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MEAN CONVERGENCE OF HERMITE-FEJÉR INTERPOLATION BASED ON THE ZEROS OF LASCENOV POLYNOMIALS

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ABSTRACT. Weighted L^p mean convergence of Hermite-Fejér interpolation based on the zeros of orthogonal polynomials with respect to the weight $|x|^{2\alpha+1}(1-x^2)^\beta$ $(\alpha, \beta > -1)$ is investigated. A necessary and sufficient condition for such convergence for all continuous functions is given. Meanwhile divergence of Hermite-Fejér interpolation in L^p with $p > 2$ is obtained. This gives a possible answer to Problem 17 of P. Turán [J. Approx. Theory, 29(1980), p. 40].

1. **Introduction.** Let $v(x) \geq 0$ be a weight function and $\{P_n(v,x)\}$ the orthonormal polynomials with respect to v. The zeros of $P_n(v,x)$ are denoted by $x_{kn} := x_{kn}(v)$ satisfying

$$1 > x_{1n} > x_{2n} > \cdots > x_{nn} > -1. \tag{1.1}$$

The Hermite-Fejér interpolation of $f \in C[-1,1]$ at the zeros (1.1) is defined by

$$H_n(v,f) := H_n(v,f,x) := \sum_{k=1}^{n} f(x_{kn})A_{kn}(v,x) \tag{1.2}$$

where

(1.3)

$$A_{kn}(v,x) := \left[1 - \frac{P_n''(v,x_{kn})}{P_n'(v,x_{kn})}(x - x_{kn})\right] \quad l_{kn}^2(v,x) := v_{kn}(v,x)l_{kn}^2(v,x), \quad k = 1,2,\ldots,n,$$

$$l_{kn}(v,x) := \frac{P_n(v,x)}{P_n'(v,x_{kn})(x - x_{kn})}, \qquad k = 1,2,\ldots,n. \tag{1.4}$$

Assume that $\alpha, \beta, a, b > -1$. Let us consider the orthogonal polynomials with respect to the weight $|x|^{2\alpha+1}(1-x^2)^\beta$, usually called Lascenov polynomials. The uniform convergence of the corresponding Hermite-Fejér interpolation was investigated by several authors [1, 6, 7]. But L^p convergence for such interpolation has never been dealt with in the literature. In this paper we present several results about weighted L^p convergence for such interpolation. The main result is the following, in which

$$\begin{aligned} w(x) &:= |x|^{2\alpha+1}(1-x^2)^\beta, \\ u(x) &:= |x|^{2a+1}(1-x^2)^b, \end{aligned} \tag{1.5}$$

$$\|f\|_{L_u^p} := \left\{\int_{-1}^{1} |f(x)|^p u(x)\,dx\right\}^{\frac{1}{p}}, \qquad 0 < p < \infty. \tag{1.6}$$

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THEOREM. *Let* $0 < p < \infty$. *Then*

$$\lim_{n\to\infty} \|H_n(w,f) - f\|_{L^p_u} = 0 \tag{1.7}$$

holds for all $f \in C[-1,1]$ *if and only if*

$$\begin{cases} p\alpha - a - 1 < 0, & \\ p\beta - b - 1 < 0, & \alpha \geq -\frac{1}{2}, \\ p\beta - b - 1 \leq p(\alpha + \frac{1}{2}), & \alpha < -\frac{1}{2}. \end{cases} \tag{1.8}$$

It is worth considering the important special case when $a = -\frac{1}{2}$ and $b = 0$. In this case for each $p > 2$ if α and β satisfy

$$\alpha < -\frac{1}{2}, \quad \beta \leq 0, \quad \beta - \alpha > \frac{1}{2} + \frac{1}{p} \tag{1.9}$$

i.e.,

$$\alpha < -\frac{1}{p} - \frac{1}{2}, \quad \alpha + \frac{1}{p} + \frac{1}{2} < \beta \leq 0, \tag{1.10}$$

then (1.8) is not valid. Hence by this theorem there must exist an $f \in C[-1,1]$ such that (1.7) does not hold. Meanwhile it is easy to see that $w(x) \geq 1, x \in [-1,1]$. Thus we have

COROLLARY. *For every* $p > 2$ *there exists a weight*

$$v(x) \geq 1 \tag{1.11}$$

and a function $f \in C[-1,1]$ *such that*

$$\limsup_{n\to\infty} \int_{-1}^{1} |H_n(v,f,x) - f(x)|^p \, dx > 0.$$

To explain the importance of this result let us state Problem 17 of P. Turán in [5, p. 40].

