Note of Newton's Theorem of Symmetric Functions.

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It can be shown as follows that Newton's Theorem can be derived from elementary considerations without making use of the idea of an equation and its roots.

Let F.S. denote indiscriminately any function of $a_1, a_2 \ldots a_r$ which can be expressed in terms of Σa_1, $\Sigma a_1 a_2$, $\Sigma a_1 a_2 a_3$, etc.

Then $\Sigma a_1 = \Sigma a_1 = \text{F.S.}$

$$\text{F.S.} = (\Sigma a_1)^2 = \Sigma a_1^2 + 2\Sigma a_1 a_2 = \Sigma a_1^2 + \text{F.S.}$$

$\therefore$ $\Sigma a_1^2 = \text{F.S.}$

$$(\Sigma a_1)^3 = (\Sigma a_1^2 + \text{F.S.})\Sigma a_1$$
$$= \Sigma a_1^3 + \Sigma a_1^2 a_2 + \text{F.S.}$$

and $\therefore$ *if* $\Sigma a_1^2 a_2 = \text{F.S.}$ we have

$$\Sigma a_1^3 = \text{F.S.}$$

and $(\Sigma a_1)^3 = \Sigma a_1^3 + \text{F.S.}$

Similarly $(\Sigma a_1)^4 = \Sigma a_1^4 + \Sigma a_1^3 a_2 + \text{F.S.}$

$= \Sigma a_1^4 + \text{F.S.}$ if $\Sigma a_1^3 a_2$ be F.S., and so on.

... ...

Assume this is true up to Σa_1^r, to show it is true for Σa_1^{r+1}.

We have $(\Sigma a_1)^{r-1} = \text{F.S.} = \Sigma a_1^{r-1} + \Sigma a_1^{r-2} a_2 + \text{F.S.}$

$$(\Sigma a_1)^r = \quad = \Sigma a_1^r + \Sigma a_1^{r-1} a_2 + \text{F.S.}$$
$$= \Sigma a_1^r + \text{F.S. by assumption,}$$

while $\Sigma a_1^{r-2} a_2$; $\Sigma a_1^{r-1} a_2$ are F.S.

Now $(\Sigma a_1)^{r+1} = \text{F.S.} \quad = (\Sigma a_1)^r \Sigma a_1 = \Sigma a_1^{r+1} + \Sigma a_1^r a_2 + \text{F.S.}$

if then we can show $\Sigma a_1^r a_2 = \text{F.S.}$ so will also Σa_1^{r+1}.

Now $(\Sigma a_1)^{r+1} = \Sigma a_1 (\Sigma a_1)^r = \Sigma a_1 \{\Sigma a_1^r + \Sigma a_1^{r-1} a_2 + \text{F.S.}\}$

$$= \Sigma a_1^{r+1} + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2^2$$
$$+ \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$
$$= (\Sigma a_1^{r+1} + \Sigma a_1^{r-1} a_2^2) + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$
$$= \Sigma a_1^{r-1} \quad \Sigma a_1^2 \quad + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$
$$= \text{F.S.} \quad + 2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 + \text{F.S.}$$

$\therefore$ $2\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 = \text{F.S.}$ ... ... (1).

Again, $(\Sigma a_1^{r-1})\Sigma a_1 a_2 = \text{F.S.}$

i.e., $\Sigma a_1^r a_2 + \Sigma a_1^{r-1} a_2 a_3 = \text{F.S.}$ ... ... (2).

From (1) and (2) by subtraction

$$\Sigma a_1^r a_2 = \text{F.S.} - \text{F.S.} = \text{F.S.}$$

$\therefore$ $\Sigma a_1^{r+1} = (\Sigma a_1)^{r+1} - \text{F.S.} - \text{F.S.}$

$$= \text{F.S.}$$

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