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Introduction

We begin with some background material. First, we need to establish the formalism and definitions for the imaginary signals we will be shining on our imaginary detectors. Second, we will describe general detector characteristics so we can judge the merits of the various types as they are discussed. This discussion introduces some common metrics: (1) the quantum efficiency, i.e., the fraction of the incoming photon stream absorbed; (2) noise and the ratio of signal to noise; (3) the fidelity of images produced by a detector array or similar arrangement; and (4) the speed of response of a detector.

1.1 Radiometry

1.1.1 Concepts and Terminology

There are some general aspects of electromagnetic radiation that need to be defined before we discuss how it is detected. Figure 1.1 illustrates schematically a photon of light with terms used to describe it. One should imagine that time has been frozen, but that the photon has been moving at the speed of light in the direction of the arrow. We often discuss the photon in terms of wavefronts, lines marking the surfaces of constant phase and hence separated by one wavelength.

As electromagnetic radiation, a photon has both electric and magnetic components, oscillating in phase perpendicular to each other and perpendicular to the direction of energy propagation. The amplitude of the electric field, its wavelength and phase, and the direction it is moving characterize the photon. The behavior of the electric field can be expressed as

$$E = E_0 cos(\omega t + \phi), \tag{1.1}$$

where E_0 is the amplitude, ω is the angular frequency, and ϕ is the phase. Alternatively, the behavior is conveniently expressed in complex notation as

$$E(t) = E_0 e^{-j\omega t} = E_0 cos(\omega t) - jE_0 sin(\omega t), \tag{1.2}$$

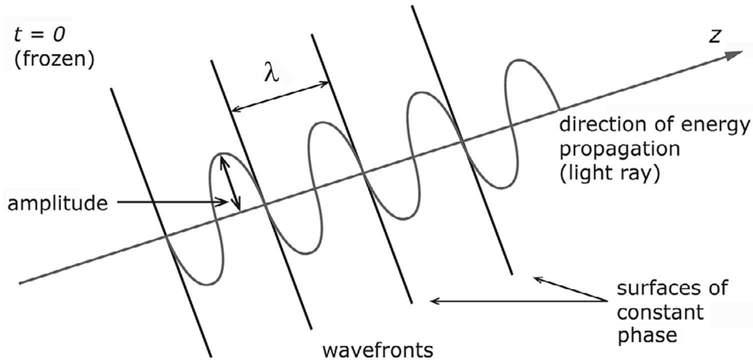

Figure 1.1 Terms describing the propagation of a photon, or a light ray.

where j is the imaginary square root of -1. In this case, the quantity is envisioned as a vector on a two-dimensional diagram with the real component along the usual x-axis and the imaginary one along the usual y-axis. The angle of this vector from the origin and relative to the positive real axis represents the phase.

Most of the time we will treat light as photons of energy; wave aspects will be important only for heterodyne detectors. A photon has an energy of

$$E_{ph} = h\nu = hc/\lambda, \tag{1.3}$$

where h ($= 6.626 \times 10^{-34}$ J s) is Planck's constant, ν and λ are, respectively, the frequency (in hertz = 1/seconds) and wavelength (in meters) of the electromagnetic wave, and c ($= 2.998 \times 10^8$ m s^{-1}) is the speed of light. In the following discussion, we define a number of expressions for the power output of sources of photons; conversion from power to photons per second can be achieved by dividing by the desired form of equation 1.3.

The spectral radiance per frequency interval, L_ν, is the power (in watts) leaving a unit projected area of the surface of the source (in square meters) into a unit solid angle (in steradians) and unit frequency interval (in hertz). The projected area of a surface element dA onto a plane perpendicular to the direction of observation is $dA\ cos\theta$, where θ is the angle between the direction of observation and the outward normal to dA; see Figure 1.2. L_ν has units of W m^{-2} Hz^{-1} ster^{-1}. The spectral radiance per wavelength interval, L_λ, has units of W m^{-3} ster^{-1}. The radiance, L, is the spectral radiance integrated over all frequencies or wavelengths; it has units of W m^{-2} ster^{-1}. The radiant exitance, M, is the integral of the radiance over solid angle, and it is a measure of the total power emitted per unit surface area in units of W m^{-2}.

We will deal only with Lambertian sources; the defining characteristic of such a source is that its radiance is constant regardless of the direction from which it is viewed. A blackbody is one example. The emission of a Lambertian source goes as

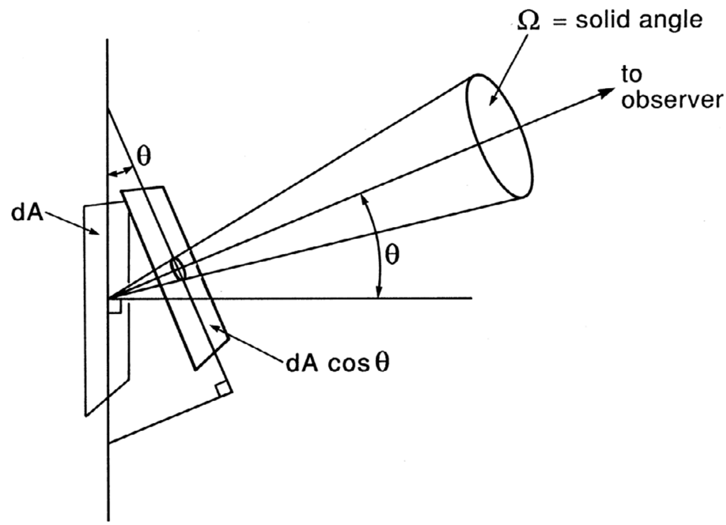

Figure 1.2 Geometry for computing radiance.

the cosine of the angle between the direction of the radiation and the normal to the source surface. From the definition of projected area in the preceding paragraph, it can be seen that this emission pattern exactly compensates for the foreshortening of the surface as it is tilted away from being perpendicular to the line of sight. That is, for the element dA, the projected surface area and the emission decrease by the same cosine factor. Thus, if the entire source has the same temperature and emissivity, every unit area of its projected surface in the plane perpendicular to the observer's line of sight appears to be of the same brightness, independent of its actual angle to the line of sight. Keeping in mind this cosine dependence, and the definition of radiant exitance, the radiance and radiant exitance are related as

$$M = \int L \, cos\,\theta \, d\Omega = 2\pi L \int_0^{\pi/2} sin\,\theta \, cos\,\theta \, d\theta = \pi L. \tag{1.4}$$