PROBLEM 17. Is condition (1.11) sufficient to assure

$$\lim_{n\to\infty} \int_{-1}^{1} [H_n(v,f,x) - f(x)]^2 dx = 0?$$

Although this problem is still open, we can now say that if the power 2 is replaced by $p > 2$ then the condition (1.11) is not sufficient to assure L^p convergence for every continuous function.

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2. **Preliminary Results.** We need the following notations. Here and later the symbols const, c, c_1, etc. denote some positive constants, not necessarily the same in different formulas.

$$\begin{cases} m := [\frac{n}{2}], s := n - 2m, \\ W(x) := (1-x)^{\alpha}(1+x)^{\beta}, \\ W_n(x) := (1-x)^s W(x), \\ U(x) := (1-x)^{\alpha}(1+x)^{b}, \end{cases} \tag{2.1}$$

$$\begin{cases} x_{km} := x_{km}(W_n) := \cos\theta_{km}, \\ x_{0m} := \cos\theta_{0m} = 1, \\ x_{m+1,m} := \cos\theta_{m+1,m} = -1, \\ |x - x_j| = \min_{0\le k\le m+1} |x - x_k|, \quad 0 \le j \le m+1. \end{cases} \tag{2.2}$$

The following results are taken from [1, 6–9].

$$P_n(w,x) = c_n x^s P_m(W_n, 1-2x^2). \tag{2.3}$$

For convenience of use, the zeros of $P_n(w,x)$ are denoted by

$$y_{kn} = -y_{-k,n} = \begin{cases} \left(\frac{1-x_{km}}{2}\right)^{\frac{1}{2}}, & k = 1,2,\ldots,m, \\ 0, & k = 0, \quad n = 2m+1, \end{cases} \tag{2.4}$$

$$l_{kn}(w,x) = \begin{cases} -\frac{x^s P_m(W_n, 1-2x^2)}{4y_{kn}^{s+1} P'_m(W_n, x_{|k|,m})(x-y_{kn})}, & k = \pm 1, \pm 2, \ldots, \pm m, \\ \frac{P_m(W_n, 1-2x^2)}{P_m(W_n, 1)}, & k = 0, \quad n = 2m+1, \end{cases} \tag{2.5}$$

$$v_{kn}(w,x) = \begin{cases} 1 + \left[4y_{kn}\frac{\alpha-\beta+(\alpha+\beta+2)x_{|k|,m}}{1-x^2_{|k|,m}} - \frac{1}{y_{kn}}\right](x - y_{kn}), & k = \pm 1, \pm 2, \ldots, \pm m, \\ 1, & k = 0, \quad n = 2m+1, \end{cases} \tag{2.6}$$

$$v_{km}(W_n, x) = 1 - \left[\frac{\alpha + s - \beta + (\alpha + s + \beta + 2)x_{km}}{1 - x_{km}^2}\right](x - x_{km}), \qquad k = 1, 2, \ldots, m, \tag{2.7}$$

$$\theta_{k+1,m} - \theta_{km} \sim \frac{1}{m}, \qquad k = 0, 1, \ldots, m, \tag{2.8}$$

$$|x - x_{km}| \sim \frac{|k-j| \min\{k+j, 2m+2-k-j\}}{m^2}, \qquad 1 \le k \le m, \quad k \ne j, \tag{2.9}$$

$$P'_m(W, x_{km}) \sim mW(x_{km})^{-1/2}(1 - x_{km}^2)^{-3/4}, \tag{2.10}$$

$$|P_m(W,x)| \le \text{const} \begin{cases} [W(x)(1-x^2)^{1/2}]^{-1/2}, & |x| \le 1 - m^{-2}, \\ m^{1/2} W(1 - m^{-2})^{-1/2}, & 1 - m^{-2} \le x \le 1, \\ m^{1/2} W(-1 + m^{-2})^{-1/2}, & -1 \le x \le -1 + m^{-2} \end{cases} \tag{2.11}$$

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uniformly for $m \geq 2$, and

$$(2.12)\quad |P_m(W,x)| \sim \begin{cases} m|x-x_{jm}|[W(x)(1-x^2)^{3/2}]^{-1/2}, & -1+x_{mm} \leq 2x \leq 1+x_{1m}, \\ m^{1/2}W(1-m^{-2})^{-1/2}, & 1+x_{1m} \leq 2x \leq 2, \\ m^{1/2}W(-1+m^{-2})^{-1/2}, & -2 \leq 2x \leq -1+x_{mm} \end{cases}$$

uniformly for $m \geq 2$.

