ON THE STRENGTH OF PACKED SPHERES

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The mechanics of a system of packed spheres has relevance to several physical disciplines. A particular case has been a recent trend among engineers to use a close-packed sphere model to aid research into the strength of cohesionless granular masses, such as sand.

The basis for the examination of the strength of sphere packings is contained in work by Rennie [3], who established a geometrical nomenclature and failure criterion for an array of close-packed spheres. These were applied in the derivation of solutions, with and without friction, under a restricted system of external stress.

It is, however, most usual practice in experimental soil mechanics to use an apparatus and loading system which gives $\sigma_1 > \sigma_2 = \sigma_3 > 0$ (taking compressive stress positive), a situation which is excluded from Rennie's original analysis. It has therefore been necessary to extend the solution to include what is, in fact, the most significant stress condition.

1. Previous work

The arrangement that constitutes the densest lattice packing of uniform rigid spheres is one in which each sphere is in tangent contact with twelve neighbours (Boerdijk [1]). This condition specifies the disposition of these twelve spheres in relation to the one they enclose.

With reference to the co-ordinate frame of Fig. 1, Rennie showed that the centres of those spheres that are in contact with one centred on the origin are given by

$$\mathbf{x} = \mathbf{P}\mathbf{n}d \tag{1}$$

where $\mathbf{P}$ is the matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$\mathbf{n}$ is a column of three integers, and d is the sphere diameter. The distance of any sphere from the origin is then

$$d\sqrt{\boldsymbol{n}'\boldsymbol{P}'\boldsymbol{P}\boldsymbol{n}}$$

which is minimal for each of the contiguous spheres, given by the row $\boldsymbol{n}'$ equal to

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (1, -1, 0),\ (1, 0, -1),\ (0, 1, -1)$$

and their negatives.

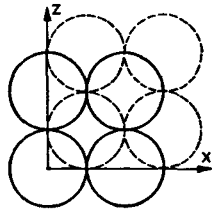

Figure 1

Under a small strain $\boldsymbol{S}$ the square of the separation distance will increase by an amount which, to a first order in $\boldsymbol{S}$, is

$$2\boldsymbol{n}'\boldsymbol{P}'\boldsymbol{S}\boldsymbol{P}\boldsymbol{n}d^2.$$

From the geometrical condition of rigid spheres that the separation distance cannot decrease during shear, Rennie showed that $\boldsymbol{S}$ is restricted by the inequalities (2)

$$\begin{aligned} &s_{11} \geqq 0 \\ &s_{11}+2\sqrt{2}s_{12}-2s_{13}+2s_{22}-2\sqrt{2}s_{23}+s_{33} \geqq 0 \\ &s_{11}+2\sqrt{2}s_{12}+2s_{13}+2s_{22}+2\sqrt{2}s_{23}+s_{33} \geqq 0 \\ &s_{33} \geqq 0 \\ &s_{11}-2\sqrt{2}s_{12}-2s_{13}+2s_{22}+2\sqrt{2}s_{23}+s_{33} \geqq 0 \\ &s_{11}-2\sqrt{2}s_{12}+2s_{13}+2s_{22}-2\sqrt{2}s_{23}+s_{33} \geqq 0 \end{aligned} \tag{2}$$

which together define a convex cone C in the strain space of the components of $\boldsymbol{S}$. The symmetry of the sphere model enabled Rennie to prove that the six inequalities (2) are linearly independent and that they may be permuted within themselves by standard transformations.

For the frictionless case, suppose that the model is subjected to a stress $\boldsymbol{F}$. It was proposed by Rennie that the model would collapse if, for some rotation $\boldsymbol{F}^*$ of $\boldsymbol{F}$, and for some geometrically possible strain $\boldsymbol{S}^*$ in C,

$$\text{Spur}\ \boldsymbol{F}^*\boldsymbol{S}^* > 0 \tag{3}$$

which requires that the strain shall be in a sense consistent with the applied stress (positive virtual work).

It was shown thereafter that the problem may be simplified in that it is only necessary to consider S^* in a one-dimensional edge of C, obtained by setting all but one of the inequalities (2) equal to zero. In this regard, Rennie considered the inequalities as the co-ordinates $x_1 \cdots x_6$ in strain space, and could then write the virtual work equation (3) as

$$\sum a_i x_i > 0 \tag{4}$$

with the condition that all $x_i \geqq 0$. This was considered to be equivalent to some $a_i > 0$ or $\sum a_i x_i > 0$ for some one co-ordinate positive and all others zero.

By symmetry, it was irrelevant which edge of C was considered, and it was convenient to take $s_{11} > 0$ with all others zero, yielding strains of the type

$$S^\dagger = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

The strain indicates relative sliding between contiguous hexagonal layers.