The flux emitted by the source, Φ, is the radiant exitance times the total surface area of the source, that is the power emitted by the entire source. For example, for a spherical source of radius R,

$$\Phi = 4\pi R^2 M = 4\pi^2 R^2 L. \tag{1.5}$$

Although there are other types of Lambertian sources, we will consider only sources that have spectra resembling those of blackbodies, for which the spectral radiance in frequency units is

$$L_\nu = \frac{\varepsilon \left[2h\nu^3/(c/\mathrm{n})^2\right]}{e^{h\nu/kT} - 1}, \tag{1.6}$$

where ε is the emissivity of the source, n is the refractive index of the medium into which the source radiates, and k ($= 1.38 \times 10^{-23}$ J K^{-1}) is the Boltzmann constant. The emissivity (ranging from 0 to 1) is the efficiency with which the source radiates compared to that of a perfect blackbody, which by definition has $\varepsilon = 1$. According to Kirchhoff's law, the absorption efficiency, or absorptivity, and the emissivity are equal for any source. In wavelength units, the spectral radiance is

$$L_\lambda = \frac{\varepsilon\left[2h(c/\mathrm{n})^2\right]}{\lambda^5\left(e^{hc/\lambda kT}-1\right)}. \tag{1.7}$$

It can be easily shown from equations 1.6 and 1.7 that the spectral radiances are related as follows:

$$L_\lambda = \left(\frac{c}{\lambda^2}\right)L_\nu = \left(\frac{\nu}{\lambda}\right)L_\nu. \tag{1.8}$$

According to the Stefan–Boltzmann law, the radiant exitance for a blackbody becomes

$$\begin{aligned} M &= \pi\int_0^\infty L_\nu d\nu = \frac{2\pi k^4T^4}{c^2h^3}\int_0^\infty \frac{x^3}{e^x-1}dx \\ &= \frac{2\pi^5k^4}{15c^2h^3}T^4 = \sigma T^4, \end{aligned} \tag{1.9}$$

where σ ($= 5.67 \times 10^{-8}$ W m^{-2} K^{-4}) is the Stefan–Boltzmann constant.

For Lambertian sources, the optical system feeding a detector will receive a portion of the source power that is determined by a number of geometric factors as illustrated in Figure 1.3. The system will accept radiation from only a limited range of directions determined by the geometry of the optical system as a whole and known as the field of view. The area of the source that is effective in producing a signal is determined by the field of view and the distance from the optical system to the source (or by the size of the source if it all lies within the field of view). This area will emit radiation with some angular dependence. Only the radiation that is emitted in directions where it is intercepted by the optical system can be detected. The range of directions accepted is determined by the solid angle, Ω, that the entrance aperture of the optical system subtends as viewed from the source. In addition, some of the emitted power may be absorbed or scattered by any medium through which it propagates to reach the optical system. For a Lambertian source, the power this system receives is then the radiance in its direction multiplied by the source area within the system field of view, multiplied by the solid angle subtended by the optical system as viewed from the source, and multiplied by the transmittance of the optical path from the source to the system.

Although a general treatment must allow for the field of view to include only a portion of the source, in many cases of interest the entire source lies within the

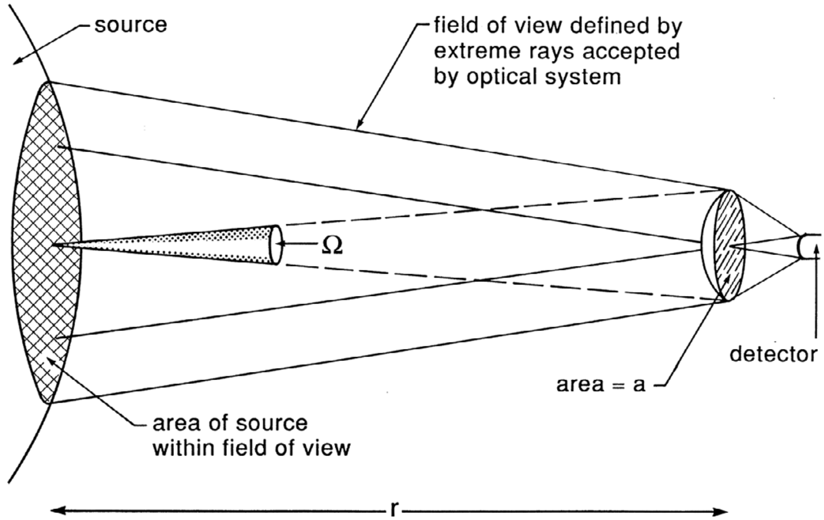

Figure 1.3 Geometry for computing power received by a detector system.

field of view, so the full projected area of the source is used in calculating the signal. For a spherical source of radius R, this area is πR^2. The solid angle subtended by the detector system is

$$\Omega = \frac{\mathrm{a}}{r^2}, \tag{1.10}$$

where a is the area of the entrance aperture of the system (strictly speaking, a is the projected area; we have assumed the system is pointing directly at the source) and r is its distance from the source. For a circular aperture,

$$\Omega = 4\pi\ sin^2(\theta/2), \tag{1.11}$$

where θ is the half-angle of the right circular cone whose base is the detector system entrance aperture, and whose vertex lies on a point on the surface of the source; r is the height of this cone.

It is particularly useful when the angular diameter of the source is small compared with the field of view of the detector system to consider the irradiance, E, which is the power in watts per square meter received at a unit surface element at some distance from the source. For the case described in the preceding paragraph, the irradiance is obtained by first multiplying the radiant exitance (from equation 1.4) by the total surface area of the source, A, to get the flux, $A\pi L$. The flux is then divided by the area of a sphere of radius r centered on the source to give

$$E = \frac{AL}{4r^2}, \tag{1.12}$$

where r is the distance of the source from the irradiated surface element. The spectral irradiance, E_ν or E_λ, is the irradiance per unit frequency or wavelength interval. It is also sometimes called the flux density, and is a very commonly used description of the power received from a source. It can be obtained from equation 1.12 by substituting L_ν or L_λ for L.

The radiometric quantities discussed above are summarized in Table 1.1. Equations are provided for illustration only; in some cases, these examples apply only to specific circumstances. The terminology and symbolism vary substantially from one discipline to another; for example, the last two columns of the table translate some of the commonly used radiometric terms into astronomical nomenclature.

1.1.2 The Detection Process

Only a portion of the power received by the optical system is passed on to the detector. The system will have inefficiencies due to both absorption and scattering of energy in its elements, and because of optical aberrations and diffraction. These effects can be combined into a system transmittance term. In addition, the range of frequencies or wavelengths to which the system is sensitive (that is, the spectral bandwidth of the system in frequency or wavelength units) is usually restricted by a spectral filter plus a combination of characteristics of the detector and other elements of the system as well as by any spectral dependence of the transmittance of the optical path from the source to the entrance aperture. A rigorous accounting of the spectral response requires that the spectral radiance of the source be multiplied point-by-point by the spectral transmittances of all the spectrally active elements in the optical path to the detector, and by the detector spectral response, and the resulting function subsequently integrated over frequency or wavelength to determine the total power effective in generating a signal.