REMARK. Neither the formula given in [8, (3.4)]

$$|x-x_k| \sim \frac{(K+J)|K-J|}{m^2}, \qquad K = \min\{k, m+1-k\}, \qquad J = \min\{j, m+1-j\}$$

nor the formula given in [4, (4.4)] $|x-x_k| \sim |k^2-j^2|/m^2$ is true, the right one should be of the form (2.9).

In their nice paper [3], Nevai and Vértesi presented several theorems about weighted mean convergence of $H_m(W,f)$. To apply their results and ideas, we first discuss relationship between $H_n(w,f)$ and $H_m(W_n,f)$.

LEMMA 1. *We have*

$$(2.13)\qquad l_{kn}(w,x) = \frac{x^s(x+y_{kn})l_{|k|,m}(W_n, 1-2x^2)}{2y_{kn}^{s+1}}, \qquad k = \pm 1, \pm 2, \ldots, \pm m.$$

PROOF. By (2.5) we get

$$l_k(w,x) = -\frac{x^s(x+y_k)P_m(W_n, 1-2x^2)}{4y_k^{s+1}P'_m(W_n, x_{|k|})(x^2-y_k^2)} = \frac{x^s(x+y_k)l_{|k|}(W_n, 1-2x^2)}{2y_k^{s+1}}.$$

Put $y := (\frac{1-x}{2})^{\frac{1}{2}}$, $F_i(x) := f(y)/y^i$, $i = 0, 1, 2, 3$. Then we have

LEMMA 2. *If $f \in C[-1,1]$ is an even function then*

$$(2.14)\qquad \|H_n(w,f)-f\|_{L^p_u} = c_1\Big\| y^{2s}\{H_m(W_n, F_{2s}, x) - F_{2s}(x)\} + sf(0)\frac{P_m(W_n,x)^2}{P_m(W_n,1)^2} + s\sum_{k=1}^{m}\frac{y^{2s}F_{2s}(x_{km})(x-x_{km})l_{km}(W_n,x)^2}{2y_{kn}^2}\Big\|_{L^p_U}.$$

If $f \in C[-1,1]$ is an odd function then

$$(2.15)\qquad \|H_n(w,f)-f\|_{L^p_u} = c_2\Big\| y^{2s+1}\Big\{H_m(W_n, F_{2s+1}, x) - F_{2s+1}(x) + \sum_{k=1}^{m}\frac{(2s+1)y^{2s+1}F_{2s+1}(x_{km})(x-x_{km})l_{km}(W_n,x)^2}{4y_{kn}^2}\Big\}\Big\|_{L^p_U}.$$

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PROOF. We give the proof of (2.15) only, the one of (2.14) being similar. Using the transformation $z = (\frac{1-x}{2})^{\frac{1}{2}}$ and noting that the weights $w(x)$ and $u(x)$ are even, by (2.13) we obtain ($\sum_{|k|=1}^{m}$ stands for $\sum_{k=-m,k\neq 0}^{m}$)

$$\begin{aligned}
&\|H_n(w,f)-f\|_{L_u^p}\\
&= 2\int_0^1\Big|\sum_{|k|=1}^{m} f(y_k)v_k(w,z)l_k(w,z)^2 - f(z)\Big|^p u(z)\,dz\\
&= c_2\int_{-1}^1\Big|\sum_{k=1}^{m}\frac{f(y_k)y^{2s}l_k(W_n,x)^2}{4y_k^{2s+2}}[v_k(w,y)(y+y_k)^2 - v_{-k}(w,y)(y-y_k)^2]\\
&\qquad - F_0(x)\Big|^p U(x)\,dx\\
&= c_2\int_{-1}^1\Big|\sum_{k=1}^{m}\frac{f(y_k)y^{2s}l_k(W_n,x)^2}{4y_k^{2s+2}}\Big[4y_k y v_k(W_n,x) + \frac{(2s+1)y(x-x_k)}{y_k}\Big] - F_0(x)\Big|^p U(x)\,dx\\
&= c_2\Big\|y^{2s+1}\Big\{H_m(W_n,F_{2s+1},x) - F_{2s+1}(x) + \sum_{k=1}^{m}\frac{(2s+1)F_{2s+1}(x_k)(x-x_k)l_k(W_n,x)^2}{4y_k^2}\Big\}\Big\|_{L_U^p}^p.
\end{aligned}$$

Next we give some estimates.