Finally, taking $S^* = S^\dagger$, it was shown that the condition of stability for smooth spheres is that the three principal stresses should all be negative (compressive) and that the greatest of them should not exceed twice the smallest. In this case the principal axes of stress and strain are coincident.

By inferring from the condition of continuity that collapse will again occur toward $S^\dagger$, Rennie extended his solution to include a small coefficient of friction, μ, between the grains. In this case, it was assumed that the intermediate principal axes of stress and strain were both coincident with the Z-axis, leaving the remaining principal stress axes to be determined.

Considering the normal and shear forces acting on a typical sphere in a lattice subjected to $S^\dagger$, the volume integral of the stress tensor through the sphere is

$$-\sum p_{ki} x_{kj} \qquad k = 1 \cdots 12$$

p_{ki} being the contact traction at point x_{ki}. This leads to a stress tensor which is a negative multiple of

$$\begin{pmatrix} \dfrac{\sqrt{3}-2\sqrt{2}\mu+2\mu\psi}{\sqrt{3}} & \psi & 0 \\ \psi & \dfrac{2\sqrt{3}+2\sqrt{2}\mu-2\mu\psi}{\sqrt{3}} & 0 \\ 0 & 0 & 1+t \end{pmatrix} \tag{6}$$

and as ψ (which is, to a first order, the angle between the principal axes of

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stress and strain) varies, the ratio of the two eigenvalues other than $1+t$ reaches a minimum of

$$\frac{3\sqrt{1+4\mu^2/3}+1+4\mu\sqrt{\frac{2}{3}}}{3\sqrt{1+4\mu^2/3}-1-4\mu\sqrt{\frac{2}{3}}}. \tag{7}$$

Equation (7), however, will be subject to the assumed condition that $1+t$ is the intermediate of the three principal stresses, where t, a contact force, cannot be negative. Since the quantity

$$\sigma_1+\sigma_3 = -3$$

is independent of μ and ψ, the lower limit of σ_2, defined by $t = 0$, is

$$(\sigma_2)_{t=0} = \frac{\sigma_1+\sigma_3}{3}. \tag{8}$$

It may be noted that it is not necessary to assume coincidence of the intermediate principal axes of stress and strain, but a greatly simplified analysis, results therefrom. Solutions of the type in equation (6) are therefore in the nature of upper bounds, as understood in the theory of plasticity.

2. Failure under compound strain

It was shown that the original solution is valid over a restricted range of the intermediate principal stress (although within this range, it is independent of the intermediate principal stress). It now remains to complete the solution, through an examination of the range

$$\frac{\sigma_1+\sigma_3}{3} > \sigma_2 \geqq \sigma_3.$$

In this case, it is necessary to admit the possibility of two non-zero terms in the virtual work equation (4). It is therefore, not sufficient to consider only one non-zero term, as was done in the original solution.

It may be recalled from (2) that any co-ordinate $x > 0$ indicates the formation of one pair of gaps at diametrically opposite points on each sphere. Clearly, it is not necessary to consider more than three $x_i > 0$, which would indicate an explosive type of failure, but the situations where two or three $x_i > 0$ require further attention.

Consider any two normals through contacts on the central sphere as

$$\mathbf{x} = P\mathbf{n}$$
$$\mathbf{y} = P\mathbf{m}$$

Ignoring 180° reflections, the scalar product

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$$|n'P'Pm| \qquad (m \neq n)$$

takes only the values $\frac{1}{2}$ and 0, indicating that, on selecting any given contact, a second can be chosen in two ways. Similarly, for the case of three $x_i > 0$, only one choice is possible if explosive failures are discounted. However, the conditions of symmetry for single contacts, enumerated by Rennie, will again apply, it being relevant which contact is initially chosen.

Thus, where multiple gap formation is envisaged, the nature of strain will be determined by the number of gaps and by their relative dispositions and magnitudes. It is, nevertheless, possible to synthesise such strains by superimposing simple sliding strains of the type $S^\dagger$ and strains derived in this way will be termed compound.

3. Case 1 - Broken contacts on perpendicular diagonals

This situation may be obtained by solving

$$s_{11} > 0$$
$$s_{33} > 0$$

of the inequalities (2). The solution for S will evidently be once indeterminate with respect to $S^\dagger$ and may be written in the form (s arbitrary)

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+s) & 0 \\ 0 & 0 & s \end{pmatrix}$$

or alternatively [1]

$$S = a\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{9}$$

where it will be considered that $a+b = 1$. A solution of strength must then be sought in terms of the variable a.

