying set $\tilde{N}$.

Definition 2.3. If $\tilde{R}$ is an arbitrary ring with unit, and for $\Delta : \tilde{P} \rightarrow \tilde{R}$ the *induced map* $\Delta : \tilde{P}^{(*)} \rightarrow \tilde{R}$ is defined by

(i) $\Delta(p_1, \ldots, p_n) = \Delta(p_1)\Delta(p_2) \ldots \Delta(p_n)$

and if

(ii) $u(\sigma) = (p_1, p_2, \ldots, p_n) \in c^{-1}(\sigma)$

is a distinguished decomposition of σ then the *generating function* for $\tilde{P}^*$ is

(iii) $\sum_{\sigma \in \tilde{P}^*} \Delta \circ u(\sigma)$ or, equivalently $\sum_{\sigma \in \tilde{P}^*} \Delta(p_1)\Delta(p_2) \ldots \Delta(p_n)$.

Definition 2.4. $\tilde{P}$ admits maximal decompositions if and only if $c^{-1}(\sigma)$ has a maximum for every $\sigma \in \tilde{P}^*$.

Accordingly if, $\tilde{P}$ admits maximal decompositions, it is natural to take $u(\sigma) = \max c^{-1}(\sigma)$. This completes the definition of a generating function based on maximal decompositions. Examples of sets of paths admitting maximal decompositions are given in the next definition. They will be used in a number of specialisations of the main theory.

Definition 2.5.
(i) $\tilde{P}_1 = \tilde{N}^+$.
(ii) $\tilde{P}_2 = \{i_1 i_2 \ldots i_n | i_{m+1} = i_m + 1\}$, the set of *increasing runs*.
(iii) $\tilde{P}_3 = \{i_1 i_2 \ldots i_n | i_1 \leqq i_2 \leqq \ldots \leqq i_n\}$, the set of *increasing sequences*.
(iv) $\tilde{P}_4 = \{i_1 i_2 \ldots i_n | i_1 < i_2 < \ldots < i_n\}$, the set of *strictly increasing sequences*.
(v) $\tilde{P}_5 = \{i_1 i_2 \ldots i_n | n \geqq 2; i_1 \text{ even}; i_n \text{ odd}; i_{m+1} = i_m + 1\}$.
(vi) $\tilde{P}_6 = \{i_1 i_2 \ldots i_n | n \geqq 1; i_n = 1\}$.

Note that $\tilde{P}^* = \tilde{N}^*$ if and only if $\tilde{N} \subset \tilde{P}$, so $\tilde{P}_5^*$ and $\tilde{P}_6^*$ do not equal $\tilde{N}^*$, The following propositions characterise a certain class of paths.

PROPOSITION 2.6. *A set of paths admitting maximal decompositions may always be obtained from a generalised successor function $s : \tilde{N} \rightarrow 2^{\tilde{N}}$ which takes letters into subsets of letters. This determines a set of paths $\tilde{P}_s = \{i_1 i_2 \ldots i_n | n \geqq 1; i_{m+1} \in s(i_m)\}$.*

Proof. Straightforward.

We can see, for example, that $\tilde{P}_3 = \tilde{P}_s$ for $s(i) = \{j | j \geqq i\}$. Now $u(\sigma)$ may

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be calculated readily by checking the conditions $i_{m+1} \in s(i_m)$ along the sequence σ. Thus, for example, if $s(i) = \{j | j \geqq i\}$ then $u(1324) = (13,24)$ since $2 \notin s(3)$.

Definition 2.7. $\tilde{P}$ *contains arbitrary subwords* if for any $p \in \tilde{P}$ then any non-empty subword of p is also a path.

PROPOSITION 2.8. *$\tilde{P}$ admits maximal decompositions and contains arbitrary subwords if and only if $\tilde{P} = \tilde{P}_s$ for some generalised successor function s.*

Proof. If $\tilde{P} = \tilde{P}_s$ for some s then $\tilde{P}$ contains arbitrary subwords. Suppose now that $\tilde{P}$ admits maximal decompositions and contains arbitrary subwords. Let s be defined as follows:

$j \in s(i)$ if and only if (ij) appears as a subword of some path in $\tilde{P}$.

Now clearly $\tilde{P} \subset \tilde{P}_s$. Further,we claim that $\tilde{P}_s \backslash \tilde{P} = \emptyset$, for if $p_s \in \tilde{P}_s \backslash \tilde{P}$ then $u(p_s) = (p_1, p_2, \ldots, p_n)$ for $n > 1$, or else it belongs to $\tilde{P}$. Let $p_1 = \bar{p}_1 a$ and $p_2 = b\bar{p}_2$ where $a, b \in N$. Since $p_s \in \tilde{P}_s$ then $b \in s(a)$, whence ab is a subword of some path in $\tilde{P}$. But the arbitrary subword property implies that $(ab) \in \tilde{P}$ as well as $\bar{p}_1 \in \tilde{P}$ and $\bar{p}_2 \in \tilde{P}$. Thus $(\bar{p}_1, (ab), \bar{p}_2, p_3, p_4, \ldots, p_n) \in c^{-1}(p_s)$ which is not comparable to $u(p_s) = (p_1, p_2, \ldots, p_n)$, which provokes a contradiction.

Note that $\tilde{P}_5$ and $\tilde{P}_6$ do not contain arbitrary subwords.

3. The main theorem. In this section the main theorem is developed, which is specialised in subsequent sections. Let $\Delta : \tilde{P} \rightarrow \tilde{R}$ be an arbitrary map from paths to a ring with identity which may be non-commutative. For those $\tilde{P}$ which admit maximal decompositions we wish to write the generating function $\sum_{\sigma \in \tilde{P}^*} \Delta \circ u(\sigma)$, which is combinatorially easy to interpret, as $\{1 - \sum_{p \in \tilde{P}} \delta(p)\}^{-1}$ which is algebraically preferable. The first form considers only the maximal decompositions, while the second one incorporates all allowable decompositions. The following theorem demonstrates that the connexion exists and describes what it is.

THEOREM 3.1 (Maximal paths). *If*

(i) *$\tilde{P}$ admits a maximal decomposition $u(\sigma)$ for any $\sigma \in \tilde{P}^*$, and*

(ii) *$\Delta : \tilde{P} \rightarrow \tilde{R}$ is arbitrary, where $\tilde{R}$ is a ring with* 1, *then there exists a unique $\delta : \tilde{P} \rightarrow \tilde{R}$ such that*

$$\Delta(p) = \sum_{d \in c^{-1}(p)} \delta(d) \quad \text{for any } p \in \tilde{P}$$

and

$$\sum_{\sigma \in \tilde{P}^*} \Delta \circ u(\sigma) = \left\{1 - \sum_{p \in \tilde{P}} \delta(p)\right\}^{-1}.$$

Proof. Informally one sets up the desired equality and attempts to find a solution for the δ's in terms of the Δ's. The following presentation, however, is more compact.

Since $c^{-1}(p) = \{d | d \leqq (p)\}$ for $p \in \tilde{P}$, the condition on δ may be rewritten as

$$\Delta(p) = \sum_{d \leqq (p)} \delta(d)$$

for any $p \in \tilde{P}$. Thus δ may be constructed by Möbius inversion with respect to each maximal element of $\tilde{P}^{(*)}$. Moreover, we have

$$\left\{1 - \sum_{p \in \tilde{P}} \delta(p)\right\}^{-1} = \sum_{j \geqq 0} \left\{\sum_{p \in \tilde{P}} \delta(p)\right\}^{j}$$

$$= \sum_{j \geqq 0} \sum_{\sigma \in \tilde{P}^*} \sum_{d} \delta(d) \quad \text{(where the summation is taken over } d \in c^{-1}(\sigma) \cap \tilde{P}^{j}\text{)}$$

$$= \sum_{\sigma \in \tilde{P}^*} \sum_{d \in c^{-1}(\sigma)} \delta(d) = \sum_{\sigma \in \tilde{P}^*} \prod_{i=1}^{n} \sum_{d \in c^{-1}(p_i)} \delta(d)$$

$$\text{(where } u(\sigma) = (p_1, p_2, \ldots, p_n)\text{)}.$$

The last equality holds because the maximal decomposition property implies that

$$c^{-1}(\sigma) = \mathop{\times}_{i=1}^{n} c^{-1}(p_i).$$

Now

$$\sum_{d \in c^{-1}(p_i)} \delta(d) = \Delta(p_i)$$

so

$$\left\{1 - \sum_{p \in \tilde{P}} \delta(p)\right\}^{-1} = \sum_{\sigma \in \tilde{P}^*} \Delta \circ u(\sigma)$$

and the proof is complete.

This generating function for $\tilde{P}^*$ can be loosely described as the translation of the monoid sum $\tilde{P}^{(*)} = \bigcup_{k \geqq 0} \tilde{P}^k$ to the sum $\sum_{k \geqq 0} (\sum_{p \in \tilde{P}} \delta(p))^k$.

4. Length and type encoding for maximal decompositions. We consider the first of the two simplifications of $\Delta : \tilde{P} \to \tilde{R}$ which allows the Möbius inversion to be performed more easily in many cases. In this section we allow only those Δ which are able to encode length and type information. The second simplification is given in Section 5.

THEOREM 4.1 (Length and type). *If*

(i) $\tilde{P}$ *admits maximal decompositions, and*

(ii) $\Delta(p) = F_j \tau(p)$ *where*

  (a) $j = |p|$ *is the length of* $p \in \tilde{P}$, *and*

  (b) τ *is a path-homomorphism* ($\tau(pq) = \tau(p)\tau(q)$ *for any two paths* p, q) *which maps to the centre of the ring* $\tilde{R}$ (*we will usually take* τ *to be an encoding of the type of* p), *then*

(iii) $\delta(p) = f_j \tau(p)$ *where* $1 + \sum_{k>0} F_k u^k = \left\{1 - \sum_{k>0} f_k u^k\right\}^{-1}$, *and*

(iv) *the generating function is* $\left\{1 - \sum_{k>0} f_k \gamma_k\right\}^{-1}$ *where* $\gamma_k = \sum_{\substack{p \in P \\ |p|=k}} \tau(p)$.

Proof. We show that $\delta(p)$ is of the form $f_j(p)$ by induction on the length of p, or alternatively, on the maximum number of subpaths into which p can be decomposed. Since $\Delta(p) = \delta(p)$ for those p of minimal length we have, in this case, that $\delta(p) = f_{|p|}\tau(p)$ where $f_{|p|} = F_{|p|}$. By the induction hypothesis for all paths shorter than p and the commutativity of $\tau(d_i)$ with every element, we have, for any p:

$$\delta(p) = \Delta(p) - \sum_{d<(p)} f_{j_1} f_{j_2} \ldots f_{j_m} \tau(d_1)\tau(d_2) \ldots \tau(d_m)$$

where $|d_k| = j_k$ and $d = (d_1, d_2, \ldots, d_m)$. Thus

$$\delta(p) = \tau(p) F_{|p|} - \tau(p) \sum_{d<(p)} f_j f_{1 j_2} \ldots f_{j_m}$$

and the result follows immediately.