LEMMA 3. *If (1.8) is valid then*

$$\lim_{m\to\infty}\Big\|\frac{P_m(W_1,x)^2}{P_m(W_1,1)^2}\Big\|_{L_U^p} = 0. \tag{2.16}$$

PROOF. By Theorem 6.3.14 in [2, p. 113] for every $0 < p < \infty$ and Jacobi weight U there exists a constant $\sigma = \sigma(p,U) > 0$ such that for every polynomial P of degree at most $2m$

$$\int_{-1}^1 |P(x)|^p U(x)\,dx \le 2\int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} |P(x)|^p U(x)\,dx. \tag{2.17}$$

Thus by (2.17) and (2.11)

$$\begin{aligned}
\Big\|\frac{P_m(W_1,x)^2}{P_m(W_1,1)^2}\Big\|_{L_U^p}^p &= \int_{-1}^1\Big|\frac{P_m(W_1,x)^2}{P_m(W_1,1)^2}\Big|^p U(x)\,dx\\
&\le 2\int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}}\Big|\frac{P_m(W_1,x)^2}{P_m(W_1,1)^2}\Big|^p U(x)\,dx\\
&\le \text{const}\, m^{-p(2\alpha+3)}\int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}}[W_1(x)(1-x^2)^{1/2}]^{-p}U(x)\,dx\\
&\le \text{const}\Big[m^{-p(2\alpha+3)} + m^{-2a-2} + m^{2(p\beta-p\alpha-p-b-1)}\Big] = o(1).
\end{aligned}$$

LEMMA 4. *Let* $0 < \sigma < 1$. *Then*

$$\begin{aligned}
&\sum_{k=1}^{m}\frac{y^{2s+t}|x-x_{km}|l_{km}(W_n,x)^2}{y_{kn}^{2s+t+1}}\\
&\le \text{const}\Big[\frac{1}{m^{2\alpha+2}} + \frac{\ln m}{mW(x)(1-x^2)^{1/2}}\Big] \quad (t=0,1)
\end{aligned} \tag{2.18}$$

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holds uniformly for $m \geq 2$ *and* $|x| \leq 1 - \sigma m^{-2}$,

$$\sum_{k=1}^{m} \frac{y^{2s}|x - x_{km}| l_{km}(W_n, x)^2}{y_{kn}^{2s+2}(1 + x_{km})} \leq \text{const}\left[1 + \frac{\ln m}{mW(x)(1 - x^2)^{1/2}}\right] \tag{2.19}$$

holds uniformly for $m \geq 2$ *and* $|x| \leq 1 - \sigma m^{-2}$, *and*

$$\sum_{k=1}^{m} \frac{y^{2s+1} l_{km}(W_n, x)^2}{y_{kn}^{2s+1}} \leq \text{const}\left[1 + \frac{\ln m}{mW(x)(1 - x^2)^{1/2}}\right] \tag{2.20}$$

holds uniformly for $m \geq 2$ *and* $|x| \leq 1 - \sigma m^{-2}$.

PROOF. We use a modification of the proof of Lemma 3.1 in [9]. We give the proofs of (2.18) and (2.20) only, the one of (2.19) being similar.

Let us show (2.18) and let S denote the left side of (2.18). Assume that $0 \leq x \leq 1 - \sigma m^{-2}$. Then $j \leq \frac{m}{2}$, where j is defined in (2.2).

First by (2.8)–(2.12) we have

$$\begin{aligned} \frac{y^{2s+t}|x - x_j| l_j(W_n, x)^2}{y_j^{2s+t+1}} &\leq \text{const} \frac{W_n(x_j)(1 - x)^{s+t/2}(1 - x_j^2)^{3/2}}{mW_n(x)(1 - x^2)(1 - x_j)^{s+(t+1)/2}} \\ &\leq \text{const} \frac{(1 + x_j)^{1/2}}{m} \leq \text{const} \frac{1}{m}. \end{aligned}$$

Next using (2.8)–(2.11) yields

$$\begin{aligned} \sum_{k \neq j} \frac{y^{2s+t}|x - x_k| l_k(W_n, x)^2}{y_k^{2s+t+1}} &\leq \text{const} \sum_{k \neq j} \frac{W_n(x_k)(1 - x)^{s+t/2}(1 - x_k^2)^{3/2}}{m^2|x - x_k| W_n(x)(1 - x^2)^{1/2}(1 - x_k)^{s+(t+1)/2}} \\ &\leq \text{const} \sum_{k \neq j} \frac{(1 - x_k)^{\alpha+1-t/2}(1 + x_k)^{\beta+3/2}}{m^2|x_k - x_j|(1 - x_j)^{\alpha+(1-t)/2}(1 + x_j)^{\beta+1/2}} \\ &\leq \text{const} \frac{1}{m^{2\beta+4} j^{2\alpha-t+1}} \sum_{k \neq j} \frac{k^{2\alpha-t+2}(m + 1 - k)^{2\beta+3}}{|k - j| \min\{k + j, 2m + 2 - k - j\}} \\ &\leq \text{const}\left[\frac{1}{mj^{2\alpha-t+1}} \sum_{\substack{k \leq \frac{3m}{4} \\ k \neq j}} \frac{k^{2\alpha-t+2}}{|k^2 - j^2|} + \frac{m^{2\alpha-t}}{j^{2\alpha-t+1}}\right]; \end{aligned}$$

