

## HARMONIC MORPHISMS BETWEEN SPACES OF CONSTANT CURVATURE

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(Received 22nd March 1991)

Let  $M$  and  $N$  be simply connected space forms, and  $U$  an open and connected subset of  $M$ . Further let  $\pi: U \rightarrow N$  be a horizontally homothetic harmonic morphism. In this paper we show that if  $\pi$  has totally geodesic fibres and integrable horizontal distribution, then the horizontal foliation of  $U$  is totally umbilic and isoparametric. This leads to a classification of such maps. We also show that horizontally homothetic harmonic morphisms of codimension one are either Riemannian submersions modulo a constant, or up to isometries of  $M$  and  $N$  one of six well known examples.

1991 *Mathematics subject classification*: Primary 58E20, Secondary 53C42.

### 0. Introduction

Harmonic morphisms  $\pi: M \rightarrow N$  between Riemannian manifolds are maps that pull back germs of harmonic functions on  $N$  to germs of harmonic functions on  $M$ . It was proved independently by Fuglede [14] and Ishihara [18], that harmonic morphisms are exactly the harmonic maps that are horizontally conformal. An alternative description of harmonic morphisms is that they map Brownian paths on  $M$  to Brownian paths on  $N$ . For this see [20], [6] and [11]. In a series of papers [2, 3, 4, 5] Baird and Wood have studied the case when  $M$  is three dimensional and  $N$  a surface. Their result, most relevant to this paper, is given by the following theorem.

**Theorem 0.1.** *Let  $M = \mathbb{R}^3, S^3$  or  $H^3$  and  $N$  be a surface. Further let  $U$  be an open and connected subset of  $M$ .*

(i) *If  $\pi: M \rightarrow N$  is a harmonic morphism, then up to isometries of  $M$ ,  $\pi$  is one of the maps given in Examples 2.4–2.7 below, followed by a weakly conformal map.*

(ii) *If  $\pi: U \rightarrow N$  is a harmonic morphism with an isolated singularity, then up to isometries of  $M$ ,  $\pi$  is a local restriction of one of the maps given in Examples 2.1–2.3 below, followed by a weakly conformal map.*

Generalizations to higher dimensions have been achieved in some cases when assuming that  $\pi$  has totally geodesic fibres. The following result is due to Kasue and Washio [19] and is of global nature, that is  $M$  is complete.

**Theorem 0.2.** *Let  $n \geq 3$  and  $\pi: \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a harmonic morphism with totally geodesic fibres. Then  $N = \mathbb{R}^n$  and  $\pi$  is an orthogonal projection, followed by a homothety.*

The next two results, were given by the author in [16]. They are of local nature, i.e.  $U$  is any open and connected subset of  $M$ .

**Theorem 0.3.** *Let  $U$  be an open and connected subset of  $\mathbb{R}^m$  and  $\pi: U \rightarrow \mathbb{R}^n$  a horizontally homothetic harmonic morphism, with totally geodesic fibres. Then  $\pi$  is the restriction of an orthogonal projection  $\hat{\pi}: \mathbb{R}^m \rightarrow \mathbb{R}^n$ , followed by a homothety.*

**Theorem 0.4.** *Let  $(M, N) = (S^m, \mathbb{R}^n), (S^m, N^n)$  or  $(\mathbb{R}^m, H^n)$  and  $U$  be an open subset of  $M$ . Then there exists no horizontally homothetic harmonic morphism  $\pi: U \rightarrow N$ .*

In this paper we establish a connection with the theory of isoparametric foliations on simply connected space forms. This enables us to use a well known classification of such foliations with totally umbilic leaves. We generalize the above mentioned results to:

**Theorem 3.5.** *Let  $(M, N) = (S^m, S^n), (\mathbb{R}^m, S^n), (H^m, S^n), (\mathbb{R}^m, \mathbb{R}^n), (H^m, \mathbb{R}^n)$  or  $(H^m, H^n)$  and  $\hat{\pi}_i: \hat{U}_i \rightarrow N$  be the corresponding harmonic morphism given below. Further let  $U$  be an open and connected subset of  $M$  and  $\pi: U \rightarrow N$  be a horizontally homothetic harmonic morphism. If  $\pi$  has totally geodesic fibres and integrable horizontal distribution, then up to isometries of  $M$  and  $N$ ,  $\pi = \hat{\pi}_i|_U$ .*