Now $\tau(p)$ may be factored from $\Delta(p) = \sum_{d \leqq (p)} \delta(d)$ to give

$$F_j = [u^j] \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} f_i u^i\right)^k$$

whence

$$1 + \sum_{k>0} F_k u^k = \left\{1 - \sum_{k>0} f_k u^k\right\}^{-1}.$$

Moreover, the generating function is

$$\left\{1 - \sum_{p \in \tilde{P}} f_{|p|}\tau(p)\right\}^{-1} = \left\{1 - \sum_{k>0} f_k \gamma_k\right\}^{-1}$$

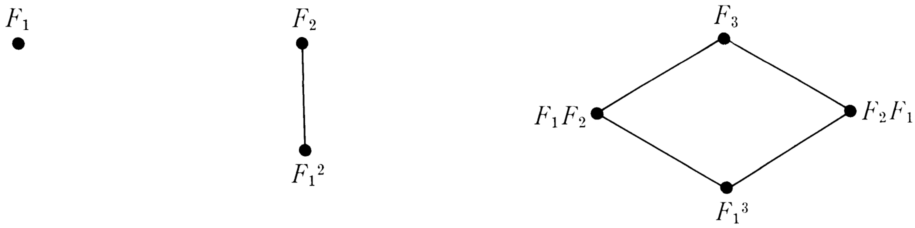
since

$$\gamma_k = \sum_{\substack{p \in \tilde{P} \\ |p|=k}} \tau(p).$$

We note that the generating function given in the above theorem involves an *umbral substitution* (see Mullin and Rota [**29**]) or, equivalently, the *umbral composition* of the series $f(u) = 1 - \sum_{k>0} f_k u^k$ and $\gamma(u) = 1 + \sum_{k>0} \gamma_k u^k$. Indeed, the generating function may be written $(f \circ \gamma)^{-1}$ where the composition is umbral. However, we do not pursue this further.

The following example demonstrates the generality of Theorem 4.1 while illustrating some of the difficulties which may attend its use. In the example, we assume that the F_k do not commute.

Example 4.2. We consider the enumeration of sequences over $\{1, 2, 3\}$ according to their type and the number and order of the increasing runs they possess. Accordingly, let $\tilde{N} = \{1, 2, 3\}$, $\tilde{P} = \tilde{P}_2 = \{1, 2, 3, 12, 23, 123\}$ (see Definition 2.5(ii)), and $\tau(i_1 i_2, \ldots, i_n) = x_{i_1} x_{i_2} \ldots x_{i_n}$. The inversion formulae and the appropriate posets of decompositions corresponding to $-[u^n]\sum_{k>0}\{-\sum_{i>0} F_i u^i\}^k$, for $n = 1, 2, 3$ are given in Figure 1.



$f_1 = F_1$ $\quad$ $f_2 = F_2 - F_1^2$ $\quad$ $f_3 = F_3 - F_2F_1 - F_1F_2 + F_1^3$

FIGURE 1

For simplicity, let $F_1 = 1$ so the generating function for the problem is, from Theorem 4.1:

$$G = \{1 - [(x_1 + x_2 + x_3) + (F_2 - 1)(x_1x_2 + x_2x_3) + (F_3 - 2F_2 + 1)x_1x_2x_3]\}^{-1}.$$

In particular, for sequences of type (2,2,1), we have

$$[x_1^2x_2^2x_3]G = 7 + 12F_2 + 4F_3 + 5F_2^2 + F_2F_3 + F_3F_2.$$

Thus, for sequences possessing exactly two 1's, two 2's and one 3, there are: 1 with a 3-run followed by a 2-run, 1 with a 2-run followed by a 3-run, 5 with two 2-runs, 4 with one 3-run, 12 with one 2-run, and 7 which are run-free. There are $5!/(2!2!) = 30$ sequences in all.

The following example uses a Dirichlet generating function in connexion with multiplicative partitions.

Example 4.3. The number of sequences over $\{1, 2, 3\}$ of type (i_1, i_2, i_3) whose maximal increasing run lengths are factorisations of n is

$$[n^{-s}][x_1^{i_1}x_2^{i_2}x_3^{i_3}]\{1 - (x_1 + x_2 + x_3) - (2^{-s} - 1)(x_1x_2 + x_2x_3) - (3^{-s} - 2.2^{-s} + 1)x_1x_2x_3\}^{-1}.$$

This follows directly from Example 4.2 with $F_k = k^{-s}$. The coefficient of $x_1x_2x_3$ is $3^{-s} + 2.2^{-s} + 3$ which agrees with the direct enumeration given in Table 1.

TABLE 1

σ	$\Delta \circ u(\sigma)$
123	$3^{-s}\, x_1x_2x_3$
213	$1\, x_1x_2x_3$
132	$1\, x_1x_2x_3$
231	$2^{-s}\, x_1x_2x_3$
312	$2^{-s}\, x_1x_2x_3$
321	$1\, x_1x_2x_3$

There are six sequences in all, a fact which emerges by putting $s = 0$.

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5. Type and initiator-terminator encoding for maximal decompositions. The following theorem presents the second simplification of Theorem 3.1. Here we allow only those Δ which are able to record the first and last elements of a maximal path along with its type.

THEOREM 5.1 (Type, initiators and terminators). *If*

(i) $\tilde{P} = \tilde{P}_s$ *for some generalised successor function* s, *such that* $s(i) \neq \emptyset$ *for any* i, *and*

(ii) $\Delta : \tilde{P} \rightarrow \tilde{R}$ *is arbitrary, where* $\tilde{R}$ *is a ring with* 1, *then*

(iii) $I + \sum_{i,j} \Delta_{ij}\bar{M}_{ij} = \left\{I - \sum_{i,j} \delta_{ij}\bar{M}_{ij}\right\}^{-1}$

where

$$\Delta_{ij} = \sum_{p \in \tilde{M}_{ij}} \Delta(p) \quad \text{and } \delta_{ij} = \sum_{p \in \tilde{M}_{ij}} \delta(p),$$

$$\tilde{M}_{ij} = \{p \in \tilde{P} | p \text{ begins with } i \text{ and ends with } j\},$$

and the matrix

$$[\bar{M}_{ij}]_{m,n} = \begin{cases} 1 & \text{if } i = m \text{ and } n \in s(j) \\ 0 & \text{otherwise,} \end{cases}$$

and I *is the identity matrix. Moreover,*

(iv) *the generating function is* $\left\{1 - \sum_{i,j} \delta_{ij}\right\}^{-1}$.

Proof. The relationship

$$\bar{M}_{ij}\bar{M}_{kl} = \bar{M}_{il} = (0) \quad \Leftrightarrow \quad \tilde{M}_{ij}\tilde{M}_{kl} \cap \tilde{P} \neq \emptyset \ (=\emptyset \text{ respectively})$$

arises because of condition (i). The matrices represent the essential properties of the monoid product on the sets $\tilde{M}_{ij}$. Now

$$\Delta(p) = \sum_{d \leqq (p)} \delta(d) \quad \text{for } p \in \tilde{P}$$

so

$$\Delta_{ij} = \sum_{p \in \tilde{M}_{ij}} \Delta(p) = \sum_{p \in \tilde{M}_{ij}} \sum_{d \leqq (p)} \delta(d) = \sum_{k \geqq 1} \sum \prod_{i=1}^{k} \delta_{m_i n_i}$$

where the second summation is over $m_1, n_1, m_2, n_2, \ldots, m_k, n_k$ such that

$$\tilde{M}_{m_1 n_1} \cdot \tilde{M}_{m_2 n_2} \cdot \ldots \cdot \tilde{M}_{m_k n_k} \cap \tilde{M}_{ij} \neq \emptyset.$$

Thus

$$\Delta_{ij} = [\bar{M}_{ij}] \sum_{k=0}^{\infty} \left(\sum_{m,n} \delta_{mn}\bar{M}_{mn}\right)^k$$

from which it follows that

$$I + \sum_{i,j} \Delta_{ij}\bar{M}_{ij} = \left\{I - \sum_{i,j} \delta_{ij}\bar{M}_{ij}\right\}^{-1}.$$

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The extraction of the coefficient is well-defined since the matrices are linearly independent. Finally, since $\tilde{P} = \dot{\bigcup}_{i,j} \tilde{M}_{ij}$ is a disjoint union we have

$$\left\{1 - \sum_{p \in \tilde{P}} \delta(p)\right\}^{-1} = \left\{1 - \sum_{i,j} \delta_{ij}\right\}^{-1}$$

and the theorem follows from Theorem 3.1.

Condition (i) of Theorem 5.1 may be replaced by

$$\bar{M}_{ij}\bar{M}_{kl} = \bar{M}_{il} = (0) \quad \Leftrightarrow \quad \tilde{M}_{ij}\tilde{M}_{kl} \cap \tilde{P} \neq \emptyset \; (= \emptyset \text{ respectively})$$

where the M_{ij} are suitable linearly independent matrices. The task of finding matrices to fit the conditions appears to be unrewarding, and we have not found an application which uses this weaker condition. The set $\tilde{P}_5$ of paths (see Definition 2.5(v)) satisfies this weaker condition, but does not admit a generalised successor function.

PART II: Applications.

6. Problems involving initiator-terminator information. This section is concerned with problems which involve initiator and terminator information, and is divided into two subsections. The first deals with a refinement of the Simon Newcomb problem, while the second deals with a variant of this problem. Clearly, there are several such variants which may be treated by the same method. Two sets of paths, $\tilde{P}_3$ (increasing sequences) and $\tilde{P}_4$ (strictly increasing sequences) are used and the section serves to illuminate the use of Theorem 5.1 in the two situations.

6.1 *Andrews' refinement of the Simon Newcomb problem.* Andrews' refinement (Andrews [**4**]) concerns the enumeration of sequences with respect to occurrences of maximal paths of $\tilde{P}_3$ (increasing sequences) and the terminator elements of each of the paths. The refinement arose in connexion with Long's Conjecture (Long [**26**; **27**]) concerning factorisations of integers, and its proof (Andrews [**5**]).

COROLLARY 6.1. *The number of sequences with a_i occurrences of i, and H_i occurrences of i as a terminator of maximal increasing subsequences is*

$$[\mathbf{x}^{\mathbf{a}}][\mathbf{v}^{\mathbf{H}}]\left\{1 - \sum_{j>0} x_j v_j \prod_{k=1}^{j} (1 - x_k(1 - v_k))^{-1}\right\}^{-1}.$$

Proof. Let $\Delta(p) = v_j \tau(p)$ whenever $p \in \tilde{M}_{ij}$, and let $\tau(i_1 i_2 \ldots i_n) = x_{i_1} x_{i_2} \ldots x_{i_n}$. The ring of weights is assumed to be commutative. Now clearly the generating function for increasing sequences beginning with i and ending with j, where $j \geqq i$, may be written down directly since a sequence on

$\{i, i+1, \ldots, j\}$ corresponds to a unique increasing sequence. Accordingly,

$$[A]_{ij} = \Delta_{ij} = \begin{cases} v_j x_j (1-x_j)^{-1} & \text{if } i = j \\ v_j x_i x_j \prod_{k=i}^{j} (1-x_k)^{-1} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$$

for an increasing path beginning with i and terminating with j. But $\tilde{P}_3 = \tilde{P}_s$ where $s(i) = \{j | j \geqq i\}$ so the representing matrices are

$$[\bar{M}_{ij}]_{mn} = \begin{cases} 1 & \text{if } m = i \leqq j \leqq n \\ 0 & \text{otherwise.} \end{cases}$$

Hence, from Theorem 5.1

$$I + \sum_{i,j} \Delta_{ij} \bar{M}_{ij} = \left\{ I - \sum_{i,j} \delta_{ij} \bar{M}_{ij} \right\}^{-1}$$

may be written

$$I + AT = (I - DT)^{-1}$$

where

$$[D]_{ij} = \delta_{ij}$$

and

$$[T]_{ij} = \begin{cases} 1 & \text{if } i \leqq j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $D = (A^{-1} + T)^{-1}$. Now, by routine but lengthy algebra we have:

$$[A^{-1}]_{mn} = \begin{cases} (1 - x_m) v_m^{-1} x_m^{-1} & \text{if } m = n \\ 0 & \text{if } m > n \\ -v_m^{-1} & \text{if } m < n \end{cases}$$

so

$$A^{-1} + T = \operatorname{diag}\,(x_1^{-1} v_1^{-1}, x_2^{-1} v_2^{-1}, \ldots) BT$$

where

$$[B]_{ij} = \begin{cases} 1 - x_i(1 - v_i) & \text{if } i = j \\ -1 & \text{if } i + 1 = j \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$D = T^{-1} B^{-1} \operatorname{diag}\,(x_1 v_1, x_2 v_2, \ldots)$$

where

$$[B^{-1}]_{ij} = \begin{cases} \prod_{k=i}^{j} \{1 - x_k(1 - v_k)\}^{-1} & \text{if } i \leqq j \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{i,j} \delta_{ij} = \sum_{j=1}^{\infty} [B^{-1} \operatorname{diag}\,(x_1 v_1, x_2 v_2, \ldots)]_{1j} = \sum_{j=1}^{\infty} x_j v_j \prod_{k=1}^{j} \{1 - x_k(1 - v_k)\}^{-1}$$

which gives the required generating function.