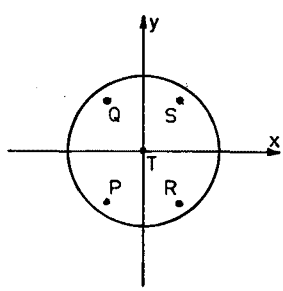

Figure 2

[1] I am grateful to Professor Rennie for pointing out this equivalence.

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Several assumptions are necessary. As the compound strain lies between the limit of $S^{\dagger}$, for which the normal contact forces are related by

$$p = s \geqq q = r \qquad \text{(sliding at } Q \text{ and } R\text{)}$$

(Rennie, 1959) and a permutation of $S^{\dagger}$ for which

$$p = s \leqq q = r \qquad \text{(sliding at } P \text{ and } S\text{)}$$

it may be assumed that $p = s$ and $q = r$ for all a. Similarly, it is assumed that all contact friction forces are equal in magnitude and are equal to μ times the smaller of p and q.

In the particular range $1 \geqq a \geqq \frac{1}{2}$, where the z-direction may be considered to be that of the intermediate principal strain, the further assumption is made that the intermediate principal axes of stress and strain will be coincident.

Consider that $p = s \geqq q = r$ and that sliding is impending at the Q and R contacts. The direction cosines of the friction forces are then derived from the product $S\mathbf{x}$ and are

$$\text{at} \qquad P(-\tfrac{1}{4}, -\tfrac{1}{2}\sqrt{\tfrac{1}{2}}, \tfrac{1}{4}) \cdots \frac{-a, \dfrac{a+b}{\sqrt{2}}, b}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}}$$

(co-ordinates in terms of d)

$$\text{at} \qquad S(\tfrac{1}{4}, \tfrac{1}{2}\sqrt{\tfrac{1}{2}}, \tfrac{1}{4}) \cdots \frac{a, -\dfrac{a+b}{\sqrt{2}}, b}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}}$$

$$\text{at} \qquad Q(-\tfrac{1}{4}, \tfrac{1}{2}\sqrt{\tfrac{1}{2}}, \tfrac{1}{4}) \cdots \frac{-a, -\dfrac{a+b}{\sqrt{2}}, b}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}}$$

$$\text{at} \qquad R(\tfrac{1}{4}, -\tfrac{1}{2}\sqrt{\tfrac{1}{2}}, \tfrac{1}{4}) \cdots \frac{a, \dfrac{a+b}{\sqrt{2}}, b}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}}.$$

Thus the shear components at the P contacts will make a contribution to the stress tensor of

$$\frac{\mu q d}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}} \begin{pmatrix} \dfrac{a}{2} & \dfrac{a}{\sqrt{2}} & -\dfrac{a}{2} \\ -\dfrac{a+b}{2\sqrt{2}} & -\dfrac{a+b}{2\sqrt{2}} & \dfrac{a+b}{2\sqrt{2}} \\ \dfrac{b}{2} & -\dfrac{b}{\sqrt{2}} & \dfrac{b}{2} \end{pmatrix}.$$

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Similar contributions result at the other contacts, such that the total contribution of shear forces to the stress tensor is

$$(10) \qquad \frac{2\mu q d}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}}\begin{pmatrix} a & 0 & 0 \\ 0 & -(a+b) & 0 \\ 0 & 0 & b \end{pmatrix}.$$

As in the original analysis, the contribution of normal forces to the stress tensor is

$$(11) \qquad -d\begin{pmatrix} 1 & \psi & 0 \\ \psi & 2 & 0 \\ 0 & 0 & 1+t \end{pmatrix}$$

but noting that the force t is zero for all $a \neq 1$.[2] The angle is again defined by $q = 1-\psi/\sqrt{2}$.

Considering now the greatest and least eigenvalues of the total stress tensor $((10)+(11))$, the sum is no longer independent of μ and ψ. (Calculations hereafter are exact in μ, for the sake of clarity, although it ought to be borne in mind that the assumptions are only reliable to a first order in μ).

$$(12) \qquad \sigma_1+\sigma_3 = 3+\frac{2b\mu(1-\psi/\sqrt{2})}{\sqrt{\frac{3}{2}a^2+ab+\frac{3}{2}b^2}}.$$

(We adopt the convention of compressive stress positive.)