Together Theorems 0.4 and 3.5 give a classification of horizontally homothetic harmonic morphisms, with totally geodesic fibres and integrable horizontal distribution, between open and connected subsets of simply connected space forms. If  $n \geq 3$ , then Theorem 3.5 is true without assuming horizontal homothety. When  $(M, N) = (\mathbb{R}^m, \mathbb{R}^n)$ , then  $\pi: U \rightarrow N$  has automatically integrable horizontal distribution, so in that case the statement of Theorem 3.5 is exactly that of Theorem 0.3. For  $\pi$  of codimension one we prove:

**Theorem 3.6.** *Let  $(M, N) = (S^m, S^{m-1}), (\mathbb{R}^m, S^{m-1}), (H^m, S^{m-1}), (\mathbb{R}^m, \mathbb{R}^{m-1}), (H^m, \mathbb{R}^{m-1})$  or  $(H^m, H^{m-1})$  and  $U$  be an open and connected subset of  $M$ . If  $\pi: U \rightarrow N$  is a horizontally homothetic harmonic morphism, then  $\pi$  has constant dilation, or up to isometries of  $M$  and  $N$ ,  $\pi = \hat{\pi}|_U$ , where  $\hat{\pi}$  is one of the maps given in Examples 2.1–2.6 below.*

We then show that if the dilation is constant, then the only possible cases are  $(M, N) = (S^m, S^n)$  or  $(\mathbb{R}^m, \mathbb{R}^n)$ , and if  $\pi: S^m \rightarrow S^n$  has totally geodesic fibres, then it must be one of the Hopf-maps.

## 1. Preliminaries

In the following we assume that all our objects, manifolds, maps etc. are smooth, that is in the  $C^\infty$ -category. Let  $(M^m, \langle \cdot, \cdot \rangle, \nabla)$  and  $(N^n, \langle \cdot, \cdot \rangle_N, \hat{\nabla})$  be two Riemannian manifolds

with their Levi-Civita connections. By  $K_M$  and  $K_N$  we denote their sectional curvatures, which we assume are constant. Harmonic morphisms  $\pi: M \rightarrow N$  are maps that pull back germs of harmonic functions on  $N$  to germs of harmonic functions on  $M$ . It can be shown that if  $m < n$ , then  $\pi$  is constant. To exclude that trivial case we assume throughout this paper that  $\pi: M \rightarrow N$  is non-constant, hence  $m \geq n$ .

The following result of Fuglede [14] and Ishihara [18] gives a more geometric description of harmonic morphisms.

**Theorem 1.1.** *A map  $\pi: M^m \rightarrow N^n$  such that  $m \geq n$  is a harmonic morphism if and only if*

- (1)  $\pi$  is a harmonic map, and
- (2)  $\pi$  is horizontally conformal.

That  $\pi$  is horizontally conformal means that  $d\pi|_{\mathcal{H}_x}$ , the restriction of  $d\pi$  to the horizontal spaces is weakly conformal. Its conformal factor  $\lambda$  is called the dilation of  $\pi$ .  $\pi$  is said to be horizontally homothetic if it is horizontally conformal and  $\lambda$  is constant along horizontal curves in  $M$ . It was shown by Fuglede in [15], that non-constant horizontally homothetic maps are submersions. The next result due to Baird and Eells [1] shows that it is of major importance whether  $N$  is a surface or not.

**Theorem 1.2.** *Let  $\pi: M \rightarrow N$  be a horizontally conformal submersion with dilation  $\lambda: M \rightarrow \mathbb{R}^+$ . If*

- (i)  $\dim N = 2$ , then  $\pi$  is harmonic if and only if  $\pi$  has minimal fibres,
- (ii)  $\dim N \geq 3$ , then two of the following conditions imply the other,
  - (1)  $\pi$  is harmonic,
  - (2)  $\pi$  has minimal fibres,
  - (3)  $\pi$  is horizontally homothetic.

To get the equivalence of  $\pi$  being harmonic and  $\pi$  having minimal fibres in higher dimensions, the condition of horizontal homothety is exactly what is needed. For a more detailed account on harmonic maps and morphisms we refer the reader to [12] and [2, 3].

In this paper we are dealing with horizontally homothetic harmonic morphisms, which are therefore submersions. The horizontal conformality of  $\pi: M \rightarrow N$  is obviously independent of homothetic changes of the metrics concerned. The same is true for the harmonicity of  $\pi$ . We are studying the case of constant curvature and can therefore without loss of generality restrict our attention to the cases when  $K_M, K_N \in \{-1, 0, 1\}$ .