Remark. This generating function differs from the one obtained by Andrews' [**4**], but yields the same coefficient as the following derivation demonstrates.

$$[\mathbf{x}^{\mathbf{a}}\mathbf{v}^{\mathbf{H}}]\left\{1-\sum_{i=1}^{n} x_i v_i \prod_{k=1}^{i}\{1-x_k(1-v_k)\}^{-1}\right\}^{-1}$$

$$= [\mathbf{x}^{\mathbf{a}}\mathbf{v}^{\mathbf{H}}] \sum_{l_1,\ldots,l_n\geqq 0} \begin{bmatrix} l_1+\ldots+l_n \\ l_1, l_2, \ldots, l_n \end{bmatrix} (x_1v_1)^{l_1}\ldots(x_nv_n)^{l_n}$$

$$\times \prod_{k=1}^{n}\{1-x_k(1-v_k)\}^{-\Sigma_{i=k}^n l_i}$$

$$= [\mathbf{v}^{\mathbf{H}}] \sum_{l_1,\ldots,l_n\geqq 0} \begin{bmatrix} l_1+\ldots+l_n \\ l_1, l_2, \ldots, l_n \end{bmatrix} (v_1^{l_1}\ldots v_n^{l_n}) \prod_{k=1}^{n} (1-v_k)^{a_k-l_k}$$

$$\times \binom{a_k-l_k+\sum_{i=k}^{n} l_i - 1}{a_k-l_k}$$

$$= \sum_{l_1,\ldots,l_n\geqq 0} \begin{bmatrix} l_1+\ldots+l_n \\ l_1, \ldots, l_n \end{bmatrix} \prod_{k=1}^{n} \binom{a_k-l_k+\sum_{i=k}^{n} l_i - 1}{a_k-l_k}$$

$$\times \binom{a_k-l_k}{H_k-l_k}(-1)^{H_k-l_k}$$

$$= \sum_{l_1,\ldots,l_n\geqq 0} (-1)^{\Sigma_{k=1}^n (H_k+l_k)} \frac{l_n}{a_n}\binom{a_n}{l_n}(l_1+\ldots+l_n)!$$

$$\times \prod_{k=1}^{n} \binom{a_k-l_k}{a_k-H_k}\left(\frac{1}{l_k!}\right).$$

$$\prod_{k=1}^{n-1}\left\{\frac{\left(a_k+\sum_{i=k+1}^{n} l_i-1\right)!}{a_k!\left(\sum_{i=k+1}^{n} l_i-1\right)!}\right\}\cdot\left\{\frac{\left(\sum_{i=k+1}^{n} l_i-1\right)!}{\left(\sum_{i=k}^{n} l_i-1\right)!}\right\}\cdot\left\{\frac{a_k!}{(a_k-l_k)!}\right\}$$

$$= \sum_{l_1,\ldots,l_n\geqq 0} (-1)^{\Sigma_{k=1}^n (H_k+l_k)} \frac{l_1+\ldots+l_n}{a_n} \prod_{k=1}^{n-1} \binom{a_k+\sum_{i=k+1}^{n} l_i-1}{a_k}$$

$$\cdot \prod_{k=1}^{n} \binom{a_k-l_k}{a_k-H_k}\binom{a_k}{l_k}$$

(which is a version of Andrews' result)

$$= \binom{a_1}{H_1}\binom{a_2}{H_2}\ldots\binom{a_n}{H_n} \sum_{l_1,\ldots,l_n\geqq 0} (-1)^{\Sigma_{k=1}^n (H_k+l_k)}$$

$$\times \frac{l_1+\ldots+l_k}{a_k}\cdot\prod_{k=1}^{n}\binom{H_k}{l_k}\cdot\prod_{k=1}^{n-1}\binom{a_k+\sum_{i=k+1}^{n} l_i-1}{a_k}.$$

No further simplification has been obtained.

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6.2 *A related problem.* There is a certain degree of freedom in the choice of the set of paths to be used in the problem stated in Section 6.1. A natural second choice is the set $\tilde{P}_4$ of strictly increasing sequences. The following corollary gives the enumeration in this case.

COROLLARY 6.2. *The number of sequences with a_i occurrences of i and H_i occurrences of i as a terminator of a maximal strictly increasing subsequence is*

$$[\boldsymbol{x}^{\boldsymbol{a}}\boldsymbol{v}^{\boldsymbol{H}}]\left\{\left(1 - \sum_{j>0} x_j v_j \prod_{k=1}^{j-1} \{1 + x_k(1 - v_k)\}\right)\right\}^{-1}.$$

Proof. The proof follows closely that of Corollary 6.1. In this case we have

$$[A]_{ij} = \begin{cases} x_i v_i & \text{if } i = j \\ v_j x_i x_j \prod_{k=i+1}^{j-1} (1 + x_k) & \text{if } i < j \\ 0 & \text{otherwise} \end{cases}$$

since the subsequences are strictly increasing. Now $\tilde{P}_4 = \tilde{P}_s$ where $s(i) = \{j | j > i\}$ so

$$[\bar{M}_{ij}]_{mn} = \begin{cases} 1 \text{ if } m = i \leqq j < n \\ 0 \text{ otherwise} \end{cases}$$

for the representing algebra. Accordingly, from Theorem 5.1,

$$I + \sum_{i,j} [A]_{ij}\bar{M}_{ij} = \left\{I - \sum_{i,j} \delta_{ij}\bar{M}_{ij}\right\}$$

becomes

$$I + AS = (I - DS)^{-1}$$

where

$$[S]_{ij} = \begin{cases} 1 \text{ if } i < j \\ 0 \text{ otherwise} \end{cases}$$

and $D = [\delta_{ij}]$ as before. Accordingly $D = (A^{-1} + S)^{-1} + Z$ where $ZS = 0$. Clearly $Z = 0$ since S is infinite dimensional.
Thus

$$D = (\text{diag } (x_1^{-1}v_1^{-1}, x_2^{-1}v_2^{-1}, \ldots)BT)^{-1}$$

where

$$[T]_{ij} = \begin{cases} 1 \text{ if } i \leqq j \\ 0 \text{ otherwise} \end{cases}$$

and

$$[B]_{ij} = \begin{cases} 1 & \text{if } i = j \\ -(1 + x_i(1 - v_i)) & \text{if } i + 1 = j \\ 0 & \text{otherwise} \end{cases}$$

whence

$$D = T^{-1}B^{-1}\operatorname{diag}(x_1v_1, x_2v_2, \ldots)$$

where

$$[B^{-1}]_{ij} = \begin{cases} \prod_{k=i}^{j-1} \{1 + x_k(1 - v_k)\} & \text{if } i \leqq j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\sum_{i,j} \delta_{ij} = \sum_{j=1}^{\infty} [B^{-1}\operatorname{diag}(x_1v_1, x_2v_2, \ldots)]_{1j} = \sum_{j=1}^{\infty} x_j v_j \prod_{k=1}^{j-1} \{1 + x_k(1 - v_k)\}$$

and the corollary follows.

7. Problems involving length information: **p-paths in sequences.** In this section, generating functions concerned with the occurrence of paths of length p are obtained, and in the next section the corresponding results for permutations are derived. The following assumptions are made for the remainder of the paper.

(i) The set of paths is $\tilde{P}_2$ (increasing runs), $\tilde{P}_3$ (increasing sequences) or $\tilde{P}_4$ (strictly increasing sequences). The results may be generalised in many cases, but this has been left to the reader.

(ii) We define

(a) $\tau(i_1i_2 \ldots i_n) = x_{i_1}x_{i_2} \ldots x_{i_n}$

and

(b) $$\gamma_k = \sum_{\substack{p \in \tilde{P} \\ |p|=k}} \tau(p) = \begin{cases} \sum_{i=0}^{\infty} x_{i+1}x_{i+2} \ldots x_{i+k} & \text{if } \tilde{P} = \tilde{P}_2 \\ [x^k] \prod_{i=1}^{\infty} (1 - xx_i)^{-1} & \text{if } \tilde{P} = \tilde{P}_3 \\ [x^k] \prod_{i=1}^{\infty} (1 + xx_i) & \text{if } \tilde{P} = \tilde{P}_4 \end{cases}$$

(iii) The ring $\tilde{R}$ of weights is commutative, and we define

$$F(x) = 1 + \sum_{k>0} F_k x^k, \quad f(x) = 1 - \sum_{k>0} f_k x^k \quad \text{and} \quad F = f^{-1}.$$

For convenience let $f_0 = -1$ and $f_k = 0$ for $k < 0$.

COROLLARY 7.1. *The generating function for sequences with exactly i p-paths is*

$$[u^i]\left\{1 - \sum_{k>0} \gamma_k f_k\right\}^{-1}$$

where

$$f_1 = 1, \quad f_2 = \begin{cases} u - 1 & \text{if } p = 2 \\ 0 & \text{if } p > 2 \end{cases}$$

and $\{f_n\}$ satisfies $f_n = uf_{n-1} + (1 - u)f_{n-p}$ for $n > 2$.

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Proof. Let $F(x) = 1 + x + x^2 + \ldots + x^{p-1} + ux^p + u^2x^{p+1} + \ldots + u^kx^{p+k-1} + \ldots$ since there are no p-paths in a sequence of length less than p and there are k p-paths in a sequence of length $p + k - 1$. Thus

$$F(x) = \frac{1 - ux + (u - 1)x^p}{(1 - x)(1 - ux)}$$

in Theorem 4.1, whence

$$f(x) = \frac{(1 - x)(1 - ux)}{1 - ux + (u - 1)x^p}$$

so

$$f(x) - uxf(x) + (u - 1)x^pf(x) = 1 - (1 + u)x + ux^2.$$

The corollary follows by comparing coefficients on either side of this relation.

COROLLARY 7.2. *The generating function for sequences with no p-paths is*

$$\left\{\sum_{k\equiv 0 (\mathrm{mod}\, p)} \gamma_k - \sum_{k\equiv 1(\mathrm{mod}\, p)} \gamma_k\right\}^{-1}$$

where $\gamma_0 = 1$.

Proof. Let $u = 0$ in Corollary 7.1. Then

$$F(x) = \frac{1 - x^p}{1 - x}$$

so

$$f(x) = \frac{1 - x}{1 - x^p} = 1 - x + x^p - x^{p+1} + \ldots$$

and the result follows.

This proves a strengthened form of the conjecture for p-runs given in Jackson [**20**].

COROLLARY 7.3. *The generating function for sequences whose longest path is of length p is* $\Psi(1) - \Psi(0)$ *where*

$$\Psi(u) = \left\{1 - \sum_{k>0} f_ku^k\right\}^{-1}$$

and for $p \geqq 2, f_1 = 1$ *and* $\{f_k\}$ *satisfies*

$$f_n = (1 - u)f_{n-p} + uf_{n-p-1}$$

for $n > 1$.

Proof. Let $F(x) = 1 + x + x^2 + \ldots + x^{p-1} + ux^p$ since there are no paths of length greater than p, and the lengths of paths less than p are not recorded.

Thus

$$F(x) = \frac{1 - (1 - u + ux)x^p}{1 - x}.$$

The recurrence for $f(x) = F^{-1}(x)$ follows provided $p \geqq 2$. Thus, from Theorem 4.1, the generating function is

$$\sum_{i=1}^{\infty} [u^i]\Psi(u) = \Psi(1) - \Psi(0).$$

8. Problems involving length information: p-paths in permutations. By a *permutation on n* we mean a permutation of $\{1, 2, \ldots, n\}$. The initial assumptions given in Section 7(i), (ii), (iii) hold in this section. Two prefatory lemmas are needed for the transition from sequences to permutations. Lemma 8.1 deals with increasing runs and Lemma 8.2 deals with increasing sequences. Theorems 8.3 and 8.4 are the enumerative theorems associated with these two cases. A collection of seven corollaries follows giving the generating functions for specific problems.