In cases where the spectral response is restricted to a range of wavelengths by a bandpass optical filter in the beam, it is generally useful to define the effective wavelength[1] of the system as

$$\lambda_0 = \frac{\int_0^\infty \lambda \, T(\lambda) \, d\lambda}{\int_0^\infty T(\lambda) \, d\lambda}, \tag{1.13}$$

where $T(\lambda)$ is the spectral transmittance of the system. Often the spectral variations of the other transmittance terms can be ignored over the restricted spectral range of the filter. The bandpass of the filter, $\Delta\lambda$, can be taken to be the full width at half maximum (FWHM) of its transmittance function (see Figure 1.4). If the filter cuts

[1] We have characterized the response using the mean wavelength; there are a number of other conventions, but for our purposes the differences are minor and unimportant.

Table 1.1 *Definitions of radiometric quantities*

Symbol	Name	Definition	Units	Equation	Alternate name	
L_ν	Spectral radiance (frequency units)	Power leaving unit projected surface area into unit solid angle and unit frequency interval	W m^{-2} Hz^{-1} ster^{-1}	(1.6)	Specific intensity (frequency units)	I_ν
L_λ	Spectral radiance (wavelength units)	Power leaving unit projected surface area into unit solid angle and unit wavelength interval	W m^{-3} ster^{-1}	(1.7)	Specific intensity (wavelength units)	I_λ
L	Radiance	Spectral radiance integrated over frequency or wavelength	W m^{-2} ster^{-1}	$L = \int L_\nu d\nu$	Intensity or specific intensity	I
M	Radiant exitance	Power emitted per unit surface area	W m^{-2}	$M = \int L(\theta) d\Omega$		
Φ	Flux	Total power emitted by source of area A	W	$\Phi = \int M\, dA$	Luminosity	L
E	Irradiance	Power received at unit surface element; equation applies well removed from the source at distance r	W m^{-2}	$E = \frac{\int M\, dA}{(4\pi r^2)}$		
E_ν, E_λ	Spectral irradiance	Power received at unit surface element per unit frequency or wavelength interval	W m^{-2} Hz^{-1}, W m^{-3}		Flux density	S_ν, S_λ

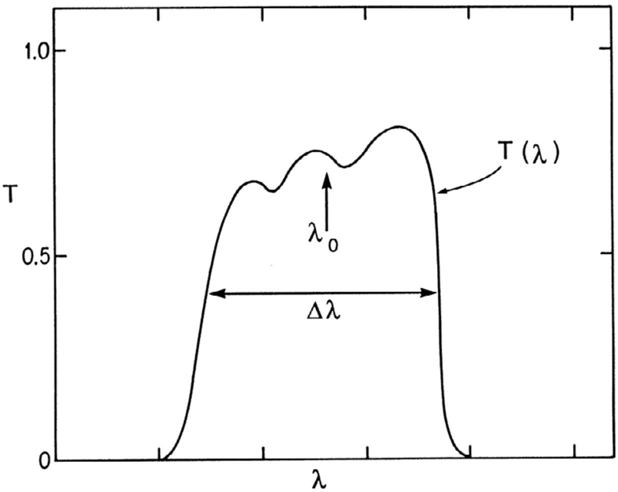

Figure 1.4 Transmittance function $T(\lambda)$ of a filter. The FWHM $\Delta\lambda$ and the effective wavelength λ_0 are indicated.

on and off sharply, its transmittance can be approximated as the average value over the range $\Delta\lambda$:

$$T_F = \frac{\int_{\Delta\lambda} T(\lambda)\, d\lambda}{\Delta\lambda}. \tag{1.14}$$

If $\Delta\lambda/\lambda_0 \leq 0.2$ and the filter cuts on and off sharply, the power effective in generating a signal can usually be estimated in a simplified manner. The signal transmitted by the bandpass filter can be approximated by taking the spectral radiance at λ_0 and multiplying it by $\Delta\lambda$ and the average filter transmittance over the range $\Delta\lambda$. Of course, to obtain the net signal that reaches the detector, this result is multiplied by the various geometric and transmittance terms already discussed for the remainder of the system. However, if λ_0 is substantially shorter than the peak wavelength of the blackbody curve (that is, one is operating in the Wien region of the blackbody) or there is sharp spectral structure within the passband, then this approximation can lead to significant errors, particularly if $\Delta\lambda/\lambda_0$ is relatively large.

Continuing with the approximation just discussed, we can derive a useful expression for estimating the power falling on the detector:

$$P_D \approx \frac{A_{proj}\ \mathrm{a}\ T_P(\lambda_0)\ T_O(\lambda_0)\ T_F\ L_\lambda(\lambda_0)\ \Delta\lambda}{r^2}. \tag{1.15}$$

Here A_{proj} is the area of the source projected onto the plane perpendicular to the line of sight from the source to the optical receiver. T_P, T_O, and T_F are the transmittances, respectively, of the optical path from the source to the receiver, of the receiver optics (excluding the bandpass filter), and of the bandpass filter. The area of the receiver entrance aperture is a, and the distance of the receiver from the source is r. An analogous expression holds in frequency units. The major underlying assumptions for equation 1.15 are (a) the field of view of the receiver includes the entire source; (b) the source is a Lambertian emitter; and (c) the spectral response of the detector is limited by a filter with a narrow or moderate bandpass that is sharply defined.

1.2 Detector Types

Nearly all detectors act as transducers that receive photons and produce an electrical response that can be amplified and converted into a form intelligible to suitably conditioned human beings. There are three basic ways that detectors carry out this function:

(a) *Photodetectors* respond directly to individual photons. An absorbed photon releases one or more bound charge carriers in the detector that may (1) modulate the electric current in the material; (2) move directly to an output amplifier; or (3) lead to a chemical change. The most common photodetectors are based on semiconducting materials and are used throughout the X-ray, ultraviolet, visible, and infrared spectral regions. Examples that we will discuss are photoconductors (Chapters 2 and 3), photodiodes (Chapter 3), charge coupled devices (CCDs) (Chapter 5), photographic materials (Chapter 6), photoemissive detectors (Chapter 6), and quantum well detectors (Chapter 6), plus some less common examples scattered about these chapters. The sheer number of types of semiconductor photodetectors provides an indication of their broad application. The unique properties of superconductors enable additional types of photodetector with applications in the submillimeter/millimeter wavelength or with the potential to provide spectral resolution within the detection process. Chapter 7 discusses two examples, microwave kinetic inductance detectors (MKIDs), and superconducting tunnel junctions (STJs).