**Definition 1.3.** Let  $M^m = S^m, \mathbb{R}^m$  or  $H^m$ , and  $f = (f_1, \dots, f_{m-n}): M^m \rightarrow \mathbb{R}^{m-n}$  be a smooth map.  $f$  is called *isoparametric* if

- (1)  $f$  has a regular value,
- (2) the functions  $\langle \text{grad}(f_i), \text{grad}(f_j) \rangle$ , and  $\Delta f_k$  are constant along the pre-images of  $f$  for all  $i, j, k$ , and
- (3)  $[\text{grad}(f_i), \text{grad}(f_j)]$  is a linear combination of  $\text{grad}(f_1), \dots, \text{grad}(f_{m-n})$  with coefficients being constant along the pre-images of  $f$ , for all  $i$  and  $j$ .

The *isoparametric foliation*  $\mathcal{F}_f$  associated to  $f$  is the decomposition of  $M$  into the pre-images of  $f$ .  $\mathcal{F}_f$  is an  $n$ -dimensional foliation with possible singularities (focal varieties). An *isoparametric submanifold* is the pre-image of a regular value of an isoparametric map. An isoparametric submanifold  $L$  of  $M$  is characterized by the following conditions:

- (1)  $L$  has flat normal bundle, and
- (2) the principal curvatures of  $L$  in the direction of any parallel normal field are constant.

Given an isoparametric submanifold  $L$  of  $M$  the corresponding isoparametric foliation  $\mathcal{F}_f$  is uniquely determined. Its leaves are simply the submanifolds of  $M$ , which are parallel to  $L$ .

The concept of an isoparametric map is a generalisation of that of an isoparametric function, first studied by Cartan in [7, 8]. The definition as above was given by Terng in [24] after Carter and West had studied the case of codimension 2 in [9, 10].

**2. Examples**

We now give examples of horizontally homothetic, harmonic morphisms with totally geodesic fibres  $\hat{\pi}: U \subset M \rightarrow N$  between open and connected subsets of simply connected space forms. They have all got integrable horizontal distributions except Example 2.7. The dilation is constant in 2.6–2.7 but non-constant for 2.1–2.5. These examples play a major role in our classification given in the next section.

**Example 2.1.**  $S^m \setminus S^0 = (\{(\cos(s), \sin(s) \cdot e) \in \mathbb{R} \times \mathbb{R}^m \mid s \in (0, \pi) \text{ and } e \in S^{m-1}\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}})$ . Let  $\hat{\pi}: S^m \setminus S^0 \rightarrow S^{m-1}$  be the projection along the longitudes onto the equatorial hypersphere, given by  $\hat{\pi}: (\cos(s), \sin(s) \cdot e) \mapsto e$ . For  $e \in S^{m-1}$  the fibre of  $\hat{\pi}$  over  $e$  is parametrized by arclength by  $\gamma_e(s) = (\cos(s), \sin(s) \cdot e)$ , where  $s \in (0, \pi)$ . Along the fibres we have  $\lambda^2(s) = 1/\sin^2(s)$ . The level hypersurfaces are small spheres  $S_{|\sin(s)|}^{m-1}$  with constant sectional curvature  $K_{S_{|\sin(s)|}^{m-1}} = 1/\sin^2(s)$ .

**Example 2.2.** Let  $\hat{\pi}: \mathbb{R}^m \setminus \mathbb{R}^0 \rightarrow S^{m-1}$  be the radial projection, given by  $\hat{\pi}: x \mapsto x/|x|$ . For  $e \in S^{m-1}$  the fibre of  $\hat{\pi}$  over  $e$  is parametrized by arclength by  $\gamma_e(s) = s \cdot e$  where  $s \in \mathbb{R}^+$ , and along the fibres  $\hat{\lambda}^2(s) = 1/s^2$ . The level hypersurfaces are spheres  $S_s^{m-1}$  with constant sectional curvature  $K_{S_s^{m-1}} = 1/s^2$ .

**Example 2.3.** Let  $H^m \setminus H^0 = (B_1^m(0) - \{0\}, (4/(1 - |x|^2)^2) \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$ , where  $B_1^m(0) := \{x \in \mathbb{R}^m \mid |x| < 1\}$ . Let  $\hat{\pi}: H^m \setminus H^0 \rightarrow S^{m-1}$  be the radial projection, given by  $\hat{\pi}: x \mapsto x/|x|$ . For  $e \in S^{m-1}$  the fibre of  $\hat{\pi}$  over  $e$  is parametrized by arclength by  $\gamma_e(s) = \tanh(s/2) \cdot e$ , where  $s \in \mathbb{R}^+$ , and along the fibres  $\hat{\lambda}^2(s) = 1/\sinh^2(s)$ . The level hypersurfaces are spheres  $S_{\tanh(s/2)}^{m-1}$  with constant sectional curvature  $K_{S_{\tanh(s/2)}^{m-1}} = 1/\sinh^2(s)$ .