LEMMA 8.1. *If*

$$\gamma_k = \sum_{i=0}^{\infty} x_{i+1}x_{i+2}\ldots x_{i+k}, \quad \gamma_0 = 1$$

and

$$q = i_1 + i_2 + \ldots + i_n,$$

then

$$[\mathbf{x}][\mathbf{y}^{\mathbf{i}}]\{\gamma_1 y_1 + \gamma_2 y_2 + \ldots + \gamma_n y_n\}^q = \begin{cases} q!\begin{bmatrix} q \\ i_1,\ldots,i_n \end{bmatrix} & \textit{if } i_1 + 2i_2 + \ldots + ni_n = n \\ 0 & \textit{otherwise} \end{cases}$$

where

$$\begin{bmatrix} q \\ i_1,\ldots,i_n \end{bmatrix} = \frac{q!}{i_1!i_2!\ldots i_n!}$$

and the notational conventions given in Section 1 *have been employed.*

Proof. The *sinister* is the number of (additive) compositions of n with q parts, since a decomposition of $x_1x_2\ldots x_n$ into products of γ_k may be obtained by marking off the i_k blocks of size k along the set $(x_1, x_2, \ldots, x_n)$. The proof follows.

LEMMA 8.2. *If*

$$\gamma_k = [x^k]\prod_{i=1}^{\infty}(1 - xx_i)^{-1}$$

and

$$q = i_1 + i_2 + \ldots + i_n$$

then

$$[\boldsymbol{x}][\boldsymbol{y}^{\boldsymbol{i}}]\{\gamma_1 y_1 + \gamma_2 y_2 + \ldots + \gamma_n y_n\}^q$$
$$= \begin{cases} \begin{bmatrix} q \\ i_1, i_2, \ldots, i_n \end{bmatrix} \dfrac{n!}{1!^{i_1} 2!^{i_2} \ldots n!^{i_n}} & \text{if } i_1 + 2i_2 + \ldots + ni_n = n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The *sinister* is the number of ordered partitions of the set $\{x_1, x_2, \ldots, x_n\}$ into unordered blocks such that there are i_k blocks of length k. This reduces immediately to the *dexter*.

For permutations, increasing sequences must be strictly increasing and identical results are accordingly obtained in Lemma 8.2 if

$$\gamma_k = [x^k] \prod_{i=1}^{\infty} (1 + xx_i),$$

the quantity associated with $\tilde{P}_4$ (see Section 7(ii)).

THEOREM 8.3 (increasing runs in permutations). *The number of permutations on n with d_i maximal increasing runs of length i is*

$$[x^n][\boldsymbol{F}^{\boldsymbol{d}}] \sum_{k=0}^{\infty} (-1)^k k! \left\{\frac{1 - F(x)}{F(x)}\right\}^k.$$

Proof. From Theorem 4.1, the generating function is

$$[\boldsymbol{F}^{\boldsymbol{d}}][\boldsymbol{x}]\left\{1 - \sum_{j>0} f_j \gamma_j\right\}^{-1}$$

where

$$F(x) = \left\{1 - \sum_{j>0} f_j x^j\right\}^{-1}$$

and

$$\gamma_k = \sum_{i=0}^{\infty} x_{i+1} x_{i+2} \ldots x_{i+k}, \quad \gamma_0 = 1.$$

But

$$[\boldsymbol{x}]\left\{1 - \sum_{j>0} f_j \gamma_j\right\}^{-1} = [\boldsymbol{x}] \sum_{k \geqq 0} \left\{\sum_{j>0} f_j \gamma_j\right\}^k$$
$$= [\boldsymbol{x}] \sum_{k \geqq 0} \sum_{i_1 + i_2 + \ldots + i_k = k} (f_1^{i_1} f_2^{i_2} \ldots f_k^{i_k})[f_1^{i_1} f_2^{i_2} \ldots f_k^{i_k}]\left(\sum_{j>0} f_j \gamma_j\right)^k$$
$$= \sum_{k \geqq 0} k! [x^n](xf_1 + x^2 f_2 + \ldots)^k$$

from Lemma 8.1 and the theorem follows.

THEOREM 8.4 (increasing subsequences in permutations). *The number of permutations on n with d_i maximal increasing subsequences of length i is*

$$\left[\frac{x^n}{n!}\right][\boldsymbol{F}^{d}]\left\{1 - \sum_{j>0} f_j \frac{x^j}{j!}\right\}^{-1}.$$

Proof. The proof is similar to that of Theorem 8.3. The generating function is

$$[\boldsymbol{x}]\left\{1 - \sum_{j>0} f_j\gamma_j\right\}^{-1}$$

where

$$\gamma_k = [x^k] \prod_{j=1}^{\infty} (1 - xx_j)^{-1}.$$

The generating function is

$$[\boldsymbol{x}] \sum_{k\geqq 0} \left(\sum_{j>0} f_j\gamma_j\right)^k$$

$$= [\boldsymbol{x}] \sum_{k\geqq 0} \sum_{i_1+i_2+\ldots+i_k=k} (f_1^{i_1}f_2^{i_2}\ldots f_k^{i_k})[f_1^{i_1}f_2^{i_2}\ldots f_k^{i_k}]\left\{\sum_{j>0} f_j\gamma_j\right\}^k$$

$$= [x^n] \sum_{k\geqq 0} k!\left(\frac{f_1x}{1!} + \frac{f_2x^2}{2!} + \ldots\right)^k$$

by Lemma 8.2, and the theorem follows.

A number of corollaries of these two theorems is now given. Clearly, variants of these problems may be treated in the same way.

COROLLARY 8.5. *The number of permutations on n with exactly i (strictly) increasing p-runs is*

$$[x^n u^i] \sum_{k\geqq 0} k!x^k\left\{\frac{1 - ux - (1 - u)x^{p-1}}{1 - ux - (1 - u)x^p}\right\}^k.$$

Proof. Following the proof of Corollary 7.1, let

$$F(x) = 1 + x + x^2 + \ldots + x^{p-1} + ux^p + u^2x^{p+1}\ldots$$

and the proof follows from Theorem 8.3.

COROLLARY 8.6. *The number of permutations on n with no increasing p-runs is*

$$[x^k] \sum_{k\geqq 0} k!x^k\left\{\frac{1 - x^{p-1}}{1 - x^p}\right\}^k.$$

Proof. Substitute $u = 0$ in Corollary 8.5.

The case $p = 2$ of Corollary 8.6 has been given by Whitworth [**39**] and the case $p = 3$ by Riordan [**30**]. The more general case has been considered by Abramson and Moser [**1**] who obtained an explicit formula for the number of permutations of $\{1, 2, \ldots, n\}$ with exactly i increasing p-runs. An independent proof of Corollary 8.6, together with a linear recurrence equation for the coefficients and tabulations of coefficients, has been given by Jackson and Reilly [**23**].

COROLLARY 8.7. *The number of permutations on n with exactly i (strictly) increasing subsequences is*

$$\left[\frac{x^n}{n!}u^i\right]\left\{1 - \sum_{i>0} \frac{f_i x^i}{i!}\right\}^{-1}$$

where

$$1 - \sum_{i>0} f_i x^i = \frac{(1-x)(1-ux)}{1 - ux + (u-1)x^p}.$$

Proof. Direct from Corollary 7.1 and Theorem 8.4.

COROLLARY 8.8. *For $S_n^{(m)}$, the Stirling numbers of the second kind, a combinatorial interpretation of the well-known identity* (Riordan [**31**])

$$\sum_{m=0}^{n} (-1)^{n-m} m! S_n^{(m)} = 1$$

is that there is exactly one permutation with no increasing 2-subsequences.

Proof. Let $u = 0$ and $p = 2$ in Corollary 8.7. Then $f_i = -(-1)^i$. Thus the number of permutations on n with no increasing 2-subsequence is

$$\left[\frac{x^n}{n!}\right]\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)^{-1} = \left[\frac{x^n}{n!}\right]e^x = 1.$$

But

$$\left[\frac{x^n}{n!}\right]e^x = \left[\frac{x^n}{n!}\right]\{1 + (e^{-x} - 1)\}^{-1} = \left[\frac{x^n}{n!}\right] \sum_{k=0}^{\infty} (-1)^k (e^{-x} - 1)^k$$
$$= \left[\frac{x^n}{n!}\right] \sum_{k=0}^{\infty} k! \sum_{j=k}^{\infty} (-1)^{k-j} S_j^{(k)} \frac{x^j}{j!}$$

and the corollary follows.

COROLLARY 8.9. *The number of permutations on n whose longest increasing run has length p is*

$$[x^n] \sum_{k\geq 0} k! x^k \left\{\left(\frac{1 - x^p}{1 - x^{p+1}}\right)^k - \left(\frac{1 - x^{p-1}}{1 - x^p}\right)^k\right\}.$$

Proof. As in the proof of Corollary 7.3, let

$$F(x) = \frac{1 - (1 - u + ux)x^p}{1 - x}.$$

The required generating function is

$$\sum_{i>0} [u^i]\Psi(u) = \Psi(1) - \Psi(0),$$

where $\Psi(u)$ is obtained from Theorem 8.3.

COROLLARY 8.10. *The number of permutations on n with a unique longest increasing run of length k is*

$$[x^n] \sum_{k \geqq 0} k.k!x^{k+p-1}(1-x)^2 \frac{(1-x^{p-1})^{k-1}}{(1-x^p)^{k+1}}.$$

Proof. In the notation of Corollary 8.9, the solution is $[u]\Psi[u]$.

COROLLARY 8.11. *The number of permutations on n with only increasing subsequences of length* 2 *is* $[x^n/n!]$ sec x.

Proof. Let $F(x) = 1 + x^2$ in Theorem 8.4.

Equivalently, Corollary 8.11 enumerates the number of permutations for even n which have an alternating sequence of rises and falls. Permutations with this property have been called "alternating". For alternating permutations of any length the exponential generating function is sec x + tan x, a result obtained by André [**2**; **3**] and examined later by Carlitz [**9**], Foata and Schützenberger [**18**] and others. In order to obtain this result, it is necessary to allow the occurrence of a single fall at the end of a permutation. More generally, it is necessary to treat the extreme maximal paths differently from the remaining paths. This is done in Part III, where André's result is given as Corollary 12.3. The result for alternating sequences is given in Corollary 11.6.

9. Maximum paths in sequences and related results. The purpose of this section is to provide an interpretation of a modified form of Theorem 9.6 given below. The latter theorem is of enumerative interest because it may be used to unify a number of earlier results obtained by Smirnov, et al. [**34**], Carlitz [**7**], Eifler *et al.* [**15**], Dillon and Roselle [**14**] and Carlitz *et al.* [**8**]. The details of the proof of Theorem 9.6 and its application to sequence enumeration problems are given in Jackson [**21**]). It may also be specialised to give the Eulerian partition identities, a number of q-binomial identities, the Ménage problem, the derangement problem and the generalised derangement problem (Even and Gillis [**16**], Jackson [**22**]). The relationship between Theorem 9.6 and its modified form, Theorem 9.7, is analogous to the relationship between the Simon Newcomb problem (Riordan [**31**]) and Andrews' refinement of it (Andrews [**4**]). Certain graph theoretic aspects of some sequence problems, including the Simon Newcomb problem, have been considered by Klarner [**24**] and by Roselle [**32**].

We begin with a basic problem involving sequences with a specified number of maximal paths. The assumptions of Section 7(i), (ii), (iii) apply throughout this section.