(b) *Thermal detectors* absorb photons and thermalize their energy. In most cases, this energy changes the electrical properties of the detector material, resulting in a modulation of the electric current passing through it. Thermal detectors have a very broad and nonspecific spectral response, but they are particularly important at infrared and submillimeter wavelengths, and as X-ray detectors. Bolometers and other thermal detectors will be discussed in Chapter 8.

(c) *Coherent detectors* respond to the electric field strength of the signal and can preserve phase information about the incoming photons. They operate by interference of the electric field of the incident photon with the electric field from a local oscillator. These detectors are primarily used in the radio and submillimeter regions but also have specialized applications in the visible and infrared.

Coherent detectors for the visible and infrared are discussed in Chapter 9, and those for the submillimeter are discussed in Chapter 10.

1.3 Performance Characteristics

Good detectors preserve a large proportion of the information contained in the incoming stream of photons. A variety of parameters are relevant to this goal:

(a) *Spectral response* – the total wavelength or frequency range over which photons can be detected with reasonable efficiency.

(b) *Spectral bandwidth* – the wavelength or frequency range over which photons are detected at any one time; some detectors can operate in one or more bands placed within a broader range of spectral response.

(c) *Linearity* – the degree to which the output signal is proportional to the number of incoming photons that were received to produce the signal.

(d) *Dynamic range* – the maximum variation in signal over which the detector output represents the photon flux without losing significant amounts of information.

(e) *Quantum efficiency* – the fraction of the incoming photon stream that is converted into signal.

(f) *Noise* – the uncertainty in the output signal. Ideally, the noise consists only of statistical fluctuations due to the finite number of photons producing the signal.

(g) *Imaging properties* – e.g., the number of detectors ("pixels") in an array. Because signal may blend from one pixel to adjacent ones, the resolution that can be realized may be less, however, than indicated just by the pixel count.

(h) *Time response* – the minimum interval of time over which the detector can distinguish changes in the photon arrival rate.

The first two items in this listing should be clear from our discussion of radiometry, and the next two are more or less self-explanatory. However, the remaining entries include subtleties that call for more discussion.

1.3.1 Quantum Efficiency

To be detected, photons must be absorbed. The absorption coefficient in the detector material is indicated as $a(\lambda)$ and conventionally has units of cm^{-1}. The absorption length is defined as $1/a(\lambda)$. The absorption of a flux of photons, S, passing through a differential thickness element dl is expressed by

$$\frac{dS}{dl} = -a(\lambda)S, \tag{1.16}$$

with the solution for the remaining flux at depth l being

$$S = S_0 e^{-a(\lambda)l}. \tag{1.17}$$

The portion of the flux absorbed by the detector divided by the flux that enters it is

$$\eta_{ab} = \frac{S_0 - S_0 e^{-a(\lambda)d_1}}{S_0} = 1 - e^{-a(\lambda)d_1}, \tag{1.18}$$

where d_1 is the thickness of the detector. The quantity η_{ab} is known as the absorption factor. The quantum efficiency, η, is the flux absorbed in the detector divided by the total flux incident on its surface. Photons are lost by reflection from the surface before they enter the detector volume, leading to a reduction in quantum efficiency below η_{ab}. Minimal reflection occurs for photons striking a nonabsorptive material at normal incidence:

$$r = \frac{(\mathrm{n} - 1)^2 + \left(\frac{a(\lambda)\ \lambda}{4\pi}\right)^2}{(\mathrm{n} + 1)^2 + \left(\frac{a(\lambda)\ \lambda}{4\pi}\right)^2}, \tag{1.19}$$

where r is the fraction of the incident flux of photons that is reflected, n is the refractive index of the material, $a(\lambda)$ is the absorption coefficient at wavelength λ, and we have assumed that the photon is incident from air or vacuum, which have a refractive index of n = 1. Reflection from the back of the detector can result in absorption of photons that would otherwise escape. If we ignore this potential gain, the net quantum efficiency is

$$\eta = (1 - r)\eta_{ab}. \tag{1.20}$$

1.3.2 Noise and Signal to Noise

The arriving photons carry a level of information that we want the detector to preserve so far as possible. We now discuss the implications of this requirement.

Ignoring minor corrections having to do with the quantum nature of photons, it can be assumed that the input photon flux follows Poisson statistics,

$$P(m) = \frac{e^{-n} n^m}{m!}, \tag{1.21}$$

where $P(m)$ is the probability of detecting m photons in a given time interval, and n is the average number of photons detected in this time interval if a large number of detection experiments is conducted. The root-mean-square noise in a number of independent events, each of which has an expected noise N, is the square root of the mean, n,

$$N_{rms} = \left\langle N^2 \right\rangle^{1/2} = n^{1/2}. \tag{1.22}$$

The errors in the detected number of photons in two experiments can usually be taken to be independent, and hence they add quadratically. That is, the noise in two measurements yielding n_1 and n_2 events, respectively, is

$$N_{rms} = \langle N^2 \rangle^{1/2} = \left[\left(n_1^{1/2} \right)^2 + \left(n_2^{1/2} \right)^2 \right]^{1/2} = (n_1 + n_2)^{1/2} . \tag{1.23}$$

From the above discussion, the signal-to-noise ratio for Poisson-distributed events is $n/n^{1/2}$, or

$$S/N = n^{1/2}. \tag{1.24}$$

This result can be taken to be a measure of the information content of the incoming photon stream.[2]

The quantum efficiency is the fraction of incoming photons converted into useful signal in the first stage of detector action. In the simplest form, if the detector converts an individual photon into a single, mobile charge carrier that is collected as the signal, the quantum efficiency is the ratio of the number of charge carriers freed to the number of photons received. For our simple detector example, photons that do not free charge carriers cannot contribute to either signal or noise; they might as well not exist. The portion of information they were carrying is therefore lost. Consequently, for n photons incident on the detector, equation 1.24 shows that the signal-to-noise ratio goes as $\eta n/(\eta n)^{1/2}$, or

$$\left(\frac{S}{N} \right)_d = (\eta n)^{1/2} \tag{1.25}$$

in the ideal case where both signal and noise are determined only by the photon statistics, where η is the quantum efficiency and the d subscript is to indicate that this value applies just to the detector itself.