**Example 2.4.** Let  $H^m = (\mathbb{R}^{m-1} \times \mathbb{R}^+, (1/x_m^2) \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$  and  $\hat{\pi}: H^m \rightarrow \mathbb{R}^{m-1}$  be the projection onto  $\mathbb{R}^{m-1}$  followed by a homothety, given by  $\hat{\pi}: (p, x) \mapsto \alpha \cdot p$ , where  $\alpha \in \mathbb{R} \setminus \{0\}$ . For

$p \in \mathbb{R}^{m-1}$  the fibre of  $\hat{\pi}$  over  $\alpha p$  is parametrized by arclength by  $\gamma_p(s) = (p, e^{-s})$ , where  $s \in \mathbb{R}$ , and along the fibres  $\hat{\lambda}^2(s) = \alpha^2 e^{-2s}$ . The level hypersurfaces are affine subspaces  $\mathbb{R}_{e^{-s}}^{m-1} := \{(p, e^{-s}) \in \mathbb{R}^{m-1} \times \mathbb{R}^+ \mid p \in \mathbb{R}^{m-1}\}$  with constant sectional curvature  $K_{\mathbb{R}_{e^{-s}}^{m-1}} = 0$ .

**Example 2.5.**  $H^m = (\mathbb{R}^{m-2} \times \mathbb{R} \times \mathbb{R}^+, (1/x_m^2) \cdot \langle, \rangle_{\mathbb{R}^m})$  and  $H^{m-1} = (\mathbb{R}^{m-2} \times \mathbb{R}^+, (1/x_{m-1}^2) \cdot \langle, \rangle_{\mathbb{R}^{m-1}})$ . Define a map  $\hat{\pi}: H^m \rightarrow H^{m-1}$  by  $\hat{\pi}: (p, x, y) \mapsto (p, \sqrt{x^2 + y^2})$ . The fibre over  $(p, r)$  is the semicircle in  $\mathbb{R}^{m-2} \times \mathbb{R}^+ \times \mathbb{R}$  with centre  $(p, 0, 0)$ , radius  $r$  and parallel to the coordinate plane  $\{(0, a, b) \mid a, b \in \mathbb{R}\}$ . The fibre is parametrized by arclength by  $\gamma_{(p,r)}(s) = (p, r \cdot \tanh(s), r/\cosh(s))$ . Geometrically this map is a projection along the geodesics of  $H^m$  orthogonal to  $H^{m-1}$ , where the latter is considered as the subset  $\mathbb{R}^{m-2} \times \{0\} \times \mathbb{R}^+$  of  $H^m$ . Along the fibre  $\hat{\lambda}^2(s) = 1/\cosh^2(s)$ . The level hypersurfaces are  $H_s^{m-1} := \{(p, r \cdot \tanh(s), r/\cosh(s)) \in H^m \mid p \in \mathbb{R}^{m-2}, r \in \mathbb{R}^+\} = \mathbb{R}^{m-2} \times \text{span}_{\mathbb{R}}\{(0, \tanh(s), 1/\cosh(s))\}$ , i.e. they are hyperbolic hyperplanes with constant sectional curvature  $K_{H_s^{m-1}} = -1/\cosh^2(s)$ .

**Example 2.6.** Let  $\hat{\pi}: \mathbb{R}^m \rightarrow \mathbb{R}^n$  be the orthogonal projection followed by a homothety, given by  $\hat{\pi}: (x_1, \dots, x_m) \mapsto \alpha \cdot (x_1, \dots, x_n)$ , where  $\alpha \in \mathbb{R} \setminus \{0\}$ . For  $p \in \mathbb{R}^n$  the fibre of  $\hat{\pi}$  over  $p$  is parameterized by  $\gamma_p(s) = (p, s)$ , where  $s \in \mathbb{R}^{m-n}$ . The dilation is constant  $\hat{\lambda}^2(s) = \alpha^2$ .