PROPOSITION 9.1. *The generating function for sequences with k maximal paths is*

$$[v^k]\left\{1 - v \sum_{j>0} (1-v)^{j-1}\gamma_j\right\}^{-1}.$$

Proof. Let $F(x) = 1 + vx + vx^2 + \ldots$ since each maximal path has weight v. Then $f_k = v(1-v)^{k-1}$ for $k \geqq 1$, and the result follows from Theorem 4.1.

LEMMA 9.2. *The generating function for sequences with exactly k maximal strictly increasing subsequences is*

$$\text{(i)}\ [v^k]\frac{1-v}{1 - v\prod_{i=1}^{\infty}\{1 + x_i(1-v)\}}.$$

The generating function for sequences with k maximal increasing subsequences is

$$\text{(ii)}\ [v^k]\frac{1-v}{1 - v\prod_{i=1}^{\infty}\{1 - x_i(1-v)\}^{-1}}.$$

Proof. (i) follows from Proposition 9.1 with

$$\gamma_k = [x^k]\prod_{j=1}^{\infty}(1 + xx_j)$$

and (ii) follows from Proposition 9.1 with

$$\gamma_k = [x^k]\prod_{j=1}^{\infty}(1 - xx_j)^{-1}.$$

We may regard a decreasing sequence as being composed of maximum strictly increasing subsequences of length one. Accordingly we have the following fact.

PROPOSITION 9.3. *A sequence has k falls or levels if and only if it has $k + 1$ maximal strictly increasing subsequences.*

Proof. Every fall or level which occurs marks the beginning of a new maximal strictly increasing subsequence.

LEMMA 9.4. *The generating function for sequences with k falls or levels is*

$$\text{(i)}\ [v^{k+1}]\frac{1-v}{1 - v\prod_{j=1}^{\infty}\{1 + x_j(1-v)\}}.$$

The generating function for sequences with k falls is

$$\text{(ii) } [v^{k+1}] \frac{1-v}{1 - v \prod_{j=1}^{\infty} \{1 - x_j(1-v)\}^{-1}}.$$

Proof. The results follow from Lemma 9.2 and Proposition 9.3.

We note that Lemma 9.4 gives the solution of the Simon Newcomb problem, and it suggests the following lemma.

LEMMA 9.5.

$$\text{(i) } \frac{1-v}{1 - v \prod_{k=1}^{\infty} \{1 + x_k(1-v)\}} = \left\{1 - v \sum_{i=1}^{\infty} x_i \prod_{k=1}^{i-1} \{1 + x_k(1-v)\}\right\}^{-1}$$

$$\text{(ii) } \frac{1-v}{1 - v \prod_{k=1}^{\infty} \{1 - x_k(1-v)\}^{-1}} = \left\{1 - v \sum_{i=1}^{\infty} x_i \prod_{k=1}^{i} \{1 - x_k(1-v)\}^{-1}\right\}^{-1}$$

Proof. (Combinatorial) By Lemma 9.2(i) (resp. (ii)), the *sinister* enumerates sequences with respect to maximal strictly (resp. non-strictly) increasing subsequences. By Corollary 6.1 (resp. Corollary 6.2)), the *dexter* does so as well.

(Algebraic) (i) By induction we may show that

$$1 + \sum_{i=1}^{n} (z_i - 1)(z_1 z_2 \ldots z_{i-1}) = z_1 z_2 \ldots z_n.$$

Let $z_i = 1 + x_i(1 - v)$ and the result follows.

(ii) Similarly, it may be shown that

$$1 + \sum_{i=1}^{n} (1 - z_i)(z_1 z_2 \ldots z_i)^{-1} = (z_1 z_2 \ldots z_n)^{-1}$$

and the result follows with $z_i = 1 - x_i(1 - v)$.

The expressions given in Lemma 9.5 are related to specialisations of the generating function given in the following theorem.

THEOREM 9.6. *The generating function for sequences with i rises, j falls and k levels is*

$$[r^{i+1} f^{j+1} l^k] 1 - rf \frac{\prod_{j=1}^{\infty} \{1 + (r-l)x_j\} - \prod_{j=1}^{\infty} \{1 + (f-l)x_j\}}{f \prod_{j=1}^{\infty} \{1 + (r-l)x_j\} - r \prod_{j=1}^{\infty} \{1 + (f-l)x_j\}}.$$

Proof. See Jackson [**21**].

The relationship between the results of this section have suggested the following theorem, which is now proved directly with the methods of Theorem 5.1 as they were applied in Section 6.1. The theorem explains, in effect, what happens when the r, l and f in Theorem 9.6 are permitted to be subscripted.

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THEOREM 9.7. *The generating function for sequences with a_i levels on the element i, b_i rises starting with i, and c_i maximal increasing subsequences terminated by i is*

$$[l^a r^b f^c]\left\{1 - \sum_{i=1}^{\infty} \frac{x_i f_i}{1 + (f_i - l_i)x_i} \prod_{k=1}^{i-1} \frac{1 + (r_k - l_k)x_k}{1 + (f_k - l_k)x_k}\right\}^{-1}.$$

Proof. Consider a sequence over $(i, i + 1, \ldots, j)$ where $i < j$. This corresponds to a unique increasing sequence, and suppose that this sequence begins with i and terminates with j. This contributes a factor of $x_i x_j$ to Δ_{ij}. But since $i < j$, i initiates a rise and j terminates an increasing subsequence so this contributes a further factor of $r_i f_j$ to Δ_{ij}. Repeated occurrences of i and j result in levels recorded by l_i and l_j. These contribute a factor of $(1 - l_i x_i)^{-1}$ $(1 - l_j x_j)^{-1}$ to Δ_{ij}. Each element k, where $i < k < j$ if it occurs, initiates a rise, marked by r_k and, of course, levels marked by l_k if k occurs more than once. For each k, $i < k < j$, the contribution to Δ_{ij} is $1 + r_k(1 - l_k x_k)^{-1}$. Accordingly, Δ_{ij} is the product of these contributions. If $i = j$, the situation is straightforward. Thus, let

$$[A]_{ij} = \Delta_{ij} = \begin{cases} x_i f_i (1 - l_i x_i)^{-1} & \text{if } i = j, \\ x_i x_j r_i f_j (1 - l_i x_i)^{-1}(1 - l_j x_j)^{-1} \prod_{k=i+1}^{j-1} \{1 + x_k r_k (1 - l_k x_k)^{-1}\} & \text{if } i < j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, following Section 6.1, we have

$$D = (A^{-1} + T)^{-1}.$$

Now, by routine but lengthy algebra we have

$$[A^{-1}]_{ij} = \begin{cases} (1 - l_i x_i) x_i^{-1} f_i^{-1} & \text{if } i = j \\ -r_i f_i^{-1} & \text{if } i < j \\ 0 & \text{otherwise.} \end{cases}$$

So $A^{-1} + T$ is of the form

$$[A^{-1} + T]_{ij} = \begin{cases} \{1 + (f_i - l_i)x_i\} x_i^{-1} f_i^{-1} = a_i, & \text{say, if } i = j \\ (f_i - r_i) f_i^{-1} = b_i, & \text{say, if } i < j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$[D]_{ij} = \begin{cases} a_j^{-1} & \text{if } i = j \\ a_j^{-1}\left\{\prod_{k=i}^{j-1} c_k - \prod_{k=i-1}^{j-1} c_k\right\} & \text{if } i < j \\ 0 & \text{otherwise} \end{cases}$$

where

$$c_k = 1 - \frac{b_k}{a_k} = \frac{1 + (r_k - l_k)x_k}{1 + (f_k - l_k)x_k}, \quad c_0 = 1.$$

The generating function is accordingly

$$\left\{1 - \sum_{i,j} [D]_{ij}\right\}^{-1} = \left\{1 - \sum_j \left(a_j^{-1} + \sum_{i=1}^{j-1} a_j^{-1}\left\{\prod_{k=1}^{j-1} c_k - \prod_{k=i-1}^{j-1} c_k\right\}\right)\right\}^{-1}$$
$$= \left\{1 - \sum_j a_j^{-1} \prod_{k=1}^{j-1} c_k\right\}^{-1}$$

and the theorem follows.

Theorem 9.7 contains Corollaries 6.1 and 6.2, and Theorem 9.6, provided the conventional rise and fall at the ends of the sequences are treated appropriately. Accordingly it unifies the results which devolve from 6.1, 6.2 and 9.6.

PART III: Extensions of the Method.

10. Two extensions of the main theorem. The remaining sections are concerned chiefly with methods of determining the number of *maxima* and *minima* in sequences and permutations. André's result [**3**] for permutations is perhaps the most familiar one in this context, but there are several others. Combinatorially, the situation presents no difficulties since we may observe that a maximal increasing subsequence must be terminated with a maximum and initiated by a minimum. The exceptions are, of course, the initial and terminal maximal subsequences, which must therefore be treated especially. However, the maximal decomposition is again applicable, and only slight adjustments to the previous analysis are necessary. For expository purposes alone we shall refer loosely to maximal paths which are treated in an exceptional fashion as *differentiated paths*. Although our chief concern lies in *differentiated initial paths* and *differentiated terminal paths* (or, collectively, *differentiated extreme paths*), in principle other paths may be differentiated. It is unclear, however, that there are situations which benefit from the latter possibility, and it has not been pursued further.

The following two theorems correspond to Theorem 3.1 (maximal paths) in the context of differentiated extreme paths. The ring $\tilde{R}$ need not be commutative.

THEOREM 10.1 (differentiated terminal maximal paths). *If*

(i) $\tilde{P}$ *admits a maximal decomposition* $u(\sigma) = (p_1, p_2, \ldots, p_n)$ *for any* $\sigma \in \tilde{P}^*$, *and*

(ii) $\Delta_1, \Delta_2 \colon \tilde{P} \to \tilde{R}$ *are arbitrary, where* $\tilde{R}$ *is a ring with* 1, *then there exist unique* $\delta_1, \delta_2 \colon \tilde{P} \to \tilde{R}$ *such that for any* $p \in \tilde{P}$

$$\text{(ii)}^{\text{a}} \quad \Delta_1(p) = \sum_{d \leqq (p)} \delta_1(d)$$

and

$$\text{(ii)}^{\text{b}} \quad \Delta_2(p) = \sum_{d \leqq (p)} \left\{\prod_{i=1}^{m-1} \delta_1(d_1)\right\} \delta_2(d_m)$$

where $d = (d_1, d_2, \ldots, d_m)$, $m \geqq 1$. *Moreover, the generating function*

$$\text{(iii) } 1 + \sum_{\sigma \in \tilde{P}^+} \left\{ \prod_{i=1}^{n-1} \Delta_1(p_i) \right\} \Delta_2(p_n)$$

(*note that the empty product is* 1, *so sequences of one path* p *contribute only* $\Delta_2(p)$ *to the sum*), *is equal to*

$$1 + \left\{ 1 - \sum_p \delta_1(p) \right\}^{-1} \sum_p \delta_2(p)$$

where $u(\sigma) = (p_1, \ldots, p_n)$.

Proof. Again we seek a connexion between the combinatorially described generating function

$$\sum_{\sigma \in \tilde{P}^+} \left\{ \prod_{i=1}^{n-1} \Delta_1(p_i) \right\} \Delta_2(p_n)$$

and the form

$$\left\{ 1 - \sum_{p \in \tilde{P}} \delta_1(p) \right\}^{-1} \sum_{p \in \tilde{P}} \delta_2(p).$$

Clearly, from (ii)[a] and (ii)[b], δ_1 and δ_2 exist and are unique by Möbius inversion. We proceed by direct expansion of the *dexter* of (iii). With $\delta_1(\epsilon) = 1$, we have

$$\left\{ 1 - \sum_p \delta_1(p) \right\}^{-1} \left\{ \sum_p \delta_2(p) \right\} = \sum_{k \geqq 1} \sum_{\sigma \in \tilde{P}^+} \sum_d \delta_1(d_1, d_2, \ldots, d_{k-1}) \delta_2(d_k)$$

(where the summation is over $d \in c^{-1}(\sigma) \cap \tilde{P}^k$)

$$= \sum_{\sigma \in \tilde{P}^+} \sum_{d \in c^{-1}(\sigma)} \delta_1(d_1, d_2, \ldots, d_{m-1}) \delta_2(d_m)$$

(where $d = (d_1, d_2, \ldots, d_m)$ and $m \geqq 1$)

$$= \sum_{\sigma \in \tilde{P}^+} \left\{ \prod_{i=1}^{n-1} \sum_{d \in c^{-1}(p_i)} \delta_1(d) \right\} \left\{ \sum_{d \in c^{-1}(p_n)} \delta_1(d_1, d_2, \ldots, d_{m-1}) \delta_2(d_m) \right\}$$

(where $d = (d_1, d_2, \ldots, d_m)$ and $u(\sigma) = (p_1, p_2, \ldots, p_n)$, $n \geqq 1$)

$$= \sum_{\sigma \in \tilde{P}^+} \left\{ \prod_{i=1}^{n-1} \Delta_1(p_i) \right\} \Delta_2(p_n)$$

which completes the proof.