Additional steps in the detection process can degrade the information present in the photon stream absorbed by the detector, either by losing signal or by adding noise. The detective quantum efficiency (*DQE*) describes this degradation succinctly in terms of the number of photons that could produce an output signal with an equivalent ratio of signal to noise if no degradation occurred. We define the detective quantum efficiency as

$$DQE = \frac{n_{out}}{n_{in}} = \frac{(S/N)^2_{out}}{(S/N)^2_{in}}, \tag{1.26}$$

[2] A more rigorous description of photon noise takes account of the Bose–Einstein nature of photons, which causes the arrival times of individual particles to be correlated. See the note at the end of the chapter for further discussion, including why this issue can usually be ignored.

where n_{in} is the actual input photon signal, and n_{out} is an imaginary input signal that would produce, with a perfect detector system, the same information content in the output signal as is received from the actual system. Converting to signal to noise, $(S/N)_{out}$ is the observed signal-to-noise ratio, while $(S/N)_{in}$ is the potential signal-to-noise ratio of the incoming photon stream, as given by equation 1.24. By substituting equations 1.24 and 1.25 into equation 1.26, it is easily shown that the *DQE* is just the quantum efficiency defined in equation 1.20 if there is no subsequent degradation of the signal to noise.

1.3.3 Imaging Properties

The resolution of an array of detectors can be most simply measured by exposing it to a pattern of alternating white and black lines (a "bar chart") and determining the minimum spacing of line pairs that can be distinguished, as illustrated in Figure 1.5. The eye can identify such a pattern if the light–dark variation is 4% or greater. The resolution of the detector array is expressed in line pairs per millimeter corresponding to the highest density of lines that produces a pattern at this threshold.

Although it is relatively easy to measure resolution in this way for the detector array alone, a resolution in line pairs per millimeter is difficult to combine with resolution estimates for other components in an optical system used with it. For example, how would one derive the net resolution for a camera with a lens and photographic film whose resolutions are both given in line pairs per millimeter? A second shortcoming is that the performance in different situations can be poorly represented by the line pairs per millimeter specification. For example, one might have two lenses, one of which puts 20% of the light into a sharply defined image core and spreads the remaining 80% widely, whereas the second puts all the light

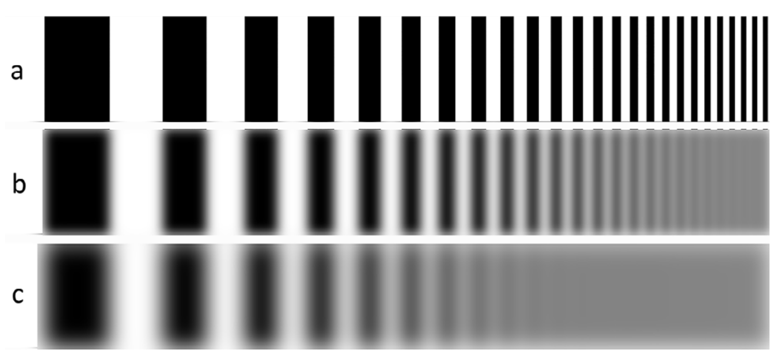

Figure 1.5 Bar chart test of resolution: (a) shows the bar chart with no degradation, while (b) and (c) show the blurring due to the optical system, obviously with lower resolution in (c) than in (b).

into a slightly less well defined core. These systems might achieve identical resolutions in line pairs per millimeter (which requires only 4% modulation), yet they would perform quite differently in other situations.

A more general concept is the modulation transfer function, or *MTF*. Imagine that the detector array is exposed to a field with a sinusoidal spatial variation of the intensity of the input, of period P and amplitude $F(x)$,

$$F(x) = a_0 + a_1 sin(2\pi f x), \tag{1.27}$$

where $f = 1/P$ is the spatial frequency, x is the distance along one axis of the array, a_0 is the mean height (above zero) of the pattern, and a_1 is its amplitude. These terms are indicated in Figure 1.6(a). The modulation of this signal is defined as

$$M_{in} = \frac{F_{max} - F_{min}}{F_{max} + F_{min}} = \frac{a_1}{a_0}, \tag{1.28}$$

where F_{max} and F_{min} are the maximum and minimum values of $F(x)$. Assuming that the resulting image output from the detector is also sinusoidal (which may be only approximately true due to nonlinearities), it can be represented by

$$G(x) = b_0 + b_1 sin(2\pi f x), \tag{1.29}$$

where x and f are the same as in equation 1.27, and b_0 and b_1 are analogous to a_0 and a_1. The modulation in the image will be

$$M_{out} = \frac{b_1}{b_0} M_{in}. \tag{1.30}$$

The modulation transfer factor is

$$MT = \frac{M_{out}}{M_{in}}. \tag{1.31}$$

A separate value of the *MT* will apply at each spatial frequency; Figure 1.6(a) illustrates an input signal that contains a range of spatial frequencies, and Figure 1.6(b) shows a corresponding output in which the modulation decreases with increasing spatial frequency. This frequency dependence of the *MT* is expressed in the modulation transfer function (*MTF*). Figure 1.7 shows the *MTF* corresponding to the response of Figure 1.6(b).

In principle, the *MTF* provides a reasonably complete specification of the imaging properties of a detector array.[3] However, one must be aware that the *MTF* may vary over the face of the array and may have color dependence. In addition, the *MTF* omits time-dependent imaging properties, such as latent images that may persist after the image of a bright source has been put on the array and removed.

[3] For optical systems in general, the complete description including phase information is provided by the optical transfer function (*OTF*); the *MTF* is the magnitude of the *OTF*, while the phase is provided by the phase transfer function (*PTF*). For simple photodetectors, the *PTF* can usually be ignored.

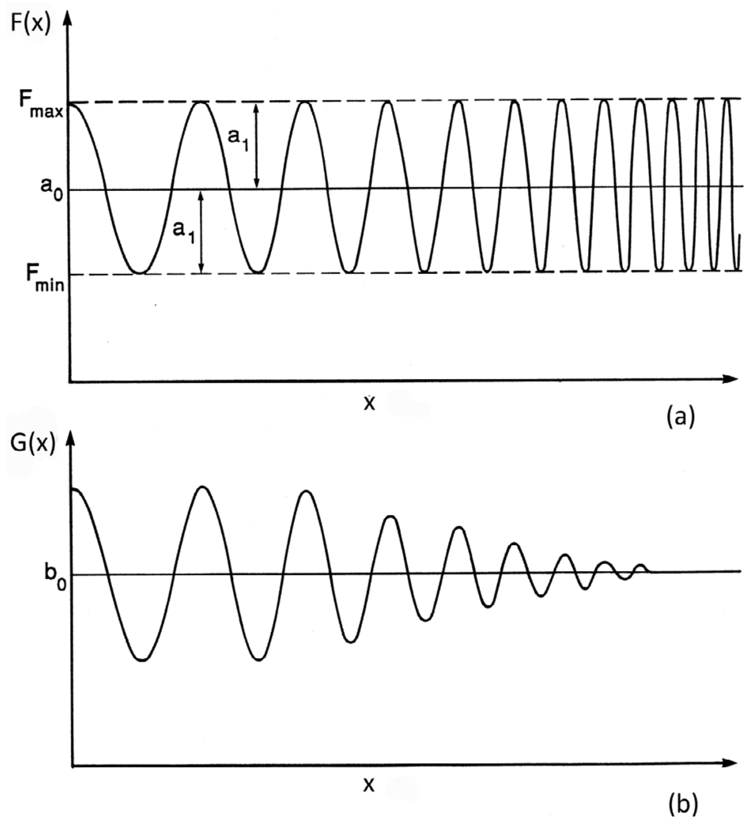

Figure 1.6 Illustration of variation of modulation with spatial frequency: (a) sinusoidal input signal of constant amplitude but varying spatial frequency; (b) how an imaging detector system might respond to this signal.