**Example 2.7.** Let  $F = \mathbb{C}, \mathbb{H}$  or  $Ca$ , i.e. the complex numbers, the quaternions or the Cayley numbers. Put  $(m, n) := (2 \cdot \dim F - 1, \dim F) = (3, 2), (7, 4)$  or  $(15, 8)$ . Define  $\pi: F \times F = \mathbb{R}^{m+1} \rightarrow \mathbb{R} \times F = \mathbb{R}^{n+1}$  by  $\pi: (x, y) \mapsto (|x|^2 - |y|^2, 2x \cdot y)$ . The restrictions of  $\pi$  to  $S^m \subset \mathbb{R}^{m+1}$  are the Hopf maps. They are harmonic morphisms  $\hat{\pi} := \pi|_{S^m}: S^m \rightarrow S^n$ , with constant dilation  $\hat{\lambda} = 2$ . The fibres are totally geodesic and therefore isometric to  $S^{m-n} \subset S^m$ . It can be shown that the horizontal distributions are nowhere integrable.

### 3. Classification

We remind the reader of the definition of the fundamental tensors of a submersion,  $A$  and  $T$ , first introduced in [21]. For two vector fields  $E, F$  on  $M$  let

$$A_E F := \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H} F, \text{ and}$$

$$T_E F := \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H} F.$$

**Lemma 3.1.** *If  $\pi: M \rightarrow N$  is a horizontally conformal submersion and  $X, Y$  horizontal vector fields, then*

$$A_X Y = \frac{1}{2} \left\{ \mathcal{V} [X, Y] - \lambda^2 \langle X, Y \rangle \text{grad}_{\mathcal{V}} \left( \frac{1}{\lambda^2} \right) \right\}$$

**Proof.** Easy calculation, using the definition of the Levi-Civita connection, see [17].

The first part of the following Lemma 3.2 is a well known result from the theory of conformal foliations. See for example [25].

**Lemma 3.2.** *Let  $M^m$  and  $N^n$  be two Riemannian manifolds and  $\pi: M \rightarrow N$  a horizontally conformal submersion. If the horizontal distribution is integrable, then any integral submanifold  $L$  is totally umbilic in  $M$ . If  $\pi$  is horizontally homothetic, then  $K_L = \lambda^2 \cdot K_N$ .*

**Proof.** Let  $X, Y$  be the two local horizontal vector fields. Since the horizontal distribution is integrable we have  $\mathcal{V}[X, Y] = 0$ , so  $A_X Y = -(\lambda^2/2)\langle X, Y \rangle \cdot \text{grad}_V(1/\lambda^2)$ . Let  $V$  be a local parallel unit normal field along one of the integral manifolds  $L$ . The corresponding shape operator  $S_V$  is given by  $S_V: X \mapsto \mathcal{H}\nabla_X V = A_X V$ . Let  $\{X_1, \dots, X_n\}$  be a local orthonormal frame for the horizontal distribution  $\mathcal{H}$ , then

$$\begin{aligned} S_V(X) &= \sum_{i=1}^n \langle A_X V, X_i \rangle X_i = - \sum_{i=1}^n \langle V, A_X X_i \rangle X_i \\ &= \sum_{i=1}^n \left\langle V, \frac{\lambda^2}{2} \langle X, X_i \rangle \text{grad}_V \left( \frac{1}{\lambda^2} \right) \right\rangle X_i = \frac{\lambda^2}{2} V \left( \frac{1}{\lambda^2} \right) X. \end{aligned}$$

This shows that  $S_V = (\lambda^2/2)V(1/\lambda^2) \cdot \text{Id}_{\mathcal{H}}$ , so  $L$  is totally umbilic in  $M$ . If  $\pi$  is horizontally homothetic, then  $\pi|_L: L \rightarrow N$  is a homothety, so therefore  $K_L = \lambda^2 \cdot K_N$ . □

Totally umbilic submanifolds  $L$  of simply connected space forms  $M^m$  have been classified. For this see for example [23]. They have constant sectional curvature  $K_L \geq K_M$ . It then follows from Lemma 3.2 that if  $M$  is a simply connected space form and  $\pi$  is horizontally homothetic, then  $N$  must have constant sectional curvature.