The following theorem deals with the case in which both ends have differentiated maximal paths. In this case we examine the relationship between the generating function

$$\sum_{\sigma \in \tilde{P}^+ \setminus \tilde{P}} \Delta_1(p_1) \left\{ \prod_{i=2}^{n-1} \Delta_2(p_i) \right\} \Delta_3(p_n) + \sum_{\sigma \in \tilde{P}} \Delta_4(\sigma)$$

where $u(\sigma) = (p_1, p_2, \ldots, p_n)$, and the formula

$$\left\{\sum_p \delta_1(p)\right\}\left\{1 - \sum_p \delta_2(p)\right\}^{-1}\left\{\sum_p \delta_3(p)\right\} + \sum_p \delta_4(p).$$

The last term in each case is due to those sequences which are exactly one path long.

THEOREM 10.2 (differentiated extreme maximal paths). *If*

(i) $\tilde{P}$ *admits a maximum decomposition* $u(\sigma) = (p_1, \ldots, p_n)$ *for any* $\sigma \in \tilde{P}^*$, *and*

(ii) $\Delta_1, \Delta_2, \Delta_3, \Delta_4 \colon \tilde{P} \to \tilde{R}$ *are arbitrary, where* $\tilde{R}$ *is a ring with* 1, *then there exist unique* $\delta_1, \delta_2, \delta_3, \delta_4 \colon \tilde{P} \to \tilde{R}$ *such that for any* $p \in \tilde{P}$

$$\text{(ii)}^{a}\ \Delta_1(p) = \sum_{d \leqq (p)} \delta_1(d_1)\delta_2(d_2, d_3, \ldots, d_m)$$

$$\text{(ii)}^{b}\ \Delta_2(p) = \sum_{d \leqq (p)} \delta_2(d)$$

$$\text{(ii)}^{c}\ \Delta_3(p) = \sum_{d \leqq (p)} \delta_2(d_1, d_2, \ldots, d_{m-1})\delta_3(d_m)$$

$$\text{(ii)}^{d}\ \Delta_4(p) = \delta_4(p) + \sum_{d \leqq (p)} \delta_1(d_1)\delta_2(d_2, d_3, \ldots, d_{m-1})\delta_3(d_m)$$

where $d = (d_1, d_2, \ldots, d_m)$, $m \geqq 1$. *Moreover, the generating function*

$$\text{(iii)}\ 1 + \sum_{\substack{\sigma \in \tilde{P}^+ \setminus \tilde{P} \\ u(\sigma) = (p_1, \ldots, p_n)}} \Delta_1(p_1)\left\{\prod_{k=2}^{n-1} \Delta_2(p_k)\right\}\Delta_3(p_n) + \sum_{\sigma \in \tilde{P}} \Delta_4(\sigma)$$

is equal to

$$1 + \left\{\sum_p \delta_1(p)\right\}\left\{1 - \sum_p \delta_2(p)\right\}^{-1}\left\{\sum_p \delta_3(p)\right\} + \sum_p \delta_4(p).$$

Proof. Clearly, from (ii)^{a}, (ii)^{b} and (ii)^{c}, δ_1, δ_2 and δ_3 exist and are unique by Möbius inversion, while δ_4 exists and is unique trivially from (ii)^{d}. Again, we proceed by direct expansion of the *dexter* of (iii). We have

$$\left\{\sum_p \delta_1(p)\right\}\left\{1 - \sum_p \delta_2(p)\right\}^{-1}\left\{\sum_p \delta_3(p)\right\} + \sum_p \delta_4(p)$$
$$= \sum_{k \geqq 2} \sum_{\sigma \in \tilde{P}^+} \sum_{d} \delta_1(d_1)\delta_2(d_2, \ldots, d_{k-1})\delta_3(d_k) + \sum_p \delta_4(p)$$

(where the summation is over $d \in c^{-1}(\sigma) \cap \tilde{P}^k$)

$$= \sum_{\sigma \in \tilde{P}^+ \setminus \tilde{P}} \sum_{d \in c^{-1}(\sigma)} \delta_1(d_1)\delta_2(d_2, \ldots, d_{m-1})\delta_3(d_m)$$
$$+ \sum_p \left\{\delta_4(p) + \sum_{d \in c^{-1}(p)} \delta_1(d_1)\delta_2(d_2, \ldots, d_{m-1})\delta_3(d_m)\right\}$$

(where $d = (d_1, d_2, \ldots, d_m)$ and $m \geqq 1$). But

$$\sigma \in \tilde{P}^+ \setminus \tilde{P} \quad \Rightarrow \quad u(\sigma) = (p_1, \ldots, p_n), \quad n \geqq 2,$$

so the generating function reduces to

$$\sum_{\sigma\in\tilde{P}^+\setminus\tilde{P}}\left(\sum_{d\leqq(p)}\delta_1(d_1)\delta_2(d_2,\ldots,d_m)\right)\left\{\prod_{k=2}^{n-1}\sum_{d\leqq(p_k)}\delta_2(d)\right\}$$
$$\times\left(\sum_{d\leqq(p_n)}\delta_2(d_1,\ldots,d_{m-1})\delta_3(d_m)\right)+\sum_{p\in\tilde{P}}\Delta_4(p)$$
$$=\sum_{\sigma\in\tilde{P}^+\setminus\tilde{P}}\Delta_1(p_1)\left\{\prod_{k=2}^{n-1}\Delta_2(p_k)\right\}\Delta_3(p_n)+\sum_{p\in\tilde{P}}\Delta_4(p)$$

which completes the proof.

The utility of these two theorems rests with the ease with which the Möbius inversion may be carried out. Clearly, there will be situations in which it will be unrewarding to attempt to cope with the algebraic details. In the next section the theorems are specialised to the case in which only length and type information is recorded.

11. Length and type encoding for differentiated maximal decomposition. The Möbius inversion may be simplified as before when $\Delta: \tilde{P}\rightarrow\tilde{R}$ encodes only length and type information. Theorems 10.1 and 10.2 are treated below in this fashion, following the development of Theorem 4.1 (length and type) from Theorem 3.1 (maximal paths). Some proofs are omitted because they follow closely the proof of Theorem 4.1. Two immediate applications of Theorem 11.1 are given.

THEOREM 11.1 (length and type with differentiated terminal maximal paths). *If*

(i) $\tilde{P}$ *admits maximum decompositions, and*

(ii) $\Delta_i(p) = F_j^{(i)}\tau(p)$ *for* $i = 1$ *and* 2, *where*

(ii)[a] $j = |p|$ *is the length of* $p \in \tilde{P}$, *and*

(ii)[b] $\tau: \tilde{P}\rightarrow\tilde{R}$, *is a path-homomorphism to the centre of the ring* $\tilde{R}$,

then

(iii) $\delta_i(p) = f_{|p|}^{(i)}\tau(p)$ *is determined by*

(iii)[a] $F^{(1)}(x) = (f^{(1)}(x))^{-1}$

(iii)[b] $F^{(2)}(x) = 1 + (f^{(1)}(x))^{-1}(1 - f^{(2)}(x))$, *or equivalently,*

$1 - f^{(2)}(x) = (F^{(1)}(x))^{-1}(F^{(2)}(x) - 1)$, *where*

(iii)[c] $F^{(i)}(x) = 1 + \sum_{k>0} F_k^{(i)}x^k$ *and* $f^{(i)}(x) = 1 - \sum_{k>0} f_k^{(i)}x^k$,

for $i = 1, 2$.

Moreover, the generating function is

$$\text{(iv) } 1+\left\{1-\sum_{k>0} f_k^{(1)}\gamma_k\right\}^{-1}\left\{\sum_{k>0} f_k^{(2)}\gamma_k\right\}$$

where

$$\gamma_k = \sum_{\substack{p\in\tilde{P}\\|p|=k}}\tau(p).$$

Proof. As in Theorem 4.1 we note that for any non-empty sequence σ we have

$$c^{-1}(\sigma) = \bigtimes_{i=1}^{n} c^{-1}(p_i) \quad \text{where } u(\sigma) = (p_1 \ldots p_n).$$

Accordingly, the generating function may be written

$$\sum_{\sigma \in \tilde{P}} \Delta_1(p_1) \ldots \Delta_1(p_{n-1})\, \Delta_2(p_n) = \sum_{\sigma \in \tilde{P}} \left\{ \prod_{i=1}^{n-1} \sum_{d \in c^{-1}(p_i)} \delta_1(d) \right\} A(p_n)$$

where

$$A(p) = \sum_{\substack{d \in c^{-1}(p) \\ d=(d_1 \ldots d_m)}} \left\{ \prod_{i=1}^{m-1} \delta_1(d_i) \right\} \delta_2(d_m).$$

To construct $f_k^{(i)}$ it is sufficient to take

i) $\Delta_1(p) = \sum_{d \leqq (p)} \delta_1(p)$

which may be treated as in Theorem 4.1 to give

$$\delta_1(p) = f_{|p|}^{(1)} \tau(p) \quad \text{and} \quad F^{(1)}(x) = \{f^{(1)}(x)\}^{-1}$$

and

ii) $\Delta_2(p) = A(p)$.

For those p of minimal length, we have $\Delta_2(p) = \delta_2(p)$ since $c^{-1}(p) = \{(p)\}$. Thus for such paths we have $\delta_2(p) = f_{|p|}^{(2)} \tau(p)$. We proceed by induction on the length of p as in Theorem 4.1. By the induction hypothesis for all paths shorter than p, the commutativity of $\tau(d_i)$, and the form for $\delta_1(p)$ we obtain, for any p:

$$\delta_2(p) = \Delta_2(p) - \sum_{d < (p)} f_{j_1}^{(1)} f_{j_2}^{(1)} \ldots f_{j_{m-1}}^{(1)} f_{j_m}^{(2)}\, \tau(d_1) \ldots \tau(d_m)$$

$$= \tau(p) \left\{ F_{|p|}^{(2)} - \sum_{d < (p)} f_{j_1}^{(1)} \ldots f_{j_{m-1}}^{(1)} f_{j_m}^{(2)} \right\}$$

where $d = (d_1 \ldots d_m)$ and $|d_k| = j_k$. Thus $\delta_2(p)$ has the form claimed, and by factoring $\tau(p)$ from $\Delta_2(p) = A(p)$ we have:

$$F_j^{(2)} = [x^j] \left\{ \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} f_i^{(1)} x^i \right)^k \right\} \left\{ \sum_{i=1}^{\infty} f_i^{(2)} x^i \right\}$$

which may be expressed in the form

$$F^{(2)}(x) = 1 + \{f^{(1)}(x)\}^{-1} \{1 - f^{(2)}(x)\}.$$

This completes the proof.