Computationally, the *MTF* can be determined by taking the absolute value of the Fourier transform, $F(u)$, of the image of a perfect point source. This image is called the point spread function. Fourier transformation is the general mathematical technique used to determine the frequency components of a function $f(x)$ (see, for example, Bracewell 2000; Press et al. 2007). $\mathbf{F}(u)$ is defined as

$$\mathbf{F}(u) = \int_{-\infty}^{\infty} f(x)e^{j2\pi ux}dx, \tag{1.32}$$

with inverse

$$f(x) = \int_{-\infty}^{\infty} \mathbf{F}(u)e^{-j2\pi xu}du, \tag{1.33}$$

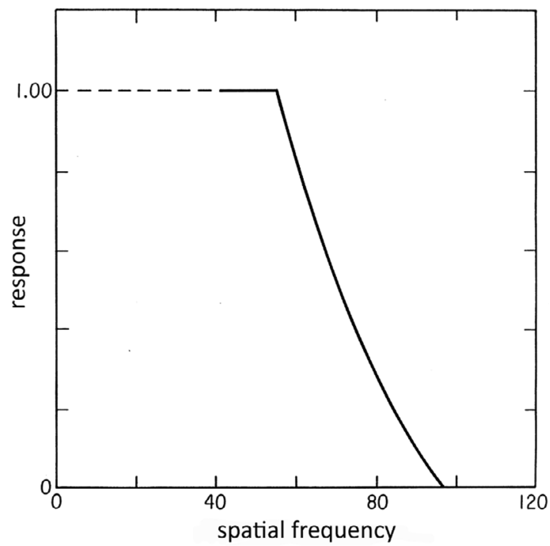

Figure 1.7 The modulation transfer function (*MTF*) for the response illustrated in Figure 1.6(b)

where j is the (imaginary) square root of -1. In the current discussion, $f(x)$ is a functional representation of the point spread function, and u is the spatial frequency. The Fourier transform can be generalized in a straightforward way to two dimensions, but for the sake of simplicity we will not do so here. The absolute value of the transform is

$$|\mathbf{F}(u)| = (\mathbf{F}(u)\mathbf{F}^*(u))^{1/2}, \tag{1.34}$$

where $\mathbf{F}^*(u)$ is the complex conjugate of $\mathbf{F}(u)$; it is obtained by reversing the sign of all imaginary terms in $\mathbf{F}(u)$.

If $f(x)$ is the point spread function, $|\mathbf{F}(u)|/|\mathbf{F}(0)|$ is the *MTF*. This formulation holds because a sharp impulse contains all frequencies equally and hence the Fourier transform of the point spread function gives the spatial frequency response of the detector. The *MTF* is normalized to unity at spatial frequency 0 by this definition. As emphasized in Figure 1.7, the response at zero frequency cannot be measured directly but must be extrapolated from higher frequencies.

Only a relatively small number of functions have Fourier transforms that are easy to manipulate. Table 1.2 contains a short compilation of some of these cases. With the use of computers, however, Fourier transformation is a powerful and very general technique.

The *MTF* of an entire linear optical system can be determined by multiplying together the *MTF*s of its constituent elements. The multiplication occurs on

Table 1.2 *Fourier transforms*

$f(x)$	$F(u)$
$F(x)$	$f(-u)$
$aF(x)$	$aF(u)$
$f(ax)$	$(1/\lvert a\rvert)F(u/a)$
$f(x)+g(x)$	$F(u)+G(u)$
1	$\delta(u)^c$
$e^{-\pi x^2}$	$e^{-\pi u^2}$
$e^{-\lvert x\rvert}$	$2/(1+(2\pi u)^2)$
$e^{-x}, x>0$	$(1-j2\pi u)/(1+(2\pi u)^2)$
$sech(\pi x)$	$sech(\pi u)$
$\lvert x\rvert^{-1/2}$	$\lvert u\rvert^{-1/2}$
$sgn(x)^a$	$-j/(\pi u)$
$e^{-\lvert x\rvert}\, sgn(x)$	$-j4\pi u/(1+(2\pi u)^2)$
$\Pi(x)^b$	$sin(\pi u)/\pi u$

$^{(a)}$ $sgn(x) = -1$ for $x < 0$ and $= 1$ for $x \geq 0$.
$^{(b)}$ $\Pi(x) = 1$ for $|x| < 1/2$ and $= 0$ otherwise.
$^{(c)}$ $\delta(u) = 0$ for $u \neq 0$, $\int \delta(u)du = 1$; that is, $\delta(u)$ is a spike at $u = 0$.

a frequency by frequency basis, that is, if the first system has $MTF_1(f)$ and the second $MTF_2(f)$, the combined system has $MTF(f) = MTF_1(f)MTF_2(f)$. The overall resolution capability of complex optical systems can be easily determined in this way. In addition, the *MTF* gives a complete description of the imaging behavior of detectors and even of many linear optical systems as opposed to single parameter descriptions that may be equivalent for systems having significantly different resolution characteristics.

1.3.4 Frequency Response

The response speed of a detector can be described very generally by specifying the dependence of its output on the frequency of an imaginary photon signal that varies sinusoidally in time. This concept is analogous to the modulation transfer function described just above with regard to imaging.

A variety of factors limit the frequency response. Many of them, however, can be described by an exponential time response, such as that of a resistor/capacitor electrical circuit. To be specific in the following, we will assume that the response is given by the *RC* time constant of such a circuit, although we will find other uses for the identical formalism later. If the capacitor is in parallel with the resistance, charge deposited on the capacitance bleeds off through the resistance with an exponential time constant

$$\tau_{RC} = RC. \tag{1.35}$$

Sometimes a "rise time" is specified rather than the exponential time constant. The rise time is the interval required for the output to change from 10% to 90% of its final value (measured relative to the initial value). For an exponential response, the rise time is 2.20 τ_{RC}.