here we use the obvious relation $\min\{k + j, 2m + 2 - k - j\} \geq \frac{1}{3}(k + j)$ for $j \leq \frac{m}{2}$. Put

$$K_1 := \{k : k \leq \frac{1}{2}j\} \quad K_2 := \{k : \frac{1}{2}j < k \leq \frac{3}{2}j, k \neq j\}$$

$$K_3 := \{k : \frac{3}{2}j < k \leq \frac{3}{4}m\},$$

$$S_i = \sum_{k \in K_i} \frac{k^{2\alpha-t+2}}{|k^2 - j^2|}, \quad i = 1, 2, 3.$$

Then

$$S_1 \le \text{const} \sum_{k\in K_1} \frac{k^{2\alpha-t+2}}{j^2} \le \text{const} j^{2\alpha-t+1},$$

$$S_2 \le \text{const} \sum_{k\in K_2} \frac{j^{2\alpha-t+1}}{|k-j|} \le \text{const} j^{2\alpha-t+1} \ln j.$$

Since $k > \frac{3}{2}j$ implies $k - j > \frac{k}{3}$, one has

$$S_3 \le \text{const} j^{-t} \sum_{k\in K_3} k^{2\alpha} \le \begin{cases} \text{const} j^{-t}, & \alpha < -\frac{1}{2}, \\ \text{const} j^{-t} m^{2\alpha+1} \ln m, & \alpha \ge -\frac{1}{2}. \end{cases}$$

Thus for $\alpha < -\frac{1}{2}$ we have $S \le \text{const}\, m^{-1} j^{-2\alpha-1} \le \text{const}\, m^{-2\alpha-2}$ and for $\alpha \ge -\frac{1}{2}$

$$S \le \text{const} \frac{\ln m}{m} \left(\frac{m}{j}\right)^{2\alpha+1} \le \text{const} \frac{\ln m}{mW(x)(1-x^2)^{1/2}}.$$

For $-1 + \sigma m^{-2} \le x \le 0$ the proof runs similarly. This proves (2.18).

Now we turn to showing (2.20). Similarly, if $0 \le x \le 1 - \sigma m^{-2}$, then a simple computation shows $y^{2s+1} l_j(W_n, x)^2 / y_j^{2s+1} \le$ const. Meanwhile

$$\sum_{k\ne j} \frac{y^{2s+1} l_k(W_n,x)^2}{y_k^{2s+1}} \le \text{const} \frac{1}{m^{2\beta+3} j^{2\alpha}} \sum_{k\ne j} \frac{k^{2\alpha+2}(m+1-k)^{2\beta+3}}{(|k-j| \min\{k+j, 2m+2-k-j\})^2}$$
$$\le \text{const}\left[\frac{1}{j^{2\alpha}} \sum_{\substack{k\le \frac{3m}{4} \\ k\ne j}} \frac{k^{2\alpha+2}}{(k^2-j^2)^2} + \text{const} \frac{m^{2\alpha-1}}{j^{2\alpha}}\right].$$

Put

$$S_i = \sum_{k\in K_i} \frac{k^{2\alpha+2}}{(k^2-j^2)^2}, \quad i = 1,2,3.$$

Again we have

$$S_1 \le \text{const} \sum_{k\in K_1} \frac{k^{2\alpha+2}}{j^4} \le \text{const} j^{2\alpha-1}, \quad S_2 \le \text{const} \sum_{k\in K_2} \frac{j^{2\alpha}}{(k-j)^2} \le \text{const} j^{2\alpha},$$

$$S_3 \le \begin{cases} \text{const} j^{-2} \sum_{k\in K_3} k^{2\alpha} \le \text{const} j^{-2}, & \alpha < -\frac{1}{2}, \\ \text{const} j^{-1} \sum_{k\in K_3} k^{2\alpha-1} \le \text{const} j^{-1}(1 + m^{2\alpha} \ln m), & \alpha \ge -\frac{1}{2}. \end{cases}$$

Hence

$$\sum_{k=1}^{m} \frac{y^{2s+1} l_k(W_n,x)^2}{y_k^{2s+1}} \le \text{const}\left[1 + \frac{\ln m}{m}\left(\frac{m}{j}\right)^{2\alpha+1}\right] \le \text{const}\left[1 + \frac{\ln m}{mW(x)(1-x^2)^{1/2}}\right]$$

and (2.20) follows.