THEOREM 11.2 (length and type with differentiated extreme maximal paths). *If*

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(i) *$\tilde{P}$ admits maximum decompositions, and*

(ii) $\Delta_i(p) = F_j^{(i)}\tau(p)$ *for* $i = 1, 2, 3$ *and* 4, *where*

(ii)[a] $j = |p|$ *is the length of* $p \in \tilde{P}$, *and*

(ii)[b] $\tau : \tilde{P} \to \tilde{R}$, *is a path-homomorphism to the centre of the ring* $\tilde{R}$,

then

(iii) $\delta_i(p) = f_{|p|}^{(i)}\tau(p)$ *is determined by*

(iii)[a] $F^{(1)}(x) = 1 + (1 - f^{(1)}(x))(f^{(2)}(x))^{-1}$

(iii)[b] $F^{(2)}(x) = (f^{(2)}(x))^{-1}$

(iii)[c] $F^{(3)}(x) = 1 + (f^{(2)}(x))^{-1}(1 - f^{(3)}(x))$

(iii)[d] $F^{(4)}(x) = 1 + (1 - f^{(4)}(x)) + (1 - f^{(1)}(x))(f^{(2)}(x))^{-1}(1 - f^{(3)}(x))$

where

$$\text{(iii)}^{\text{e}}\ F^{(i)}(x) = 1 + \sum_{k>0} F_k^{(i)}x^k \quad \textit{and} \quad f^{(i)}(x) = 1 - \sum_{k>0} f_k^{(i)}x^k$$

for $i = 1, 2, 3$ *and* 4.

Moreover, the generating function is

$$\text{(iv)}\ 1 + \sum_{k>0} f_k^{(4)}\gamma_k + \left(\sum_{k>0} f_k^{(1)}\gamma_k\right)\left\{1 - \sum_{k>0} f_k^{(2)}\gamma_k\right\}^{-1}\left(\sum_{k>0} f_k^{(3)}\gamma_k\right)$$

where

$$\gamma_k = \sum_{\substack{p\in\tilde{P}\\ |p|=k}} \tau(p).$$

Proof. Straightforward, and similar to the proofs of Theorems 4.1 and 11.1.

We now establish a connexion between Theorem 9.6, which enumerates sequences according to the number of rises, levels and falls, and Theorem 11.1, which enumerates with respect to maximal paths. The following theorem gives the enumerators $F^{(1)}(x)$ and $F^{(2)}(x)$ which achieve this connexion by means of strictly increasing subsequences (in $\tilde{P}_4$).

THEOREM 11.3. *Let*

$$F^{(1)}(x) = 1 + lx + rfx^2 + rf(f - l + r)x^3 + rf(f - l + r)^2x^4 + \ldots$$

and

$$F^{(2)}(x) = 1 + rfx + rf(f - l + r)x^2 + rf(f - l + r)^2x^3 + \ldots$$

Then

$$\Theta = \Phi(f, r, l)$$

where

$$\text{i)}\ \Phi(f, r, l) = 1 - rf\,\frac{\prod_{j=1}^{\infty}\{1 + (r - l)x_j\} - \prod_{j=1}^{\infty}\{1 + (f - l)x_j\}}{f\prod_{j=1}^{\infty}\{1 + (r - l)x_j\} - r\prod_{j=1}^{\infty}\{1 + (f - l)x_j\}} \quad \textit{and}$$

ii) $\Theta = 1 + \left\{1 - \sum_{k>0} f_k^{(1)}\gamma_k\right\}^{-1}\left\{\sum_{k>0} f_k^{(2)}\gamma_k\right\}$, *where*

iii)[a] $F^{(1)}(x) = \{f^{(1)}(x)\}^{-1}$; $F^{(2)}(x) = 1 + \{f^{(1)}(x)\}^{-1}\{1 - f^{(2)}(x)\}$,

iii)[b] $f^{(i)}(x) = 1 - \sum_{k>0} f_k^{(i)}x^k$, $i = 1, 2$,

iii)[c] $\sum_{k\geqq 0} \gamma_k x^k = \prod_{j=1}^{\infty} (1 + xx_j)$, $\gamma_0 = 1$.

Proof.

$$\sum_{k>0} f_k^{(2)}\gamma_k = -(f-r)^{-1}rf\left\{\prod_{j=1}^{\infty}\{1 + (r-l)x_j\} - \prod_{j=1}^{\infty}\{1 + (f-l)x_j\}\right\},$$

and

$$1 - \sum_{k>0} f_k^{(1)}\gamma_k$$
$$= (f-r)^{-1}\left\{f\prod_{j=1}^{\infty}\{1 + (r-l)x_j\} - r\prod_{j-1}^{\infty}\{1 + (f-l)x_j\}\right\}.$$

Then

$$f_k^{(2)} = (f-r)^{-1}rf\{(f-l)^k - (r-l)^k\}, \quad k \geqq 1$$

so

$$f^{(2)}(x) = 1 - rfx\{1 - (r-l)x\}^{-1}\{1 - (f-l)x\}^{-1}.$$

Also

$$f_k^{(1)} = -(f-r)^{-1}\{f(r-l)^k - r(f-l)^k\}, k \geqq 1$$

so

$$f^{(1)}(x) = \{1 - (f+r-l)x\}\{1 - (r-l)x\}^{-1}\{1 - (f-l)x\}^{-1}.$$

Thus

$$\begin{aligned} F^{(1)}(x) &= \{1 - (r-l)x\}\{1 - (f-l)x\}\{1 - (f+r-l)x\}^{-1} \\ &= 1 + lx + rfx^2 + rf(f-l+r)x^2 + \ldots \end{aligned}$$

and

$$\begin{aligned} F^{(2)}(x) &= 1 + rfx\{1 - (f+r-l)x\}^{-1} \\ &= 1 + rfx + rf(f-l+r)x^2 + \ldots \end{aligned}$$

and the theorem follows.

No combinatorial interpretation for the enumerators $F^{(1)}(x)$ and $F^{(2)}(x)$ of Theorem 11.3 has been discovered, and the connexion remains at present a purely formal algebraic one. The theorem is now used in a formal fashion to give the enumeration of sequences with respect to rises, non-rises and maxima, where a maximum is a rise followed by a non-rise.

COROLLARY 11.4. *The generating function for the number of sequences with a specified number of rises (marked by u), non-rises (marked by v) and maxima (marked by w) is*

$$\Phi(\phi_1(u, v, w), \phi_2(u, v, w), \phi_3(u, v, w))$$

where

$$\phi_1(u, v, w) = \tfrac{1}{2}\{u + v - \{(u + v)^2 - 4uvw\}^{1/2}\}$$
$$\phi_2(u, v, w) = \tfrac{1}{2}\{u + v + \{(u + v)^2 - 4uvw\}^{1/2}\}$$
$$\phi_3(u, v, w) = v$$

and $\Phi(f, r, l)$ is the generating function (given in Theorem 9.6) for the number of rises (marked by r), levels (marked by l) and falls (marked by f).

Proof. We decompose the sequences into strictly increasing subsequences. Accordingly, Theorem 11.1 is applicable. Now a non-rise must follow each non-terminal maximum, by definition. A maximal subsequence of length $n \geqq 1$ has $n - 1$ rises. Thus the enumerator for a non-terminal maximal subsequence is given by

$$F^{(1)}(x) = 1 + vx + uvwx^2 + u^2vwx^3 + \ldots$$

since an increasing subsequence of length 1 has no maxima or rises. For the terminal maximal subsequence the enumerator is given by

$$F^{(2)}(x) = 1 + uvwx + u^2vwx^2 + \ldots$$

since the sequence is terminated by a conventional maximum and a conventional rise. Accordingly, set

$$l = v, \quad rf = uvw, \quad f + r - l = u$$

so

$$l = v, \quad rf = uvw \quad \text{and} \quad f + r = u + v.$$

Thus $f = \phi_1(u, v, w)$, $r = \phi_2(u, v, w)$ and $l = \phi_3(u, v, w)$. This identifies the enumerators with those of Theorem 11.3. The result follows from Theorems 11.1 and 11.3.

COROLLARY 11.5. *The exponential generating function for the number of permutations of specified length (marked by t) and a specified number of rises (marked by u), falls (marked by v) and maxima (marked by w) is*

$$\Psi(\psi_1(u, v, w), \psi_2(u, v, w), t)$$

where

$$\psi_1(u, v, w) = \tfrac{1}{2}\{u + v + \{(u + v)^2 - 4uvw\}^{1/2}\}$$
$$\psi_2(u, v, w) = \tfrac{1}{2}\{u + v - \{(u + v)^2 - 4uvw\}^{1/2}\}$$

and

$$\Psi(x, y, t) = -xy \frac{e^{yt} - e^{xt}}{xe^{yt} - ye^{xt}}.$$

Proof. Let $c_n(i, j)$ be the number of permutations over $\{1, 2, \ldots, n\}$ with i rises and j falls. Then from Theorem 9.6 we have

$$\begin{aligned} c_n(i, j) &= -[x^i y^i][x_1 x_2 \ldots x_n] xy \frac{\prod_{k=1}^{n} \{1 + yx_k\} - \prod_{k=1}^{n} \{1 + xx_k\}}{x \prod_{k=1}^{n} \{1 + yx_k\} - y \prod_{k=1}^{n} \{1 + xx_k\}} \\ &= -[x^i y^i][x_1 x_2 \ldots x_n] xy \frac{e^{yt} - e^{xt}}{xe^{yt} - ye^{xt}} \quad \text{where } t = x_1 + x + \ldots + x_n. \end{aligned}$$

Let

$$G(t) = -xy \frac{e^{yt} - e^{xt}}{xe^{yt} - ye^{xt}} = \sum_{k=0}^{\infty} g_k t^k$$

Then

$$[x_1 x_2 \ldots x_n] G(t) = \sum_{k=0}^{\infty} g_k [x_1 x_2 \ldots x_n] t^k = n! g_n = \left[\frac{t^n}{n!}\right] G(t)$$

which identifies $G(t)$ as the exponential generating function for $\{c_n(i, j)\}$. The result follows from Theorem 11.3.

The generating function $-xy(e^{yt} - e^{xt})/(xe^{yt} - ye^{xt})$ is given in Foata and Schützenberger [**18**]. Corollary 11.5 appears as Theorem 2 in Carlitz and Scoville [**10**]. A final example of the use of Theorem 11.1 is given in the following corollary.

COROLLARY 11.6. *The generating function for the number of alternating sequences $i_1 i_2 \ldots$ (with the property that $i_1 < i_2 \geqq i_3 < i_4 \geqq i_5 \ldots$) beginning with a rise is*

$$\left\{1 + \sum_{k=0}^{\infty} (-1)^k \gamma_{2k+1}\right\} \left\{\sum_{k=0}^{\infty} (-1)^k \gamma_{2k}\right\}^{-1}$$

where

$$\sum_{k=0}^{\infty} \gamma_k x^k = \prod_{j=1}^{\infty} (1 + xx_j).$$

Proof. For alternating sequences only maximal strictly increasing subsequences of length two are permitted. Thus the enumerator for non-terminal maximal paths is

$$F^{(1)}(x) = 1 + x^2.$$

The enumerator for terminal maximal paths is

$$F^{(2)}(x) = 1 + x + x^2$$

since the terminal path may be a rise or a fall. Thus, from Theorem 11.1

$$f^{(1)}(x) = (1 + x^2)^{-1} \quad \text{and} \quad 1 - f^{(2)}(x) = \frac{x + x^2}{1 + x^2}$$

whence

$$f^{(2)}(x) - f^{(1)}(x) = -x(1 + x^2)^{-1}$$

If $k \geqq 1$, then

$$f_k^{(2)} - f_k^{(1)} = [x^{k-1}](1 + x^2)^{-1} = \begin{cases} 0 & \text{if } k \text{ even} \\ (-1)^p & \text{if } k = 2p + 1, p \geqq 0. \end{cases}$$

The result follows directly from Theorem 11.1.

12. Differentiated p-paths and permutations. In the case of permutations on n further simplification is possible. We consider only one case, that of increasing subsequences ($\tilde{P} = \tilde{P}_4$). The transition from sequences to permutations is again accomplished by means of Lemmas 8.2 and 8.3. The ring $\tilde{R}$ is now assumed to be commutative.