It was noted by Baird and Eells in [1] that a non-constant dilation of a horizontally homothetic harmonic morphism of codimension one is an isoparametric function. We generalize this further to higher codimensions by the following:

**Theorem 3.3.** *Let  $M$  and  $N$  be simply connected space forms and  $U$  an open and connected subset of  $M$ . Let  $\pi: U \rightarrow N$  be a horizontally homothetic harmonic morphism with totally geodesic fibres and integrable horizontal distribution. Further let  $\mathcal{F}_{\mathcal{H}}$  be the horizontal foliation, whose leaves are the integral submanifolds. Then  $\mathcal{F}_{\mathcal{H}}$  is a totally umbilic isoparametric foliation.*

**Proof.** It follows from Lemma 3.2 that  $L \in \mathcal{F}_{\mathcal{H}}$  is totally umbilic in  $U$ . To show that  $L$  is isoparametric we must prove that its normal bundle  $\nu(L)$  is flat, and that the principal curvatures of  $L$  in the direction of any parallel normal field are constant. Let  $V$  and  $U$  be two local parallel unit normal fields along  $L$ , then follows from the proof above, that  $S_U S_V - S_V S_U = 0$ . Since  $M$  has constant sectional curvature, it follows from Proposition 2.1.2 of [22], that the normal bundle is flat. Let  $X$  be a local horizontal vector field, then

$$X \left( \frac{\lambda^2}{2} V \left( \frac{1}{\lambda^2} \right) \right) = \frac{\lambda^2}{2} X V \left( \frac{1}{\lambda^2} \right) = \frac{\lambda^2}{2} \left( [X, V] \left( \frac{1}{\lambda^2} \right) + \underbrace{V X \left( \frac{1}{\lambda^2} \right)}_{=0} \right) = 0$$

$$\begin{aligned}
 &= \frac{\lambda^2}{2} \left\langle \text{grad}_V \left( \frac{1}{\lambda^2} \right), \nabla_X V - \nabla_V X \right\rangle \\
 &= \frac{\lambda^2}{2} \left( \underbrace{\left\langle \text{grad}_V \left( \frac{1}{\lambda^2} \right), \underbrace{V \nabla_X V}_{=0} \right\rangle}_{=0} + \underbrace{\left\langle \nabla_V \text{grad}_V \left( \frac{1}{\lambda^2} \right), X \right\rangle}_{=0} \right) = 0.
 \end{aligned}$$

since  $V$  is parallel and  $\pi$  has totally geodesic fibres. The principal curvatures of  $L$  in the direction of any parallel normal field are therefore constant along  $L$ . □

Let  $M = S^m, \mathbb{R}^m$  or  $H^m$  and  $\mathcal{F}$  be an  $n$ -dimensional isoparametric foliation of  $M$  with possible singularities (focal varieties). If the regular leaves of  $\mathcal{F}$  are totally umbilic in  $M$ , then they have constant sectional curvatures, which all have the same sign. Given this sign, the foliation is uniquely determined up to an isometry of  $M$ .  $\mathcal{F}$  is known to exist if and only if  $(M, K_{\text{leaves}}) = (H^m, > 0), (S^m, \geq 1), (\mathbb{R}^m, > 0), (\mathbb{R}^m, = 0), (H^m, = 0), (H^m, \in [-1, 0))$ . Furthermore  $\mathcal{F}$  has singularities exactly if  $K_{\text{leaves}} > 0$ .

**Corollary 3.4.** *Let  $(M, N) = (S^m, S^n), (\mathbb{R}^m, S^n)$  or  $(H^m, S^n)$ , then there exists no horizontally homothetic harmonic morphism  $\pi: M \rightarrow N$  with totally geodesic fibres and integrable horizontal distribution.*

**Proof.** If  $\pi$  existed, then it would define an isoparametric foliation on the whole of  $M$ . But since  $S^n$  has positive constant sectional curvature this is not possible. □

It follows directly from the definition, that the composition of two harmonic morphisms  $\tilde{\pi}: (M, \langle, \rangle) \rightarrow (M, \langle, \rangle_{\bar{M}})$  and  $\bar{\pi}: (\bar{M}, \langle, \rangle_{\bar{M}}) \rightarrow (N, \langle, \rangle_N)$  is again a harmonic morphism. It is easily checked, that if  $\tilde{\pi}$  and  $\bar{\pi}$  are horizontally homothetic with totally geodesic fibres, so is  $\pi = \bar{\pi} \circ \tilde{\pi}$ . The same is true for the integrability of the horizontal distributions. We now use these facts to construct the above mentioned unique totally umbilic isoparametric foliations. They are given as the horizontal foliations of the following compositions of Examples 2.1–2.6.