If ξ is the ratio

$$\left\{\frac{\sigma_1+\sigma_3}{\sigma_1-\sigma_3}\right\}^2$$

$$\beta = \psi/\sqrt{2} \quad \text{and} \quad F = \sqrt{\tfrac{3}{2}a^2+ab+\tfrac{3}{2}b^2},$$

then

$$\xi = \frac{9F^2+12\mu bF(1-\beta)+4\mu^2b^2(1-\beta)^2}{F^2+4\mu F(2a+b)(1-\beta)+4\mu^2(2a+b)^2(1-\beta)^2+8F^2\beta^2}$$

which is maximum when $d\xi/d\beta = 0$. A trivial root

$$3F+2\mu b(1-\beta) = 0$$

is determined by inspection and is equivalent to $\sigma_1+\sigma_3 = 0$. The only remaining root is given by

$$(13) \qquad \beta = \frac{\mu(3a+b)\{F+2\mu(2a+b)\}}{6F^2+4\mu bF+2\mu^2(3a+b)(2a+b)}.$$

The value of β according to (13) will be termed critical, since it specifies the worst orientation of model in relation to the applied stress. The critical

[2] In fact, b and t may be related in the form $bt = 0$.

strength will then be

$$\frac{\sigma_1}{\sigma_3} = \frac{\sqrt{\xi}+1}{\sqrt{\xi}-1} \tag{14}$$

at a specific value of the ratio σ_2/σ_3 to be determined.

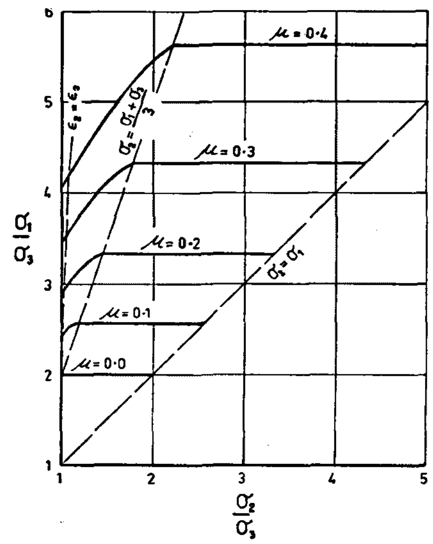

Figure 3

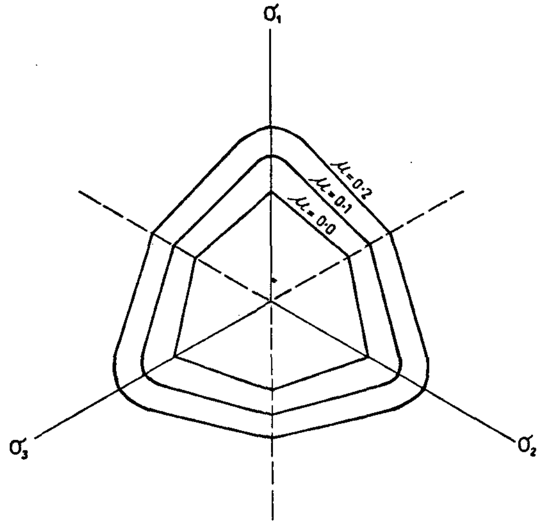

Figure 4

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From (10) and (11), and dividing by (12)

$$\frac{\sigma_2}{\sigma_3} = \frac{F-2b\mu(1-\beta)}{3F+2b\mu(1-\beta)}\left(\frac{\sigma_1}{\sigma_3}+1\right). \tag{15}$$

Equations (13), (14), and (15) have been solved by digital computer and the solution is depicted in Fig. 3 and again in Fig. 4 in stress space, together with Rennie's original linear solution. It should be noted that the solution is only valid to a first order in μ, as pointed out by Rennie, but the computations are exact for purposes of clarity.

It might also be noted that the solution $\varepsilon_2 = \varepsilon_3$ is reached at a very small distance from the axis (Fig. 3), and thereafter the intermediate principal axes of stress and strain no longer coincide. Although contrary to the original assumption, however, this does not conflict with any essential physical requirement.

For the particular case $\sigma_2 = \sigma_3$ (for which $a \approx b$), the condition of stability is that

$$\frac{\sigma_1}{\sigma_3} \leqq 2+4\mu+O(\mu^2) \tag{16}$$

whereas the equivalent expression for the case solved by Rennie is

$$\frac{\sigma_1}{\sigma_3} \leqq 2+2\mu\sqrt{6}+O(\mu^2). \tag{17}$$

4. Other failure modes

There remain two other failure modes to be examined (section 2). The first of these (two broken contacts, normals intersecting at 60°) has been studied elsewhere (Parkin [2]), but leads to higher strengths and does not rate further mention. Only the case of three broken contacts remain to be solved and must be the subject of further investigation. However, since the support conditions are less stable and since there is a higher potential energy requirement for failure, it may be anticipated that the solution will be less critical than the one advanced above.

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