Let a voltage impulse be deposited on the capacitor,

$$v_{in}(t) = v_0 \delta(t), \tag{1.36}$$

where v_0 is a constant and $\delta(t)$ is the delta function (see footnotes to Table 1.2). We can observe this event in two ways. First, we might observe the voltage across the resistance and capacitance directly, for example with an oscilloscope. It will have the form

$$v_{out}(t) = \left[\begin{array}{rl} 0, & t < 0, \\ \frac{v_0}{\tau_{RC}} e^{-t/\tau_{RC}}, & t \geq 0. \end{array} \right. \tag{1.37}$$

The same event can be analyzed in terms of the effect of the circuit on the input frequencies rather than on the time dependence of the voltage. To do so, we convert the input and output voltages to frequency spectra by taking their Fourier transforms. The delta function contains all frequencies at equal strength, that is, from Table 1.2,

$$V_{in}(f) = v_0 \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = v_0. \tag{1.38}$$

Since the frequency spectrum of the input is flat ($V_{in}(f)$ = constant), any deviations from a flat spectrum in the output must arise from the action of the circuit. That is, the output spectrum gives the frequency response of the circuit directly. Again from Table 1.2, it is

$$\begin{aligned} V_{out}(f) &= \int_{-\infty}^{\infty} v_{out}(t) e^{-j2\pi f t} dt \\ &= v_0 \left[\frac{1 - j2\pi f \tau_{RC}}{1 + (2\pi f \tau_{RC})^2} \right]. \end{aligned} \tag{1.39}$$

The imaginary part of $V_{out}(f)$ represents phase shifts that can occur in the circuit. For a simple discussion, we can ignore the phase and describe the strength

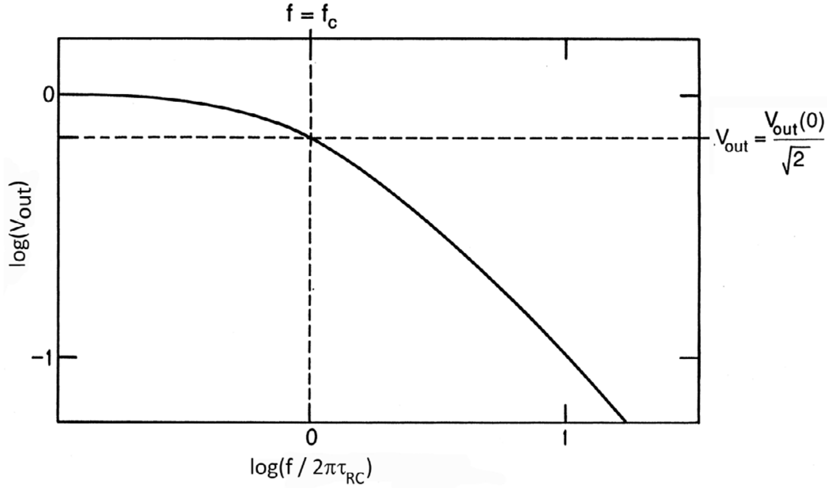

Figure 1.8 Frequency response of an *RC* circuit. The cutoff frequency is illustrated.

of the signal only in terms of the frequency dependence of its amplitude. The amplitude can be determined by taking the absolute value of $V_{out}(f)$:

$$
\begin{aligned}
|V_{out}(f)| &= \left(V_{out}V^{*}{}_{out}\right)^{1/2} \\
&= \frac{v_0}{\left[1+(2\pi f\tau_{RC})^2\right]^{1/2}},
\end{aligned}
\tag{1.40}
$$

where V^{*}_{out} is the complex conjugate of V_{out}. This function is plotted in Figure 1.8. As with the *MTF*, the effects of different circuit elements on the overall frequency response can be determined by multiplying their individual response functions together. The frequency response is often characterized by a cutoff frequency

$$f_c = \frac{1}{2\pi\tau_{RC}}, \tag{1.41}$$

at which the amplitude drops to $1/\sqrt{2}$ of its value at $f = 0$, or

$$|V_{out}(f_c)| = \frac{1}{\sqrt{2}}|V_{out}(0)|. \tag{1.42}$$

1.4 Radiometry Example

A 1000 K spherical blackbody source of radius 1 m is viewed in air by a detector system from a distance of 1000 m. The entrance aperture of the system has a radius of 5 cm, and the optical system has a field of view half-angle of 0.1°. The detector operates at a wavelength of 1 μm with a spectral bandpass of 1%, and its optical system is 50% efficient. Compute the spectral radiances in both frequency and wavelength units. Calculate the corresponding spectral irradiances at the detector entrance aperture, and the power received by the detector. Compare the usefulness of radiances and irradiances for this situation. Compute the number of photons hitting the detector per second. Describe how these answers would change if the blackbody source were 10 m in radius rather than 1 m.

The refractive index of air is n $\sim$ 1, so the spectral radiance in frequency units is given by equation 1.6 with $\varepsilon = \mathrm{n} = 1$. From equation 1.3, the frequency corresponding to 1 μm is $\nu = c/\lambda = 2.998 \times 10^{14}$ Hz. Substituting into equation 1.6, we find that

$$L_\nu = 2.21 \times 10^{-13}\,\mathrm{W\,m^{-2}Hz^{-1}ster^{-1}}. \tag{1.43}$$

Alternatively, we can substitute the wavelength of 1×10^{-6} m into equation 1.7 to obtain

$$L_\lambda = 6.62 \times 10^{7}\,\mathrm{W\,m^{-3}ster^{-1}}. \tag{1.44}$$

The solid angle subtended by the detector system as viewed from the source is given by equation 1.10. The area of the entrance aperture is 7.854×10^{-3} m^2, so

$$\Omega = 7.854 \times 10^{-9}\mathrm{ster}. \tag{1.45}$$

The 1% bandwidth corresponds to $0.01 \times 2.998 \times 10^{14}$ Hz $= 2.998 \times 10^{12}$ Hz, or to $0.01 \times 1 \times 10^{-6}$ m $= 1 \times 10^{-8}$ m. The radius of the area accepted into the beam of the detector system at the distance of the source is 1.745 m, and, since it is larger than the radius of the source, the entire visible area of the source will contribute to the signal. The projected area of the source is 3.14 m^2 (since it is a Lambertian emitter, no further geometric corrections are required for its effective emitting area). Then, computing the power at the entrance aperture of the detector system by multiplying the spectral radiances by the source area (projected), spectral bandwidth, and solid angle received by the system, we obtain $P = 1.63 \times 10^{-8}$ W.