$$\hat{\pi}_1: S^m \setminus S^{m-(n+1)} \xrightarrow{2.1} \dots \xrightarrow{2.1} S^{n+1} \setminus S^0 \xrightarrow{2.1} S^n$$

$$\hat{\pi}_2: \mathbb{R}^m \setminus \mathbb{R}^{m-(n+1)} \xrightarrow{2.6} \mathbb{R}^{n+1} \setminus \mathbb{R}^0 \xrightarrow{2.2} S^n$$

$$\hat{\pi}_3: H^m \setminus H^{m-(n+1)} \xrightarrow{2.5} \dots \xrightarrow{2.5} H^{n+1} \setminus H^0 \xrightarrow{2.3} S^n$$

$$\hat{\pi}_4: H^m \xrightarrow{2.4} \mathbb{R}^{m-1} \xrightarrow{2.6} \mathbb{R}^n$$

$$\hat{\pi}_5: H^m \xrightarrow{2,5} H^{m-1} \xrightarrow{2,5} \dots \xrightarrow{2,5} H^{n+1} \xrightarrow{2,5} H^n$$

$$\hat{\pi}_6: \mathbb{R}^m \xrightarrow{2,6} \mathbb{R}^n$$

**Theorem 3.5.** *Let  $(M, N) = (S^m, S^n), (\mathbb{R}^m, S^n), (H^m, S^n), (H^m, \mathbb{R}^n), (H^m, H^n)$  or  $(\mathbb{R}^m, \mathbb{R}^n)$ , and  $\hat{\pi}_i: \hat{U}_i \rightarrow N$  be the corresponding harmonic morphism given above. Further let  $U$  be an open and connected subset of  $M$  and  $\pi: U \rightarrow N$  be a horizontally homothetic harmonic morphism. If  $\pi$  has totally geodesic fibres and integrable horizontal distribution, then up to isometries of  $M$  and  $N$ ,  $\pi = \hat{\pi}_i|_U$ .*

**Proof.** It follows from Theorem 3.3 that  $\pi$  determines a totally umbilic isoparametric foliation  $\mathcal{F}_U$  on  $U$  without singularities. This foliation is up to isometries of  $M$  uniquely determined. This means that there exists a foliation preserving isometric embedding  $\sigma: (U, \mathcal{F}_U) \rightarrow (\hat{U}_i, \mathcal{F}_{\hat{U}_i})$ , where  $\mathcal{F}_{\hat{U}_i}$  is the foliation given by  $\hat{\pi}_i: \hat{U}_i \rightarrow N$ .  $\sigma$  is foliation preserving, so there exists a map  $\hat{\sigma}: N \rightarrow N$ , such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & \hat{U}_i \\ \pi \downarrow & & \downarrow \hat{\pi}_i \\ N & \xrightarrow{\hat{\sigma}} & N \end{array}$$

The maps  $\pi$  and  $\hat{\pi}_i$  are horizontally conformal and  $\sigma$  is an isometry, so  $\hat{\sigma}$  is conformal. If  $\lambda_{\hat{\sigma}}^2$  is the corresponding conformal factor, then  $\pi^* \lambda_{\hat{\sigma}}^2: U \rightarrow \mathbb{R}^+$  satisfies

$$\lambda^2 \cdot \pi^* \lambda_{\hat{\sigma}}^2 = \sigma^* \hat{\lambda}_i^2.$$

$\sigma^* \hat{\lambda}_i^2$  and  $\lambda^2$  are horizontally constant, so  $\pi^* \lambda_{\hat{\sigma}}^2$  is.  $\pi^* \lambda_{\hat{\sigma}}^2$  is a pull back via  $\pi$  and therefore vertically constant, so  $\hat{\sigma}$  is a homothety. If  $N = S^n$  or  $H^n$ , then  $\lambda_{\hat{\sigma}}^2$  is obviously 1. But if  $N = \mathbb{R}^n$  then  $\lambda_{\hat{\sigma}}^2$  can be any  $\alpha \in \mathbb{R}^+$ . □

Together Theorems 0.4 and 3.5 give a classification for horizontally homothetic harmonic morphisms with totally geodesic fibres and integrable horizontal distribution between open and connected subsets of simply connected space forms.

To see that the condition of integrability is necessary, consider the following horizontally homothetic harmonic morphisms.

$$\hat{\pi}_8: \mathbb{R}^4 \setminus \mathbb{R}^1 \xrightarrow{2,6} \mathbb{R}^3 \setminus \{0\} \xrightarrow{2,2} S^2,$$

and



$$\hat{\pi}_9: \mathbb{R}^4 \setminus \mathbb{R}^1 \xrightarrow{2:2} S^3 \setminus \{N, S\} \xrightarrow{2:2} S^2.$$

They both have totally geodesic fibres, but their corresponding vertical foliations are fundamentally different, so the maps must be different. It is the non-integrability of the horizontal distribution of  $\hat{\pi}_9$  that is responsible for this.