LEMMA 5. *If (1.8) is valid then ($\|\cdot\|$ stands for the uniform norm)*

$$\|H_n(w)\|_{L^\infty\to L^p_u} := \sup_{\|f\|=1} \|H_n(w,f)\|_{L^p_u} \le \text{const.} \tag{2.21}$$

PROOF. Since every function f may be written as a sum of an even function and an odd one, it suffices to show

$$\sup_{\substack{\|f\|=1\\ f(x)=f(-x)}} \|H_n(w,f)\|_{L^p_u} \le \text{const}$$

and

$$\sup_{\substack{\|f\|=1\\ f(x)=-f(-x)}} \|H_n(w,f)\|_{L^p_u} \le \text{const.} \tag{2.22}$$

Let us show the latter, the former is similar. Let N denote the left part in (2.22). Then by (2.15)

$$N^p \le \text{const} \sup_{\substack{\|f\|=1\\ f(x)=-f(-x)}} \int_{-1}^{1} \left| \sum_{k=1}^{m} \left[\frac{f(y_k)v_k(W_n,x)l_k(W_n,x)^2}{y_k^{2s+1}} + \frac{(2s+1)f(y_k)(x-x_k)l_k(W_n,x)^2}{4y_k^{2s+3}} \right] \right|^p U_1(x)\,dx,$$

where $U_1(x) := (1-x)^{p(s+1/2)}U(x)$. According to (2.17) with $\sigma = \sigma(p,U_1) > 0$, we obtain

$$\begin{aligned} N^p &\le \text{const} \sup_{\substack{\|f\|=1\\ f(x)=-f(-x)}} \int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} \left| \sum_{k=1}^{m} \left[\frac{f(y_k)v_k(W_n,x)l_k(W_n,x)^2}{y_k^{2s+1}} + \frac{(2s+1)f(y_k)(x-x_k)l_k(W_n,x)^2}{4y_k^{2s+3}} \right] \right|^p U_1(x)\,dx \\ &\le \text{const} \int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} \left\{ \sum_{k=1}^{m} \left[\frac{y^{2s+1}|v_k(W_n,x)|l_k(W_n,x)^2}{y_k^{2s+1}} + \frac{y^{2s+1}|x-x_k|l_k(W_n,x)^2}{y_k^{2s+3}} \right] \right\}^p U(x)\,dx. \end{aligned}$$

It follows from (2.7) that

$$\frac{y^{2s+1}|v_k(W_n,x)|l_k(W_n,x)^2}{y_k^{2s+1}} \le \text{const} \left[\frac{y^{2s+1}l_k(W_n,x)^2}{y_k^{2s+1}} + \frac{y^{2s+1}|v_k(W_n,x)|l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)} \right].$$

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Meanwhile

$$\frac{y^{2s+1}|x-x_k|l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)} \le \frac{y^{2s}|(y-y_k)(x-x_k)|l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)} + \frac{y^{2s}|x-x_k|l_k(W_n,x)^2}{y_k^{2s+2}(1+x_k)}$$
$$\le \frac{y^{2s}(x-x_k)^2 l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)|y+y_k|} + \frac{y^{2s}|x-x_k|l_k(W_n,x)^2}{y_k^{2s+2}(1+x_k)}.$$

Thus using the inequality

$$(|A|+|B|)^p \le 2^p(|A|^p+|B|^p) \tag{2.23}$$

yields

$$N^p \le \text{const} \int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} \Big\{\sum_{k=1}^{m} \frac{y^{2s}|x-x_k|l_k(W_n,x)^2}{y_k^{2s+2}(1+x_k)}\Big\}^p U(x)\,dx$$
$$+ \text{const} \int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} \Big\{\sum_{k=1}^{m} \frac{y^{2s+1}l_k(W_n,x)^2}{y_k^{2s+1}}\Big\}^p U(x)\,dx$$
$$+ \text{const} \int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} \Big\{\sum_{k=1}^{m} \frac{y^{2s}(x-x_k)^2 l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)|y+y_k|}\Big\}^p U(x)\,dx$$
$$:= N_1 + N_2 + N_3.$$

By (2.19) and (1.8)

$$N_1 \le \text{const} \int_{-1+\sigma m^{-2}}^{1-\sigma m^{-2}} \Big\{1 + (l_n m)\big[mW(x)(1-x^2)^{1/2}\big]^{-1}\Big\}^p U(x)\,dx$$
$$\le \text{const}\Big\{1 + (l_n m)^p m^{2(p\alpha-a-1)} + m^{2(p\beta-b-1)}\Big\} \le \text{const}.$$

Similarly $N_2 \le$ const.