Because the angular diameter of the source is less than the field of view, it is equally convenient to use the irradiance. The surface area of the source is 12.57 m^2. Using equation 1.12 and frequency units, we obtain

$$E_\nu = 6.945 \times 10^{-19}\,\mathrm{W\,m^{-2}\,Hz^{-1}}. \tag{1.46}$$

Similarly for wavelength units,

$$E_\lambda = 2.08 \times 10^{2}\,\mathrm{W\,m^{-3}}. \tag{1.47}$$

Multiplying by the bandpass and entrance aperture area yields a power of 1.63×10^{-8} W, as before.

The power received by the detector is reduced by optical inefficiencies to 50% of the power incident on the entrance aperture, so it is 8.2×10^{-9} W. The energy per photon can be computed from equation 1.3 to be 1.99×10^{-19} J. The detector therefore receives 4.12×10^{10} photons s^{-1}.

If the blackbody source were 10 m in radius, the spectral radiances, L_ν and L_λ, would be unchanged. The irradiances, E_ν and E_λ, would increase in proportion to the surface area of the source, so they would be 100 times larger than computed above. The field of view of the optical system, however, no longer includes the entire source; therefore, the power at the system entrance aperture is most easily computed from the spectral radiances, where the relevant surface area is that within the field of view and hence has a radius of 1.745 m. The power at the entrance aperture therefore increases by a factor of only 3.05, giving $P = 4.97 \times 10^{-8}$ W, as do the power falling on the detector (2.48×10^{-8} W) and the photon rate (1.25×10^{11} photons s^{-1}).

1.5 Problems

1.1 A spherical blackbody source at 300 K and of radius 0.1 m is viewed from a distance of 1000 m by a detector system with an entrance aperture of radius 1 cm and field of view half angle of 0.1 degree.

(a) Compute the spectral radiances in frequency units at 1 and 10 μm.
(b) Compute the spectral irradiances at the entrance aperture.
(c) For spectral bandwidths 1% of the wavelengths of operation and assuming that 50% of the incident photons are absorbed in the optics before they reach the detector, compute the powers received by the detector.
(d) Compute the numbers of photons hitting the detector per second.

1.2 Consider a detector with an optical receiver of entrance aperture 2 mm diameter, optical transmittance (excluding bandpass filter) of 0.8, and field of view 1° in diameter. This system views a blackbody source of 1000 K with an exit aperture of diameter 1 mm and at a distance of 2 m. The signal out of the blackbody is interrupted by a shutter at a temperature of 300 K. The receiver system is equipped with two bandpass filters, one with $\lambda_0 = 20\ \mu m$ and $\Delta\lambda = 1\ \mu m$ and the other with $\lambda_0 = 2\ \mu m$ and $\Delta\lambda = 0.1\ \mu m$; both have transmittances of 0.8. The transmittance of the air between the source and receiver is 1 at both wavelengths. Compute the net signal at the detector, that is compute the change in power incident on the detector as the shutter is opened and closed.

1.3 For blackbodies, the wavelength of the maximum spectral irradiance times the temperature is a constant, or

$$\lambda_{max} T = C. \tag{1.48}$$

This expression is known as the Wien displacement law; derive it. For wavelength units, show that $C \sim 0.3$ cm K.

1.4 Show that for $h\nu/kT \ll 1$ (setting $\varepsilon = \text{n} = 1$),

$$L_\nu = 2kT\nu^2/c^2. \tag{1.49}$$

This expression is the Rayleigh–Jeans law and is a useful approximation at long wavelengths. For a source temperature of 100 K, compute the shortest wavelength for which the Rayleigh–Jeans law is within 20% of the result given by equation 1.6. Compare with λ_{max} from Problem 1.3.

1.5 Derive equation 1.11. Note the particularly simple form for small θ.

1.6 Consider a bandpass filter that has a transmittance of zero outside the passband $\Delta\lambda$ and a transmittance that is the same for all wavelengths within the passband. Compare the estimate of the signal passing through this filter when the signal is determined by integrating the source spectrum over the filter passband with that where only the effective wavelength and FWHM bandpass are used to characterize the filter. Assume a source radiating in the Rayleigh–Jeans regime. Show that the error introduced by the simple effective wavelength approximation is a factor of

$$1 + \frac{5}{6}\left(\frac{\Delta\lambda}{\lambda_0}\right)^2 \tag{1.50}$$

plus terms of order $(\Delta\lambda/\lambda_0)^4$ and higher. Evaluate the statement in the text that the approximate method usually gives acceptable accuracy for $\Delta\lambda/\lambda_0 \leq 0.2$.

1.7 From equation 1.51, show that the Bose–Einstein correction to the rms photon noise $\langle N^2\rangle^{1/2}$ is less than 10% if $(5\varepsilon\tau\eta kT/h\nu) < 1$. Consider a blackbody source at $T = 1000$ K viewed by a detector system with optical efficiency 50% and quantum efficiency 50%. Calculate the wavelength beyond which the correction to the noise would exceed 10%. Compare this wavelength with that at the peak of the source output.

1.8 Compute the Fourier transform of $f(x) = H(x) + sech(10x)$, where $H(x) = 0$ for $x < 0$ and $= 1$ for $x \geq 0$.

1.6 Note

This note discusses the correction to simple noise estimates due to the boson nature of photons. Equation 1.24 is derived using the assumption that the particles arrive completely independently; the bunching of Bose–Einstein particles increases the noise above this estimate. The full description of photon noise shows it to be

$$\langle N^2\rangle = n\left[1 + \frac{\varepsilon\tau\eta}{e^{h\nu/kT} - 1}\right], \tag{1.51}$$

where $\langle N^2 \rangle$ is the mean square noise, n is the average number of photons detected, h is Planck's constant, ε is the emissivity of the source of photons, τ is the transmittance of the system optics, η is the detector quantum efficiency, ν is the photon frequency, k is Boltzmann's constant, and T is the absolute temperature of the photon source (see van Vliet 1967). Comparing with equation 1.24, it can be seen that the term in square brackets in equation 1.51 is a correction factor for the increase in noise from the Bose–Einstein behavior. It becomes important only at frequencies much lower than that of the peak emission of the blackbody, and then only for highly efficient detector systems. In most cases of interest, particularly with realistic instrument efficiencies, this correction factor is sufficiently close to unity that it can be ignored. Moreover, the entire theory of this noise behavior is rather complex and the predicted phenomena are yet to be observed (Lee and Talghader 2018).

1.7 Further Reading

Boreman (2001) – good introduction to the *MTF* and its applications
Bukshtab (2019) – advanced and very extensive treatment of photometry and radiometry
Grant (2011) – short and practical guide to practice of radiometry
Grum and Becherer (1979) – classic description of radiometry
McCluney (2014) – a good introduction to radiometry
Palmer and Grant (2009) – excellent introduction to radiometry, starting from first principles
Press et al. (2007) – thorough and practical general description of numerical methods, including Fourier transformation