From Theorem 1.2 we see that the condition of horizontal homothety on  $\pi$  in Theorem 3.5 can be dropped in the case when  $n \geq 3$ . It also follows from Proposition 2.6 of [16], that if  $(M, N) = (\mathbb{R}^m, \mathbb{R}^n)$ , then  $\pi: U \rightarrow N$  has automatically integrable horizontal distribution. In this case the statement of Theorem 3.5 is exactly that of Theorem 0.3. For the case when  $\pi$  has codimension one we have the following:

**Theorem 3.6.** *Let  $(M, N) = (S^m, S^{m-1}), (\mathbb{R}^m, S^{m-1}), (H^m, S^{m-1}), (\mathbb{R}^m, \mathbb{R}^{m-1}), (H^m, \mathbb{R}^{m-1})$  or  $(H^m, H^{m-1})$  and  $U$  be an open and connected subset of  $M$ . If  $\pi: U \rightarrow N$  is a horizontally homothetic harmonic morphism, then  $\pi$  has constant dilation, or up to isometries of  $M$  and  $N$ ,  $\pi = \hat{\pi}|_U$ , where  $\hat{\pi}$  is one of the maps given in Examples 2.1–2.6.*

**Proof.** If the dilation  $\lambda$  is not constant, there exists a point  $p \in U$ , such that  $\text{grad}(\lambda^2) \neq 0$  on an open neighbourhood  $W$  of  $p$ . It follows from  $\text{grad}_{\mathcal{H}}(\lambda^2) = 0$  that  $\mathcal{H}$  is integrable on  $W$  and its integral manifolds are the level hypersurfaces of  $\lambda$ . Since  $\pi$  has codimension one its fibres are totally geodesic. We can therefore apply Theorem 3.5 on  $W$ .  $\pi$  is analytic and extends therefore uniquely onto the whole of  $U$ . □

**Proposition 3.7.** *Let  $M^m$  and  $N^n$  be simply connected space forms, and  $U$  an open and connected subset of  $M$ . If  $\pi: U \rightarrow N$  is a harmonic morphism with constant dilation and totally geodesic fibres, then  $(M, N) = (S^m, S^n)$  or  $(\mathbb{R}^m, \mathbb{R}^n)$ .*

**Proof.** The cases  $(M, N) = (S^m, \mathbb{R}^n), (S^m, H^n)$  and  $(\mathbb{R}^m, H^n)$  is excluded by Theorem 0.4. Let  $X, Y \in \mathcal{H}$  and  $V \in \mathcal{V}$  be local vector fields, such that  $|X| = |Y| = |V| = 1$  and  $\langle X, Y \rangle = 0$ . Modifying O'Neill's well known curvature equations for Riemannian submersions (see [21]), one gets:

- (1)  $K_M(X \wedge V) = |A_X V|^2$ , and
- (2)  $K_M(X \wedge Y) = \lambda^2 \cdot K_N(\check{X} \wedge \check{Y}) - \frac{3}{4} |\mathcal{V}[X, Y]|^2$ .

For this see [17]. Equation (1) now excludes the cases  $(M, N) = (H^m, S^n), (H^m, \mathbb{R}^n)$  and  $(H^m, H^n)$ . If  $M = \mathbb{R}^m$ , then  $A_X E = 0$  for all  $E \in TM$ , so the horizontal distribution is integrable. This makes the case when  $(M, N) = (\mathbb{R}^m, S^n)$  impossible by (2). □

For the case when  $(M, N) = (S^m, S^n)$  we have the following conjecture.

**Conjecture 3.8.** *Let  $U$  be an open and connected subset of  $S^m$  and  $\pi: U \rightarrow S^n$  a harmonic morphism with constant dilation. If  $\pi$  has totally geodesic fibres, then up to isometries of  $S^m$  and  $S^n$   $\pi = \hat{\pi}|_U$ , where  $\hat{\pi}$  is one of the Hopf-maps given in Example 2.7.*

This conjecture is true if  $U = S^m$ , by the classification of Riemannian submersions between spheres due to Escobales, see [13].

**Acknowledgements.** This paper has benefitted from some valuable suggestions made by J. C. Wood and P. Baird. I am very grateful for their interest in this work. I would also like to thank S. Carter, G. Thorbergsson and A. West who advised me on the theory of isoparametric systems. The research leading to this paper was supported by the ORS and FCO award schemes.

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