Since $x \le 0$ means $y \ge 2^{-1/2}$, we have

$$N_3 \le \text{const} \int_{-1+\sigma m^{-2}}^{0} \Big\{\sum_{k=1}^{m} \frac{y^{2s}(x-x_k)^2 l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)}\Big\}^p U(x)\,dx$$
$$+ \text{const} \int_{0}^{1-\sigma m^{-2}} \Big\{\sum_{k=1}^{m} \frac{y^{2s}(x-x_k)^2 l_k(W_n,x)^2}{y_k^{2s+3}(1+x_k)|y+y_k|}\Big\}^p U(x)\,dx$$
$$:= I_1 + I_2.$$

A simple computation using (1.8)–(2.11) gives

$$I_1 \le \text{const}\Big[\sum_{k=1}^{m} \frac{1}{y_k^{2s+3}(1+x_k)P_m'(W_n,x_k)^2}\Big]^p \int_{-1+\sigma m^{-2}}^{0} |P_m(W_n,x)|^{2p}(1+x)^b\,dx$$
$$\le \text{const}[m^{-2\alpha-2} \sum_{k\le \frac{m}{2}} k^{2\alpha} + m^{-1}]^p[1 + m^{2p\beta+p-2b-2}] \le \text{const}.$$

If $\alpha \ge -\frac{1}{2}$ then

$$I_2 \le \text{const}\Big[\sum_{k=1}^{m} \frac{1}{y_k^{2s+4}(1+x_k)P_m'(W_n,x_k)^2}\Big]^p \int_{0}^{1-\sigma m^{-2}} |P_m(W_n,x)|^{2p}(1-x)^{a+ps}\,dx$$
$$\le \text{const}[m^{-2\alpha-1} \sum_{k\le \frac{m}{2}} k^{2\alpha-1} + m^{-1}]^p[1 + m^{2p\alpha+p-2a-2}] \le \text{const}.$$

If $\alpha < -\frac{1}{2}$ then

$$I_2 \le \text{const}\Big[\sum_{k\le \frac{m}{2}} \frac{1}{y_k^{2s+3}P_m'(W_n,x_k)^2}\Big]^p \int_0^{1-\sigma m^{-2}} |P_m(W_n,x)|^{2p}(1-x)^{a+p(s-\frac{1}{2})}\,dx$$

$$+\text{const}\Big[\sum_{k>\frac{m}{2}} \frac{1}{(1+x_k)P_m'(W_n,x_k)^2}\Big]^p \int_0^{1-\sigma m^{-2}} |P_m(W_n,x)|^{2p}(1-x)^{a+ps}\,dx$$

$$\le \text{const}\, m^{-2p(\alpha+1)}[1+m^{2(p\alpha+p-a-1)}] + \text{const}\, m^{-p}[1+m^{2p\alpha+p-2a-2}] \le \text{const}.$$

This proves (2.22). ∎

3. **Proof of Theorem.** Assume that (1.8) is true. Put $f_i := x^i, i = 0, 1, 2$.

Let $p \ge 1$. This formula (1.7) obviously holds for f_0. By the same argument used in Lemma 2 it follows from (2.19) that

$$\|H_n(w,f_1) - f_1\|_{L_u^p} = \Big\|\sum_{|k|=1}^m (z-y_k)l_k(w,z)^2\Big\|_{L_u^p}$$

$$\le \text{const}\Big\|\sum_{k=1}^m \frac{y^{2s+1}(x-x_k)l_{km}(W_n,x)^2}{y_k^{2s+2}}\Big\|_{L_U^p} = o(1).$$

Similarly

$$\|H_n(w,f_2) - f_2\|_{L_u^p} = \Big\|2\sum_{|k|=1}^m y_k(z-y_k)l_k(w,z)^2\Big\|_{L_u^p}$$

$$\le \text{const}\Big\|\sum_{k=1}^m \frac{y^{2s}(x-x_k)l_k(W_n,x)^2}{y_k^{2s}}\Big\|_{L_U^p} = o(1).$$

By Theorem 5 in [4] for every polynomial P of degree at most $2n-1$

$$\|H_n(w,P)-P\|_{L_u^p} \le (\|P'\| + \|P''\|)\|H_n(w,f_1)-f_1\|_{L_u^p} + \frac{1}{2}\|P''\|\,\|H_n(w,f_2)-f_2\|_{L_u^p} = o(1).$$

By means of the well known Banach theorem, (1.7) holds for all $f \in C[-1,1]$.