THEOREM 12.1 (permutations with increasing subsequences and differentiated terminal maximal subsequences). *The number of permutations on n with d_i non-terminal maximal increasing subsequences of length i and the terminal one of length j is*

$$\left[\frac{x^n}{n!}\right][(\boldsymbol{F}^{(1)})^{\boldsymbol{d}}][F_j^{(2)}]\, 1 + \left\{1 - \sum_{k>0} f_k^{(1)} \frac{x^k}{k!}\right\}^{-1} \left\{\sum_{k>0} f_k^{(2)} \frac{x^k}{k!}\right\}$$

where conditions (iii)^a, (iii)^b, (iii)^c *of Theorem* 11.1 *are met.*

Proof. Similar to the proof of Theorem 8.4, by means of Theorem 11.1.

THEOREM 12.2 (permutations with increasing subsequences and differentiated extreme maximal subsequences). *The generating function for number of permutations on n with d_i non-extreme maximal increasing subsequences of length i is*

$$\left[\frac{x^n}{n!}\right][(\boldsymbol{F}^{(2)})^{\boldsymbol{d}}]\, 1 + \sum_{k>0} f_k^{(4)} \frac{x^k}{k!} + \left\{\sum_{k>0} f_k^{(1)} \frac{x^k}{k!}\right\} \left\{1 - \sum_{k>0} f_k^{(2)} \frac{x^k}{k!}\right\}^{-1} \times \left\{\sum_{k>0} f_k^{(3)} \frac{x^k}{k!}\right\}$$

where conditions (iii)^a, (iii)^b, (iii)^c, (iii)^d *of Theorem* 11.2 *are met.*

Proof. Similar to the proof of Theorem 8.4, and using Theorem 11.2.

The following well-known result concerning alternating permutations may now be proved.

COROLLARY 12.3 (André [**3**]). *The number of alternating permutations on n is* $[x^n/n!]$ (sec x + tan x).

Proof. The increasing subsequences are each of length 2, so $F^{(1)}(x) = 1 + x^2$. Since there may be a single fall at the end, we have $F^{(2)}(x) = 1 + x + x^2$. Thus, from Theorem 12.1, we have the following generating function for the problem

$$1 + \left\{1 - \sum_{k>0} f_k^{(1)} \frac{x^k}{k!}\right\}^{-1} \left\{\sum_{k>0} f_k^{(2)} \frac{x^k}{k!}\right\}$$
$$= 1 + \sec x(\sin x + 1 - \cos x) = \sec x + \tan x$$

and the result follows.

The following two corollaries concern the enumeration of permutations with respect to strict maxima and minima. A strict maximum is a rise followed by a fall, and a strict minimum is a fall followed by a rise. Related results have been given by Carlitz and Scoville [**10**], and Carlitz and Vaughan [**11**].

COROLLARY 12.4. *The number of permutations on n with i strict maxima and j strict minima is*

$$\left[M^i m^j \frac{x^n}{n!}\right] GH^{-1}$$

where

$$G = 1 - (1 - M)(1 - m)(1 - Mm)^{-1}$$
$$\times \{x(1 - \alpha^{-1} \tanh \alpha x) + 2(1 - Mm)^{-1}(1 - \operatorname{sech} \alpha x) - \alpha^{-1} \tanh \alpha x\},$$
$$H = 1 - \alpha^{-1} \tanh \alpha x, \quad \textit{and} \quad \alpha = (1 - Mm)^{1/2}.$$

Proof. The initial maximal increasing subsequence has a strict maximum terminating it only it if is longer than one element. Thus

$$F^{(1)}(x) = 1 + x + Mx^2 + Mx^3 + \ldots = \{1 - (1 - M)x^2\}(1 - x)^{-1}.$$

Similarly,

$$F^{(3)}(x) = \{1 - (1 - m)x^2\}(1 - x)^{-1}.$$

Each non-terminal maximal increasing subsequence is initiated and terminated by a strict minimum and a strict maximum respectively, so

$$F^{(2)}(x) = \{1 - (1 - Mm)x^2\}(1 - x)^{-1}.$$

Permutations consisting of exactly one increasing sequence have no strict

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maxima or minima, so $F^{(4)}(x) = (1 - x)^{-1}$. Thus, from Theorem 12.2,

$$\sum_{k>0} f_k^{(1)} \frac{x^k}{k!} = \alpha^{-1} \operatorname{sh} \alpha x - (1 - M)\alpha^{-2}(\operatorname{ch} \alpha x - 1)$$

$$1 - \sum_{k>0} f_k^{(2)} \frac{x^k}{k!} = \operatorname{ch} \alpha x - \alpha^{-1} \operatorname{sh} \alpha x$$

$$\sum_{k>0} f_k^{(3)} \frac{x^k}{k!} = \alpha^{-1} \operatorname{sh} \alpha x - (1 - m)\alpha^{-2}(\operatorname{ch} \alpha x - 1)$$

$$\sum_{k>0} f_k^{(4)} \frac{x^k}{k!} = x - (m + M - 2)\alpha^{-3}(\operatorname{sh} \alpha x - \alpha x)$$

and the result follows immediately.

COROLLARY 12.5. *The number of permutations on n with i strict maxima is*

$$\left[M^i \frac{x^n}{n!}\right](1 - \alpha^{-1} \tanh \alpha x)^{-1}$$

where $\alpha = (1 - M)^{1/2}$.

Proof. Put $m = 1$ in Corollary 12.4.

We observe that the number of permutations on n is given by

$$\left[\frac{x^n}{n!}\right] \lim_{M \to 1} \{1 - (1 - M)^{-1/2} \tanh (1 - M)^{1/2} x\}^{-1} = \left[\frac{x^n}{n!}\right](1 - x)^{-1} = n!$$

as required. Also the number of permutations on n with no strict maxima is

$$\left[\frac{x^n}{n!}\right](1 - \tanh x)^{-1} = 2^{n-1},$$

a result which may be obtained trivially by induction on n.

13. Concluding remarks. The enumerative method which has been described above has certain limitations. For example, problems such as the Davenport-Shinzel problem (Davenport and Shinzel [**13**]) and the Terquem problem (Moser and Abramson [**28**]), both of which involve positional information, cannot be treated by this method. However, a certain class of such problems has already been considered by Stanley [**36**] by means of *binomial posets*. In addition, no way has been found for treating by this method problems which involve both maximal increasing paths and maximal decreasing paths occurring together, since no usable unique maximal decomposition exists in this case. The Erdös-Szekeres problem (Erdös and Szekeres [**17**]), involving embedded increasing and decreasing subsequences of length $n + 1$ in a permutation on $n^2 + 1$, is an example of such a problem. However, this problem already admits an elegant solution by means of plane partitions (Schenstead [**33**]; see also Stanley [**35**]).

No attempt has been made to apply this enumerative method in an exhaustive fashion. Probably, several specialisations and applications of the theorems remain. In particular, the "partition trick", namely the substitution $x_i = q^i$, leads to a variety of expressions in terms of Eulerian generating functions.

To a certain extent the theory presented here may be a service in establishing the positivity of the coefficients in the expansion of certain rational functions. Certainly positivity is established provided a combinatorial interpretation to the rational function is found. If the rational function is symmetric it is reasonable to seek, in the first instance, a combinatorial interpretation involving the enumeration of sequences. Accordingly, by reversing the application of Theorem 4.1, for example, we may construct an enumerator $F(x)$ from a knowledge of $\{1 - \sum_{k>0} f_k \gamma_k\}^{-1}$. In the cases where the coefficients in $F(x)$ are non-negative, the problem admits a combinatorial interpretation. In this context, the contribution of the theory presented here lies chiefly in the construction of the combinatorial problem, rather than in demonstrating positivity since the latter may be more readily proved by other means. The following remark demonstrates the principle.

Remark 13.1. Let

$$A(\boldsymbol{x}) = \{1 - 2(x_1 + x_2 + x_3) + (x_1x_2 + x_2x_3 + x_1x_3)\}^{-1}$$
$$= \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}}$$

Then i) $a_{\boldsymbol{i}}$ is the sum of the weights $\Delta \circ u(\sigma)$ over all sequences σ in $\{1, 2, 3\}^*$ of type $\boldsymbol{i}$ where

$$\Delta \circ u(\sigma) = (|p_1| + 1)(|p_2| + 1) \ldots (|p_n| + 1)$$

and $u(\sigma) = (p_1 p_2 \ldots p_n)$, the maximal decomposition of σ into strictly increasing subsequences (in $\tilde{P}_4$).

Also ii) $a_{\boldsymbol{i}} \geqq 0$.

Proof. Let $\tilde{P} = \tilde{P}_4$ (see Definition 2.5(iv)). Then from Section 7(ii) we have

$$\gamma_0 = 1, \quad \gamma_1 = x_1 + x_2 + x_3, \quad \gamma_2 = x_1x_2 + x_2x_3 + x_1x_3$$

Thus

$$A(\boldsymbol{x}) = (1 - f_1\gamma_1 - f_2\gamma_2)^{-1} \quad \text{where} \quad f(x) = 1 - f_1x - f_2x^2 = (1 - x)^2$$

Then

$$F(x) = \{f(x)\}^{-1} = (1 - x)^{-2} = 1 + 2x + 3x^2 + \ldots$$

and the result follows from Theorem 4.1.

ii) Immediate, since the coefficients in the expansion of $F(x)$ are positive.

The following example shows how the weights are determined.

Example 13.2. By direct expansion of $A(\boldsymbol{x})$ we have $a_{(1,1,1)} = 36$. The following table lists the sequences σ of type (1, 1, 1) together with their maximal decompositions $p_1p_2 \ldots p_n$ and their weights $\Delta \circ u(\sigma)$.

TABLE 2

σ	$p_1p_2 \ldots p_n$	$\Delta \circ u(\sigma)$
123	(123)	4
132	(13)(2)	3.2
213	(2)(13)	2.3
231	(23)(1)	3.2
312	(3)(12)	2.3
321	(3)(2)(1)	2.2.2

Thus

$$\sum_{\tau(\sigma)=(1,1,1)} \Delta \circ u(\sigma) = 4 + 6 + 6 + 6 + 6 + 8 = 36 = a_{(1,1,1)}$$

as asserted.

No such interpretation exists for the function

$$\{(1 - x_1)(1 - x_2) + (1 - x_2)(1 - x_3) + (1 - x_3)(1 - x_1)\}^{-1}$$

considered by Friedrichs and Lewy since the corresponding enumerator $F(x)$ has some negative coefficients. Positivity has been proved by Szegö [**37**] by an argument relying on special functions. No combinatorial interpretation to the coefficients in the expansion of this function has been discovered.

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Appendix. The following table lists some of the more common enumeration problems which may be treated by the methods described here. The generating functions are given in the cited corollaries. The list does not exhaust the possible applications.

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TABLE 3

Corollary	permutation (P) sequence (S)	Path Type	Configurations recognised	Conditions on p-paths
6.1	S	increasing sequence	terminators	
6.2	S	strictly increasing sequence	terminators	
7.1	S	arbitrary	p-path	exactly i
7.2	S	arbitrary	p-path	exactly 0
7.3	S	arbitrary	longest path	
8.5	P	increasing run	p-path	exactly i
8.6	P	increasing run	p-path	exactly 0
8.7	P	strictly increasing sequence	p-path	exactly i
8.9	P	increasing run	longest path	
8.10	P	increasing run	longest path	unique length
8.11	P even length	strictly increasing sequence		2 (alternating)
11.4	S	strictly increasing sequence	rise, non-rise, maximum	
11.5	P	strictly increasing sequence	rise, non-rise, maximum	
11.6	S	strictly increasing sequence		length 2 (alternating)
12.3	P	strictly increasing sequence		length 2 (alternating)
12.4	P	strictly ncreasing sequence	$\left\{\begin{array}{l}\text{strict maximum}\\\text{strict minimum}\end{array}\right.$	
12.5	P	strictly increasing sequence	strict maximum	

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