For $0 < p < 1$ applying Hölder's inequality

$$\|g\|_{L_u^p}^p = \|\,|g|^p u\|_{L^1} \le \|\,|g|u\|_{L^1}^p\,\|u\|_{L_1}^{1-p}$$

we can also get (1.7).

Conversely, assume that (1.7) holds. First, it is clear that each polynomial of x may be written as

$$F(x) = \sum_{k=0}^r a_k(1-x)^k = \sum_{k=0}^r a_k 2^k\left(\frac{1-x}{2}\right)^k = \sum_{k=0}^r a_k 2^k y^{2k} = f(y),$$

which is an even polynomial of y. Hence by (1.7) and (2.14)

$$\|H_m(W,F) - F\|_{L_U^p} \le \text{const}\,\|H_{2m}(w,f) - f\|_{L_u^p} = o(1).$$

By Theorem 4 in [3], this implies $W^{-1} \in L^p_U$, which is equivalent to the first two inequalities in (1.8).

Next, in order to prove the last inequality in (1.8) suppose to the contrary that $p\beta - b - 1 > p(\alpha + \frac{1}{2}), \alpha < -\frac{1}{2}$. It follows from (2.23) that

$$|A - B|^p \geq 2^{-p}|A|^p - |B|^p.$$

Applying this inequality, (2.6), and (2.7) by the same argument as in Lemma 2 we have

$$\begin{aligned}
&\|H_{2m}(w)\|^p_{L^\infty \to L^p_u} \\
&\geq \int_0^1 \Big|\sum_{|k|=1}^m (\operatorname{sgn} y_k) v_k(w,z) l_k(w,z)^2\Big|^p u(z)\,dz \\
&= c_1 \int_{-1}^1 \Big|\sum_{k=1}^m \Big\{\frac{y(x-x_k) l_k(W,x)^2}{y_k}\Big[\frac{\beta+1}{2(1-y_k^2)} - \frac{2\alpha+1}{4y_k^2}\Big] - \frac{y l_k(W,x)^2}{y_k}\Big\}\Big|^p U(x)\,dx \\
&\geq 2^{-p} c_1 \int_{-1}^{\frac{x_m-1}{2}} \Big|\sum_{k=1}^m \frac{|2\alpha+1| y(x-x_k) l_k(W,x)^2}{4y_k^3}\Big|^p U(x)\,dx \\
&\quad - c_1 \int_{-1}^1 \Big|\sum_{k=1}^m \frac{y l_k(W,x)^2}{y_k}\Big|^p U(x)\,dx \\
&:= Q_1 - Q_2
\end{aligned}$$

By (2.8)–(2.11) we get

$$\begin{aligned}
Q_1 &\sim \Big[\sum_{k=1}^m \frac{P_m(W,-1)^2}{y_k^3(1+x_k)P'_m(W,x_k)^2}\Big]^p \int_{-1}^{\frac{x_m-1}{2}} (1+x)^b\,dx \\
&\sim \Big[\sum_{k\leq\frac{m}{2}} m^{2\beta+1} \frac{k^{2\alpha+3}}{m^{2\alpha+5}} \frac{m^3}{k^3} + \sum_{k>\frac{m}{2}} m^{2\beta+1} \frac{(m-k+1)^{2\beta+3}}{m^{2\beta+5}} \frac{m^2}{(m-k+1)^2}\Big]^p m^{-2b-2} \\
&\sim \Big[m^{2\beta-2\alpha-1} \sum_{k\leq\frac{m}{2}} k^{2\alpha} + m^{-2} \sum_{k>\frac{m}{2}} (m-k+1)^{2\beta+1}\Big]^p m^{-2b-2} \\
&\sim m^{p(2\beta-2\alpha-1)-2b-2}.
\end{aligned}$$

Meanwhile we have shown by (2.20) in Lemma 5 that $Q_2 \leq \text{const} N_2 \leq \text{const}$ for $s = 0$. Thus as $m \longrightarrow \infty$

$$\|H_{2m}(w)\|^p_{L^\infty \to L^p_u} \longrightarrow \infty.$$

This contradiction proves the last inequality in (1.8).

This completes the proof.

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